ALGEBRAS AND MATRICES FOR ANNOTATED LOGICS*

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Abstract

We study the matrices, reduced matrices and algebras associated to the systems $\mathcal{SAL}_\tau$ of structural annotated logics. In previous papers, these systems were proven algebraizable in the finitary case and the class of matrices analyzed here was proven to be a matrix semantics for them.

We prove that the equivalent algebraic semantics associated with the systems $\mathcal{SAL}_\tau$ are proper quasivarieties, we describe the reduced matrices, the subdirectly irreducible algebras and we give a general decomposition theorem. As a consequence we obtain a decision procedure for these logics.

1 Introduction

Annotated logics were introduced in [14] by V. S. Subrahmanian as logical foundations for computer programming. Blair and Subrahmanian [2] proved that these systems were paraconsistent and showed that they could form the basis of a programming language for reasoning about data bases that contain inconsistent information. Subsequently, numerous applications in Artificial Intelligence, like inheritance networks, object oriented data bases, etc. have been developed.

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A complete study of these systems, from the model theoretical and proof theoretical points of view has been done in [1] and [8]. These papers show that most classical basic results in model theory can be adapted to these systems. However, since there are several kinds of axioms (some for complex formulas, others for atomic formulas and still others for arbitrary formulas), these systems are not structural in the sense that their consequence relation is not closed under substitutions. One of the difficulties with non-structural systems is that they do not have a natural algebraic counterpart.

In [11] we introduced a structural version S$P\mathcal{T}$ of annotated logics built with the purpose of obtaining systems as closely related as possible to the original systems $P\mathcal{T}$ of [8], but to which the techniques of algebraic logic could be applied. The cost of this process was that we had to introduce several unary operation symbols to the language of $P\mathcal{T}$ and the corresponding axioms to control these operations. Then we proved that they are equivalent to the original $P\mathcal{T}$ systems, in the sense that there is a translation from the language of $P\mathcal{T}$ into that of $SP\mathcal{T}$ such that a formula is provable in $P\mathcal{T}$ if and only if it is provable in $SP\mathcal{T}$ using some additional premises. For details, see [11]. Finally we studied the algebraizability of these systems using as main framework the theory of algebraization of deductive systems developed by Blok and Pigozzi in [3]. The main result proven in [11] is that these systems are algebraizable. Moreover, we proved that an annotated logic is finitely algebraizable (i.e., it has a finite system of congruence formulas) if and only if the lattice of annotation constants is finite.

All axioms and inference rules of $SP\mathcal{T}$ are either (a translation of) an axiom or a rule of the systems $P\mathcal{T}$, or are necessary to define the new operations or to obtain the above mentioned equivalence of the systems. Even so, in the finite case, in $\mathcal{SAL}_\tau$ we had to add a new axiom for negations, axiom $(\neg_4)$, because one of the weaknesses of $P\mathcal{T}$ is that the axioms and rules pay little or no attention to the peculiarities of the lattice $\mathcal{T}$ or of the particular function $\sim$ that is being used.

In order to take these into account in [13] we proposed the systems $\mathcal{SAL}_\tau$, *structural annotated logic based on* $\mathcal{T}$, (for finite $\mathcal{T}$,) which are axiomatic extensions of the $SP\mathcal{T}$. This immediately implies that the systems $\mathcal{SAL}_\tau$ are algebraizable. An easy verification shows that an equivalence of translations similar to that between $SP\mathcal{T}$ and $P\mathcal{T}$ also holds for $\mathcal{SAL}_\tau$ and $P\mathcal{T}$. We then defined a special class of matrices and proved that it is a matrix semantics for $\mathcal{SAL}_\tau$.

In close connection with these systems, [12] proposes a (family of) semantically defined annotated logic. This system differs from $\mathcal{SAL}_\tau$ in the treatment of the paraconsistent elements and in the effect of the annotation operators over the atomic formulas. Even though it is an unpublished paper, [12] antecedes the present one, and some of the ideas, particularly the axioms (τ_7) and (τ_8), are already present there.

In this paper we study $\mathcal{SAL}_\tau$'s equivalent algebraic semantics $\mathbf{SAL}_\tau$. In Section 2 we introduce the system $\mathcal{SAL}_\tau$ and give some motivations for the connectives added and for the new axioms. In section 3 we define the set of *nice matrices*, a subset of the class of all $\mathcal{SAL}_\tau$–matrices. In Section 4 we characterize the reduced nice matrices and give two examples that illustrate the process. Using the reduced nice matrices, in Section 5 we begin the study of $\mathbf{SAL}_\tau$. The main results are that $\mathbf{SAL}_\tau$ is a proper quasi–variety, that it is generated by a single algebra and a characterization of the subdirectly irreducible algebras is also presented.

2 The Systems $\mathcal{SAL}_\tau$

In this Section we will define structural annotated deductive systems $\mathcal{SAL}_\tau$ that are axiomatic extensions of the systems SP$\mathcal{T}$ introduced in [11].

Let $\mathcal{T}$ be a (fixed) finite lattice and $\sim : \mathcal{T} \longmapsto \mathcal{T}$ an arbitrary function. Let $\bot$, $\top$ denote the least and the greatest elements of the lattice.

2.1 The Language

The language of $\mathcal{SAL}_\tau$ will consist of the binary logical symbols $\wedge$, $\vee$, $\rightarrow$ and the unary symbols $\neg$, $^\circ$ and f_λ, for each $\lambda \in \mathcal{T}$, a denumerable set $\mathcal{P} = \{p_i : i \in \omega\}$ of propositional letters and parentheses. The set $\mathcal{F}m$ of formulas is defined recursively as usual.

For any $m \in \omega$, define $\neg^n$ recursively: $\neg^0 p = p$ and $\neg^{n+1} p = \neg(\neg^n p)$. The formulas of the form $\neg^{k_n} f_{\lambda_n} \neg^{k_{n-1}} f_{\lambda_{n-1}} \cdots \neg^{k_1} f_{\lambda_1} \neg^{k_0} p_i$, where for $0 \leq j \leq n$, $k_j \in \omega$, $\lambda_j \in \mathcal{T}$ and $i \in \omega$, are called *hyperliterals*. Observe that for $n = 0$, p_i and $\neg^k p_i$ are hyperliterals. A formula that is not a hyperliteral will be called a *complex formula*.

The elements of $\mathcal{T}$ may be thought of as confidence factors, degrees of belief or degrees of credibility. In this last sense, the formula $f_\lambda p$ stands for "the proposition p has credibility greater or equal than λ." They correspond to the annotated variables p_λ of [8] (or $p : \lambda$ in other papers on the subject.) Formulas like $f_\mu f_\lambda p$ may be considered updatings of the credibility of p.

The intended meaning for the formula p° is "p is well behaved with regard to negations." This unary operation is not unnatural for the intended applications of these logics and was introduced in order to get a structural system in a smooth way.

The function $\sim$, intended to act as a negation for annotated formulas, is arbitrary. For $m \in \omega$ we define $\sim^m$ recursively in the same way as $\neg^m$.

Remark 1. Since the lattice $\mathcal{T}$ has finite cardinality $|\mathcal{T}|$, for any $\lambda \in \mathcal{T}$, the set $\{\sim^m \lambda : m \in \omega\}$, is finite. So for any λ, there exists a least integer t_λ such that for some $s \in \omega \setminus \{0\}$, $\sim^{t_\lambda} \lambda = \sim^{t_\lambda + s} \lambda$. Of course, the least such an integer $s \in \omega$ also exists. Call it s_λ.

Observe that if $r \geq t_\lambda$ and m is a multiple of s_λ, then $\sim^r \lambda = \sim^{r+m} \lambda$, so defining

$$\begin{aligned} t &= max\{t_\lambda : \lambda \in \mathcal{T}\}, \\ s &= l.c.m.\{2\} \cup \{s_\lambda : \lambda \in \mathcal{T}\}, \end{aligned}$$

we have, for any $\lambda \in \mathcal{T}$, $\sim^t \lambda = \sim^{t+s} \lambda$.

In the definition of s above, we have included 2 in the least common multiple in order to have a single negation axiom $\neg_4$ both for hyperliterals and for complex formulas.

2.2 Axioms and Inference Rules

The axioms and inference rules for $\mathcal{SAL}_\tau$ are as follows:

1. Axioms for binary connectives:

   ($\rightarrow_1$) $p \rightarrow (q \rightarrow p)$,

   ($\rightarrow_2$) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$,

   ($\rightarrow_3$) $((p \rightarrow q) \rightarrow p) \rightarrow p$,

   ($\wedge_1$) $(p \wedge q) \rightarrow p$,

   ($\wedge_2$) $(p \wedge q) \rightarrow q$,

   ($\wedge_3$) $p \rightarrow (q \rightarrow (p \wedge q))$,

   ($\vee_1$) $p \rightarrow (p \vee q)$,

($\vee_2$) $q \to (p \vee q)$,

($\vee_3$) $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$.

2. Axioms for negation:

($\neg_1$) $(p^\circ \wedge q^\circ) \to ((p \to q) \to ((p \to \neg q) \to \neg p))$,

($\neg_2$) $p^\circ \to (p \to (\neg p \to q))$,

($\neg_3$) $p^\circ \to (p \vee \neg p)$,

($\neg_4$) $\neg^{t+s}\, p \leftrightarrow \neg^t\, p\,$,

where t and s are the numbers defined in Remark 1.

3. Axioms for $^\circ$:

($\circ_1$) $p^\circ \leftrightarrow (\neg p)^\circ$,

($\circ_2$) $p^\circ \leftrightarrow (f_\lambda p)^\circ$,

($\circ_3$) $(p^\circ)^\circ$, $(p \vee q)^\circ$,
$(p \wedge q)^\circ$, $(p \to q)^\circ$.

4. Axioms for annotated formulas:

(τ_1) $\neg(p^\circ) \to f_\perp p\,$,

(τ_2) $\neg(p^\circ) \to (\neg^k f_\lambda p \leftrightarrow \neg^{k-1} f_{\sim\lambda} p)\,$, for $k \geq 1$,

(τ_3) $f_\mu p \to f_\lambda p\,$, for $\mu \geq \lambda$,

(τ_4) $f_\mu f_\lambda p \leftrightarrow f_\mu p\,$,

(τ_5) $\neg(p^\circ) \to (f_\lambda p \leftrightarrow f_\lambda \neg p)$,

(τ_6) $p^\circ \to (f_\lambda p \leftrightarrow p)$.

For the next two axioms we will need the following definitions.

$$S_\kappa(p) = \begin{cases} f_\kappa p \wedge \bigwedge_{\lambda \not\leq \kappa} \neg(f_\lambda p \wedge f_\lambda p) & \text{if } \kappa \neq \top, \\ \neg(p^\circ) \wedge f_\top p & \text{if } \kappa = \top \end{cases}$$

and

$$T_{(\lambda,\kappa)}(p) = \bigwedge_{\substack{m<t+s \\ \sim^m \lambda \leq \kappa}} \neg^m p \wedge \bigwedge_{\substack{m<t+s \\ \sim^m \lambda \not\leq \kappa}} \neg(\neg^m p \wedge \neg^m p).$$

(τ_7) $\neg(p^\circ) \to \bigvee_{\kappa\in\mathcal{T}} S_\kappa(p)$,

(τ_8) $S_\kappa(p) \to \bigvee_{\lambda\in\mathcal{T}} T_{(\lambda,\kappa)}(p)$.

5. Inference Rules

(R1) Modus Ponens

$$\frac{p\ ,\ p \to q}{q}$$

(R2)

$$\frac{p \to f_\lambda q\ ,\ p \to f_\mu q}{p \to f_{\lambda\vee\mu} q.}$$

Remark 2. 1. The first three axioms for negations are tautologies of classical propositional calculus relativized to "well behaved formulas." Axiom ($\neg_4$) codes into the system the fact that all functions over a finite set are eventually periodic. Together with (τ_2), (τ_5), (τ_7) and (τ_8) it handles the negation for annotated formulas.

2. The axioms for $\circ$ control this new unary operation. The third one states that complex formulas are well behaved. The other two state that the behaviour of a formula is not changed by negations or by the operations f_λ.

3. The first three groups of axioms guarantee that the set of complex formulas behave classically with respect to the Boolean operations.

4. The first three axioms for annotated formulas are relativized translations of those of P$\mathcal{T}$. The next three were introduced to control the action of the unary operations f_λ. Essentially, they say that the degrees of credibility apply only to the "core" p of the hyperliterals.

5. For each lattice $\mathcal{T}$, $\mathcal{SAL}_\tau$ is an axiomatic extension of the corresponding system SP$\mathcal{T}$ of [11] obtained by adding axioms τ_7 and τ_8. As we said in the introduction, in the axiomatization of SP$\mathcal{T}$there is only a crude consideration of the specific lattice $\mathcal{T}$ and the function $\sim$ that we are using. Since $\mathcal{T}$ and $\sim$ are arbitrary, these axioms take a rather cumbersome form, nevertheless they are not complicated in spirit.

6. Intuitively, the formula $S_\kappa(p)$ is true only if the maximum degree of credibility of the formula p is κ. Axiom τ_7 states that every hyperliteral has a certain maximum degree of credibility.

7. The formula $T_{(\lambda,\kappa)}(p)$ codes some of the finer aspects of the behavior of the negations of the degree of credibility λ with respect to a given degree of credibility κ. Axiom τ_8 states that if a formula has maximum credibility κ, then its negations will behave like some given degree of credibility λ with respect to κ.

3 Nice Annotated Matrices

In this Section a special class $\boldsymbol{M}_\tau$ of $\mathcal{SAL}_\tau$–matrices is defined; these matrices are a slight simplification of the ones that were introduced in [13], where we proved that they form a matrix semantics for the systems $\mathcal{SAL}_\tau$.

We first observe that since $\mathcal{T}$ is finite, all its ideals are principal, that is, if $\mathcal{I}$ is an ideal, then for some $\kappa \in \mathcal{T}$, $\mathcal{I} = \{\lambda \in \mathcal{T} \,:\, \lambda \leq \kappa\}$. This ideal will be denoted by $\mathcal{I}_\kappa$.

For $I \subseteq \omega$ and $\mathbf{0}, \mathbf{1} \notin \mathcal{T}$ define

$$L = (I \times \mathcal{T}) \cup \{\mathbf{0}, \mathbf{1}\} ,$$

and for any I–indexed family $\langle \kappa_i : i \in I \rangle$ of elements of $\mathcal{T}$, let

$$D = \bigcup_{i\in I} (\{i\} \times \mathcal{I}_{\kappa_i}) \;\cup\; \{\mathbf{1}\}.$$

The set $\boldsymbol{M}_\tau$ of *nice* matrices is

$$\boldsymbol{M}_\tau = \{\mathcal{M} = \langle \boldsymbol{L}, D \rangle \,:\, I \subseteq \omega\,,\ \langle \kappa_i : i \in I \rangle\} ,$$

where L and D are defined as above and the operations of $\boldsymbol{L}$ are defined as follows.

$$a \vee b = \begin{cases} \mathbf{1} & \text{if } a \in D \text{ or } b \in D, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$a \wedge b = \begin{cases} \mathbf{1} & \text{if } a \in D \text{ and } b \in D, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$a \to b = \begin{cases} \mathbf{1} & \text{if } a \notin D \text{ or } b \in D, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$a^\circ = \begin{cases} \mathbf{0} & \text{if } a \in I \times \mathcal{T}, \\ \mathbf{1} & \text{if } a \in \{\mathbf{0}, \mathbf{1}\}. \end{cases}$$

$$f_\lambda a = \begin{cases} \langle i, \lambda \rangle & \text{if } a = \langle i, \mu \rangle, \\ a & \text{if } a \in \{\mathbf{0}, \mathbf{1}\}. \end{cases}$$

$$\neg a = \begin{cases} \mathbf{0} & \text{if } a = \mathbf{1}, \\ \mathbf{1} & \text{if } a = \mathbf{0}, \\ \langle i, \sim \mu \rangle & \text{if } a = \langle i, \mu \rangle. \end{cases}$$

A special case of these matrices is when $I = \{i\}$ is a singleton. In this case we simply drop the ordered pairs and identify $\{i\} \times \mathcal{T}$ with $\mathcal{T}$. We call these *elementary* matrices.

The main theorem proven in [13] is the following completeness theorem.

Theorem 1.

$$\Gamma \vdash A \ \textit{ if and only if } \ \Gamma \vDash_{\boldsymbol{M}_\tau} A.$$

4 Reduced Nice Matrices

Reduced matrices play an important role in the study of the classes of algebras that arise in the algebraization process. A well known result is that for any algebraizable system $\mathcal{S}$ whose class of reduced $\mathcal{S}$–matrices is $\boldsymbol{M}^*$, the equivalent quasivariety semantics $\boldsymbol{Q}$ is

$$\boldsymbol{Q} = \{A \ : \langle A, F \rangle \in \boldsymbol{M}^*, \ \text{ for some } \mathcal{S}\text{–filter } F\},$$

that is, the algebras in the equivalent algebraic semantics are the algebra reducts of the reduced matrices.

There is one further interesting result that we will use in this section, Theorem I.14 in [6]. Using the same notations of the previous paragraph this theorem may be stated as follows: if $\mathcal{S}$ is algebraizable and $\boldsymbol{K} \subseteq \boldsymbol{M}^*$ is a matrix semantics for $\mathcal{S}$, (the wording in [6] is $\boldsymbol{K}$ *is strongly adequate for* $\mathcal{S}$,) then

$$\boldsymbol{M}^* = \mathbb{SPP}_U(\boldsymbol{K}),$$

where $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_U$ are the usual submatrix, direct product and ultraproduct matrix–theoretic operators.

This theorem states that $\boldsymbol{M}^*$ is the smallest matrix quasivariety containing $\boldsymbol{K}$.

4.1 The Leibniz Operator

One of the main tools in algebraic logic is the so called Leibniz operator, which is extensively studied in [3, 5, 9, 10], and other places. We give here the main definitions and properties for future reference.

For any algebra $\boldsymbol{A}$ and $D \subseteq A$, we define the *Leibniz congruence relation on* $\boldsymbol{A}$ *over* D

$$\begin{aligned}\boldsymbol{\Omega_A}(D) = \{\langle a,b\rangle : \varphi^{\boldsymbol{A}}(a,c_1,\ldots,c_n) \in D \ \text{ iff } \ \varphi^{\boldsymbol{A}}(b,c_1,\ldots,c_n) \in D,\\ \text{for any formula } \ \varphi(x,y_1,\ldots,y_n) \in \mathcal{F}m \text{ and } c_1,\ldots,c_n \in A\}.\end{aligned}$$

The function $\boldsymbol{\Omega_A}$ whose domain is $\mathcal{P}(A)$ is called the *Leibniz operator on* $\boldsymbol{A}$.

The most useful characterization of the Leibniz operator is probably Theorem 1.5 in [3]. We say that the congruence θ on an algebra $\boldsymbol{A}$ is *compatible* with a subset F of A, if $a\ \theta\ b$ and $b \in F$, then $a \in F$.

Theorem 2. *The Leibniz operator* $\boldsymbol{\Omega_A}$ *on* $\boldsymbol{A}$ *assigns to each* $X \subseteq A$ *the largest congruence* $\boldsymbol{\Omega_A}(X)$ *of* $\boldsymbol{A}$ *that is compatible with* X*.*

4.2 Reduced Matrices

A matrix $\mathcal{M} = \langle \boldsymbol{A}, F\rangle$ is *reduced* if $\boldsymbol{\Omega_A}(F)$ is the identity relation on A.

Lemma 3. *Let* $\mathcal{M} = \langle \boldsymbol{L}, D\rangle \in \boldsymbol{M}_\tau$ *, where* $D = \mathcal{I}_\kappa \cup \{\mathbf{1}\}$*, for some* $\kappa \in \mathcal{T}$*, be an elementary matrix. Then the following are equivalent, for all* μ*,* $\nu \in \mathcal{T}$*.*

1. *For all* $m < s+t$, $\sim^m \mu \leq \kappa$ *if and only if* $\sim^m \nu \leq \kappa$ *,*
2. $T^{\boldsymbol{L}}_{(\mu,\kappa)}(\nu) = \mathbf{1}$ *,*
3. $T^{\boldsymbol{L}}_{(\nu,\kappa)}(\mu) = \mathbf{1}$ *.*

Proof. If for all $m < s+t$, $\sim^m \mu \leq \kappa$ if and only if $\sim^m \nu \leq \kappa$, then $T_{(\mu,\kappa)}(p) = T_{(\nu,\kappa)}(p)$. Also, for all $m < s+t$, $\sim^m \mu \in D$ if and only if $\sim^m \nu \in D$, so

$$T^{\boldsymbol{L}}_{(\mu,\kappa)}(\nu) = \mathbf{1} = T^{\boldsymbol{L}}_{(\nu,\kappa)}(\mu).$$

This proves that 1 implies 2 and 3.

On the other hand, assuming $T^{\boldsymbol{L}}_{(\mu,\kappa)}(\nu) = \mathbf{1}$, the first part of the formula $T_{(\mu,\kappa)}(p)$ implies that for all $m < s+t$, if $\sim^m \mu \leq \kappa$, then $\sim^m \nu \in D$, so $\sim^m \nu \leq \kappa$. The second part of the formula $T_{(\mu,\kappa)}(p)$ implies that for all $m < s+t$, if $\sim^m \mu \nleq \kappa$, then $\neg(\neg^m\nu \wedge \neg^m\nu) \in D$ and since $\neg^m\nu \wedge \neg^m\nu$ is either 0 or 1, this is equivalent to $\neg^m\nu \wedge \neg^m\nu = \mathbf{0}$, or equivalently, $\sim^m \nu \notin D$ or $\sim^m \nu \nleq \kappa$. This proves that 2 implies 1.

The equivalence of 1 and 3 is proven similarly. □

Theorem 4. *Let* $\mathcal{M} = \langle \boldsymbol{L}, D\rangle \in \boldsymbol{M}_\tau$, *where* $D = \bigcup_{i\in I}(\{i\} \times \mathcal{I}_{\kappa_i}) \cup \{\mathbf{1}\}$. *Then*

$$a\, \mathbf{\Omega}_{\boldsymbol{L}}(D)\, b \;\; \textit{iff} \;\; \begin{cases} a = b \qquad\qquad\qquad \textit{or} \\ \\ a = \langle i,\mu\rangle,\ b = \langle j,\nu\rangle,\ \kappa_i = \kappa_j = \kappa \quad \textit{for some } \kappa \\ \textit{and } \ T^{\boldsymbol{L}}{}_{(\mu,\kappa)}(\nu) = \mathbf{1} = T^{\boldsymbol{L}}{}_{(\nu,\kappa)}(\mu)\,. \end{cases}$$

Proof. It is straightforward that the relation Θ defined on the right hand side is an equivalence relation that preserves the binary connectives and the unary connectives f_λ and $^\circ$. The preservation of negations is obtained by the condition $T^{\boldsymbol{L}}{}_{(\mu,\kappa)}(\nu) = \mathbf{1} = T^{\boldsymbol{L}}{}_{(\nu,\kappa)}(\mu)$, which states that the negations of μ and ν have the same behavior with respect to κ. See Remark 2., 7.

The condition $\kappa_i = \kappa_j = \kappa$ implies $\mathcal{I}_{\kappa_i} = \mathcal{I}_{\kappa_j}$ which insures compatibility with D, since by Lemma 3 and the definition of D

$$\langle i,\mu\rangle \in D \text{ iff } \mu \in \mathcal{I}_\kappa \text{ iff } \nu \in \mathcal{I}_\kappa \text{ iff } \langle j,\nu\rangle \in D.$$

To prove that the congruence Θ defined on the right hand side above is the largest congruence compatible with D, let θ be any other congruence compatible with D. If $\langle i,\mu\rangle\ \theta\ \langle j,\nu\rangle$, then

$$\langle i,\kappa_i\rangle = f_{\kappa_i}\langle i,\mu\rangle \quad \theta \quad f_{\kappa_i}\langle j,\nu\rangle = \langle j,\kappa_i\rangle$$

and since $\langle i, \kappa_i \rangle \in D$, by compatibility, $\langle j, \kappa_i \rangle \in D$, so $\kappa_i \leq \kappa_j$. Similarly $\kappa_j \leq \kappa_i$, so $\kappa_i = \kappa_j$, and thus $\mathcal{I}_{\kappa_i} = \mathcal{I}_{\kappa_j} = \mathcal{I}_\kappa$ for some κ.

Finally since $\neg^m \langle i, \mu \rangle \ \theta \ \neg^m \langle j, \nu \rangle$, we get $\langle i, \sim^m \mu \rangle \ \theta \ \langle j, \sim^m \nu \rangle$, so by compatibility,

$$\sim^m \mu \in \mathcal{I}_\kappa \ \text{ if and only if } \ \sim^m \nu \in \mathcal{I}_\kappa,$$

which by Lemma 3 is equivalent to $T^{\boldsymbol{L}}_{(\mu,\kappa)}(\nu) = \mathbf{1} = T^{\boldsymbol{L}}_{(\nu,\kappa)}(\mu)$.

This proves that $\theta \subseteq \Theta$, so the latter is maximal compatible with D and thus equals $\boldsymbol{\Omega}_{\boldsymbol{L}}(D)$. □

In order to simplify notations, we will first build the elementary reduced matrices. So let us consider the matrix $\mathcal{M} = \langle \boldsymbol{A}, D_\kappa \rangle = \langle \mathbf{2} \cup \mathcal{T}, \mathcal{I}_\kappa \cup \{\mathbf{1}\} \rangle$.

If we let $[a]_\kappa$ be the class of a modulo $\boldsymbol{\Omega}_A(D_\kappa)$ and $\mathcal{T}_\kappa$ the set of equivalence classes, then

$$\begin{aligned} [\mathbf{0}]_\kappa &= \{\mathbf{0}\}, \\ [\mathbf{1}]_\kappa &= \{\mathbf{1}\}, \\ [\lambda]_\kappa &= \{\mu \in \mathcal{T} : T^{\boldsymbol{A}}_{\lambda,\kappa}(\mu) = \mathbf{1}\}. \end{aligned}$$

$\mathcal{T}_\kappa$ should always be understood as associated to the obvious reduced ideal, namely,

$$\mathcal{I}^*_\kappa = \{[\lambda]_\kappa : \lambda \leq \kappa\} \quad \text{and} \quad D^*_\kappa = \mathcal{I}^*_\kappa \cup \{[\mathbf{1}]_\kappa\}.$$

So the reduction $\mathcal{M}^*$ of matrix $\mathcal{M}$ is (isomorphic to)

$$\mathcal{M}^* = \langle A^*, D^*_\kappa \rangle = \langle \mathbf{2} \cup \mathcal{T}_\kappa, D^*_\kappa \rangle.$$

We let $\mathbf{2}_{\langle \kappa \rangle}$ be the algebra reduct of $\mathcal{M}^*$.

Again for notational convenience, we will assume that for different κ, the corresponding $\mathcal{T}_\kappa$ are disjoint.

The following theorem follows easily from Theorem 4 if we observe that if a matrix contains repeated copies of the same ideal of $\mathcal{T}$, then these copies are identified under the Leibniz congruence.

Theorem 5. *Let $\langle \boldsymbol{L}, D \rangle \in \boldsymbol{M}_\tau$ and let $\mathcal{I}_{\kappa_1}, \ldots, \mathcal{I}_{\kappa_m}$ be all the distinct ideals that appear in D. Then the reduced matrix*

$$\langle \boldsymbol{L}, D \rangle^* = \langle \boldsymbol{L}/\boldsymbol{\Omega}_L(D), D/\boldsymbol{\Omega}_{\boldsymbol{L}}(D) \rangle$$

is such that

$$\boldsymbol{L}^* \cong \{\mathbf{0},\mathbf{1}\} \cup \bigcup_{i=1}^{m} \mathcal{T}_{\kappa_i}$$

and

$$D^* \cong \bigcup_{i=1}^{m} \mathcal{I}^*_{\kappa_i} \cup \{\mathbf{1}\}.$$

We let $\mathbf{2}_{\langle\kappa_1,\ldots,\kappa_m\rangle}$ be the algebra reduct of $\langle \boldsymbol{L}^*, D^*\rangle$. If $m = |\mathcal{T}|$, that is, all possible ideals appear in D, we let $\mathbf{2}_{\tau} = \mathbf{2}_{\langle\kappa_1,\ldots,\kappa_{|\tau|}\rangle}$.

Corollary 6. *There are* $2^{|\mathcal{T}|}$ *reduced matrices in* $\boldsymbol{M}_{\tau}$.

Proof. Since all ideals are principal, there are as many ideals as elements in the lattice, so there are as many reduced matrices as subsets of $\mathcal{T}$. □

The last corollary provides us with a decision procedure for annotated logics: we just have to check $2^{|\mathcal{T}|}$ finite matrices. As a matter of fact, we will see in the next theorem that we only need to check a single matrix.

Let us consider the reduced matrix $\mathfrak{M}$ whose algebra reduct is $\mathbf{2}_{\tau}$.

The following lemma is straightforward.

Lemma 7. *Every reduced matrix in* $\boldsymbol{M}_{\tau}$ *is a submatrix of* $\mathfrak{M}$.

Theorem 8. *The class* $\{\mathfrak{M}\}$ *is a matrix semantics for system* $\mathcal{SAL}_{\tau}$, *that is,*

$$\Gamma \vdash A \ \textit{ if and only if } \ \Gamma \vDash_{\mathfrak{M}} A.$$

4.3 Two Examples

We will illustrate the process determining the reduced nice matrices for two classes of annotated logics which appear in the literature.

Example 1.

Let $\mathcal{T} = \{0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1\}$ with the usual order and let $\sim$ be defined by $\sim a = 1 - a$, that is, $\mathcal{T}$ is the $(m+1)$–element chain with Lukasiewicz negation. There are two cases, the first of which is when m is even, that is $|\mathcal{T}|$ is odd. In this case $\frac{1}{2} \in \mathcal{T}$. Then there are four possibilities.

1. $\kappa = 1$. Then for all $\lambda \in \mathcal{T}$, $[\lambda]_1 = \mathcal{T}$, so $\mathcal{T}_1 = \{[0]_1\} = \mathcal{I}^*_1$.

2. $\kappa = \frac{j}{m} > \frac{1}{2}$. Then there are three equivalence classes.

$$\begin{aligned}
[0]_{\frac{j}{m}} &= \{0, \frac{1}{m}, \dots, \frac{m-j-1}{m}\}, \\
[\frac{j}{m}]_{\frac{j}{m}} &= \{\frac{m-j}{m}, \dots, \frac{j}{m}\} = [\frac{1}{2}]_{\frac{j}{m}}, \\
[1]_{\frac{j}{m}} &= \{\frac{j+1}{m}, \dots, 1\},
\end{aligned}$$

so

$$\mathcal{T}_{\frac{j}{m}} = \{[0]_{\frac{j}{m}}, [\frac{1}{2}]_{\frac{j}{m}}, [1]_{\frac{j}{m}}\} \quad \text{and} \quad \mathcal{I}^*_{\frac{j}{m}} = \{[0]_{\frac{j}{m}}, [\frac{1}{2}]_{\frac{j}{m}}\}.$$

3. $\kappa = \frac{1}{2}$.

$$\begin{aligned}
[0]_{\frac{1}{2}} &= \{0, \frac{1}{m}, \dots, \frac{m-2}{2m}\}, \\
[\frac{1}{2}]_{\frac{1}{2}} &= \{\frac{1}{2}\}, \\
[1]_{\frac{1}{2}} &= \{\frac{m+2}{2m}, \dots, 1\}.
\end{aligned}$$

so

$$\mathcal{T}_{\frac{1}{2}} = \{[0]_{\frac{1}{2}}, [\frac{1}{2}]_{\frac{1}{2}}, [1]_{\frac{1}{2}}\} \quad \text{and} \quad \mathcal{I}^*_{\frac{1}{2}} = \{[0]_{\frac{j}{m}}, [\frac{1}{2}]_{\frac{j}{m}}\}.$$

4. $\kappa = \frac{j}{m} < \frac{1}{2}$.

$$\begin{aligned}
[0]_{\frac{j}{m}} &= \{0, \frac{1}{m}, \dots, \frac{j}{m}\}, \\
[\frac{j+1}{m}]_{\frac{j}{m}} &= \{\frac{j+1}{m}, \dots, \frac{m-j-1}{m}\} = [\frac{1}{2}]_{\frac{j}{m}}, \\
[1]_{\frac{j}{m}} &= \{\frac{m-j}{m}, \dots, 1\},
\end{aligned}$$

so

$$\mathcal{T}_{\frac{j}{m}} = \{[0]_{\frac{j}{m}}, [\frac{1}{2}]_{\frac{j}{m}}, [1]_{\frac{j}{m}}\} \quad \text{and} \quad \mathcal{I}^*_{\frac{j}{m}} = \{[0]_{\frac{j}{m}}\}.$$

One should observe that for $\kappa \neq 1$, even though as ordered sets the $\mathcal{T}_\kappa$'s are isomorphic, the operations in particular the f_λ's are not defined in the same way. For example, if $m > 2$, for any x

$$f_{\frac{1}{m}}[x]_{\frac{1}{2}} = [\frac{1}{m}]_{\frac{1}{2}} = [0]_{\frac{1}{2}} \quad \text{while} \quad f_{\frac{1}{m}}[x]_0 = [\frac{1}{m}]_0 = [\frac{1}{2}]_0 \,.$$

The second case is when m is odd. Then $\frac{1}{2} \notin \mathcal{T}$ and there are two "central points", namely, $\frac{m-1}{2m}$ and $\frac{m+1}{2m}$.

If $\kappa > \frac{1}{2}$ or $\kappa < \frac{m-1}{2m}$, $\mathcal{T}_\kappa$ is like in the first case. The only difference arises when $\kappa = \frac{m-1}{2m}$ for then

$$\begin{aligned}
[0]_{\frac{m-1}{2m}} &= \{0, \frac{1}{m}, \ldots, \frac{m-1}{2m}\}, \\
[1]_{\frac{m-1}{2m}} &= \{\frac{m+1}{2m}, \ldots, 1\},
\end{aligned}$$

so

$$\mathcal{T}_{\frac{m-1}{2m}} = \{[0]_{\frac{m-1}{2m}}, [1]_{\frac{m-1}{2m}}\} \quad \text{and} \quad \mathcal{I}^*_{\frac{m-1}{2m}} = \{[0]_{\frac{m-1}{2m}}\}\,.$$

In the first case $|\mathbf{2}_\mathcal{T}| = |\{0,1\} \cup \bigcup \mathcal{T}_{\frac{j}{m}}| = 2 + 1 + 3m = 3(m+1)$, so the algebra $\mathbf{2}_\mathcal{T}$ has $3(m+1)$ elements.

In the second case the algebra $\mathbf{2}_\mathcal{T}$ has $3m+2$ elements.

In case $m = 2$, then $\mathcal{T} = \{0, \frac{1}{2}, 1\}$ and $\mathbf{2}_\mathcal{T}$ has 9 elements, namely $\{\mathbf{1}, \mathbf{0}\} \cup \{0_1, 0_a, 0_0, a_a, a_0, 1_a, 1_0\}$ and $D = \{\mathbf{1}, 0_1, 0_a, 0_0, a_a\}$, where in order to simplify notation we use a instead of $\frac{1}{2}$ and we drop the brackets. The following is a table of the implication and of the unary operations. The first five elements belong to D.

$\to$	$\mathbf{1}$	0_1	0_a	0_0	a_a	a_0	1_a	1_0	$\mathbf{0}$	$\neg$	f_0	f_a	f_1	$\circ$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
0_1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	0_1	0_1	0_1	0_1	$\mathbf{0}$
0_0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	1_0	0_0	a_0	1_0	$\mathbf{0}$
0_a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	1_a	0_a	a_a	1_a	$\mathbf{0}$
0_0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	1_0	0_0	a_0	1_0	$\mathbf{0}$
a_a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	a_a	0_a	a_a	1_a	$\mathbf{0}$
a_0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	a_0	0_0	a_a	1_0	$\mathbf{0}$
1_a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0_a	0_a	a_a	1_a	$\mathbf{0}$
1_0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0_0	0_0	a_0	1_0	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$

Example 2. Let $\mathcal{T}$ be the bilattice $\boldsymbol{FOUR} = \{\bot, \boldsymbol{f}, \boldsymbol{t}, \top\}$. $\boldsymbol{FOUR}$ is ordered as the four element Boolean algebra with greatest element $\top$ and least element $\bot$. The negation $\sim$ is defined by $\sim \bot = \bot$, $\sim \boldsymbol{f} = \boldsymbol{t}$, $\sim \boldsymbol{t} = \boldsymbol{f}$, $\sim \top = \top$.

1. $\kappa = \top$. Then for all $\lambda \in \mathcal{T}$, $[\lambda]_\top = \mathcal{T}$, so
$$\mathcal{T}_\top = \{[\bot]_\top\} = \mathcal{I}^*_1 .$$

2. $\kappa = \boldsymbol{t}$ or $\kappa = \boldsymbol{f}$. Then for all $\lambda \in \mathcal{T}$, $[\lambda]_{\boldsymbol{t}} = \{\lambda\} = [\lambda]_{\boldsymbol{f}}$, so
$$\mathcal{T}_{\boldsymbol{t}} = \{[\bot]_{\boldsymbol{t}}, [\boldsymbol{t}]_{\boldsymbol{t}}, [\boldsymbol{f}]_{\boldsymbol{t}}, [\top]_{\boldsymbol{t}}\}, \quad \text{and} \quad \mathcal{I}^*_{\boldsymbol{t}} = \{[\bot]_{\boldsymbol{t}}, [\boldsymbol{t}]_{\boldsymbol{t}}\} .$$
and
$$\mathcal{T}_{\boldsymbol{f}} = \{[\bot]_{\boldsymbol{f}}, [\boldsymbol{t}]_{\boldsymbol{f}}, [\boldsymbol{f}]_{\boldsymbol{f}}, [\top]_{\boldsymbol{f}}\}. \quad \text{and} \quad \mathcal{I}^*_{\boldsymbol{f}} = \{[\bot]_{\boldsymbol{f}}, [\boldsymbol{f}]_{\boldsymbol{f}}\} .$$
Observe that in these two cases the lattices $\mathcal{T}_\kappa$ with their associated ideals $\mathcal{I}^*_\kappa$ are isomorphic to $\mathcal{T}$ with the ideal $\mathcal{I}_\kappa$.

3. $\kappa = \bot$. Then
$$[\bot]_\bot = \{\bot\},$$
$$[\top]_\bot = \{\top, \boldsymbol{t}, \boldsymbol{f}\},$$
so
$$\mathcal{T}_\bot = \{[\bot]_\bot, [\top]_\bot\} \quad \text{and} \quad \mathcal{I}^*_\bot = \{[\bot]_\bot\} .$$

In this example the algebra $\mathbf{2}_\mathcal{T}$ has 13 elements.

5 The Quasi–variety $\mathbf{SAL}_\tau$ of $\mathcal{SAL}_\tau$–algebras

In this Section we will describe the quasivariety of $\mathcal{SAL}_\tau$–algebras and find some of its properties.

The theory of algebraization of deductive systems developed in [3] provides us with a standard axiomatization for the quasivariety that arises in the process. This is the content of Theorem 2.17 in that monograph.

In order to apply these results in our case, we observe that $\mathcal{SAL}_\tau$ is an axiomatic extension of the algebraizable system SP$\mathcal{T}$, so it is algebraizable too. Moreover, the same defining equations and equivalence formulas used in [11] for the algebraization of systems SP$\mathcal{T}$ can be used for $\mathcal{SAL}_\tau$. These are the following.

The single defining equation is $p \wedge p \approx p \to p$ and the equivalence formulas are

$$\begin{array}{rcll} \Delta_\circ(p,q) & = & p^\circ \leftrightarrow q^\circ, & \\ \Delta_k(p,q) & = & \neg^k p \leftrightarrow \neg^k q, & \text{for } 0 \le k < t+s, \\ \Delta_\lambda(p,q) & = & f_\lambda p \leftrightarrow f_\lambda q, & \text{for all } \lambda \in \mathcal{T}. \end{array}$$

If we let $J = \{\circ\} \cup \{0, 1, \ldots, t+s-1\} \cup \mathcal{T}$, we write $p \Delta q$ as an abbreviation of the conjunction $\bigwedge_{j\in J} \Delta_j(p,q)$ of the equivalence formulas.

We will introduce two new constant symbols to the language, $\mathbf{1}$ to stand for the class of all $\mathcal{SAL}_\tau$–theorems and $\mathbf{0}$ to stand for the class of all negations of a $\mathcal{SAL}_\tau$–theorem.

Theorem 9. *The class of all $\mathcal{SAL}_\tau$–algebras is the quasivariety of all algebras $\boldsymbol{A} = \langle A, \vee, \wedge, \to, \neg, {}^\circ, f_\lambda, \mathbf{0}, \mathbf{1}\rangle_{\lambda\in\tau}$ axiomatized by the following identities*

$$\sigma \approx \mathbf{1} \tag{1}$$

for every axiom σ of $\mathcal{SAL}_\tau$, together with the following quasi–identities

$$p \wedge p \approx \mathbf{1} \ \ \& \ \ p \to q \approx \mathbf{1} \quad \Rightarrow \quad q \wedge q \approx \mathbf{1} \tag{2}$$

$$p \to f_\lambda q \approx \mathbf{1} \ \ \& \ \ p \to f_\mu q \approx \mathbf{1} \ \Rightarrow \quad x \to f_{\lambda\vee\mu} q \approx \mathbf{1} \tag{3}$$

$$p\Delta q \approx \mathbf{1} \quad \Rightarrow \quad p \approx q. \tag{4}$$

Proof. It should be noted that this axiomatization is an obvious simplification of the one obtained by direct application of [3], Theorem 2.17 taking into account that if σ is complex then $\vdash (\sigma \wedge \sigma)\, \Delta\, \sigma$. □

Given an algebra $\boldsymbol{A} \in \mathbf{SAL}_{\mathcal{T}}$, it is immediate that if we let $\mathcal{B} = \{b \in A : b^\circ = \mathbf{1}\}$, $\mathcal{B}$ is a Boolean algebra with unary operations defined by $f_\lambda(b) = b$ and $b^\circ = \mathbf{1}$, for all $b \in \mathcal{B}$ and all $\lambda \in \mathcal{T}$. $\mathcal{B}$ will be referred to as *the Boolean part of* $\boldsymbol{A}$.

We will let $\mathcal{H} = \{b \in A : b^\circ \neq \mathbf{1}\}$ and call it the *hyperliteral part of* $\boldsymbol{A}$, thus any $\mathcal{SAL}_{\mathcal{T}}$–algebra is of the form

$$\boldsymbol{A} = \langle \mathcal{B} \cup \mathcal{H}, \vee, \wedge, \rightarrow, \neg, {}^\circ, f_\lambda, \mathbf{0}, \mathbf{1}\rangle_{\lambda \in \mathcal{T}}.$$

We will see that there is a close connection between the Boolean and hyperliteral parts.

The next theorem is the most important application of the theory to the reduced nice matrices of the previous Section. It is a consequence of the general theoretical remarks at the beginning of Section 4 and in Theorem 8.

Theorem 10. *The class* $\mathbf{SAL}_{\mathcal{T}}$ *of* $\mathcal{SAL}_{\mathcal{T}}$ *–algebras is given by*

$$\mathbf{SAL}_{\mathcal{T}} = \mathbb{SP}(\{\mathbf{2}_{\mathcal{T}}\}).$$

5.1 $\mathbf{SAL}_{\mathcal{T}}$ is not a Variety

Theorem 11. *The class of all* $\mathcal{SAL}_{\mathcal{T}}$ *algebras is not a variety.*

Proof. Let us consider the reduced algebra $\mathbf{2}_{\langle\top\rangle}$ whose universe is $\mathbf{2} \cup \{a\}$, for which the operations are defined as follows:

$\vee$	0	a	1
0	0	1	1
a	1	1	1
1	1	1	1

$\wedge$	0	a	1
0	0	0	0
a	0	1	1
1	0	1	1

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	1
1	0	1	1

$\neg$	$\circ$	f_λ
1	1	0
a	0	a
0	1	1

Let $\boldsymbol{B} = \langle\{\alpha, \top\}; \vee, \wedge, \rightarrow, \neg, {}^\circ, f_\lambda, \top, \top\rangle$ with operations defined by

$\vee$	α	$\top$
α	$\top$	$\top$
$\top$	$\top$	$\top$

$\wedge$	α	$\top$
α	$\top$	$\top$
$\top$	$\top$	$\top$

$\rightarrow$	α	$\top$
α	$\top$	$\top$
$\top$	$\top$	$\top$

$\neg$	$\circ$	f_λ
α	$\top$	α
$\top$	$\top$	$\top$

Finally define

$$\begin{array}{rcccc} f & : & \mathbf{2}_{\langle\top\rangle} & \longrightarrow & \boldsymbol{B} \\ & & 0 & \longmapsto & \top \\ & & a & \longmapsto & \alpha \\ & & 1 & \longmapsto & \top \end{array}$$

Then f is a homomorphism, so $\boldsymbol{B}$ is a homomorphic image of $\mathbf{2}_{\langle\top\rangle} \in \mathbf{SAL}_{\tau}$ but it is not a $\mathcal{SAL}_{\tau}$–algebra since it fails to verify axiom (4). $\square$

5.2 Subdirectly Irreducibles in $\mathbf{SAL}_{\tau}$

Theorem 12. *A $\mathcal{SAL}_{\tau}$ –algebra is subdirectly irreducible if and only if its Boolean part is* $\mathbf{2}$.

Proof. Let $\boldsymbol{A}$ be a subdirectly irreducible $\mathcal{SAL}_{\tau}$-algebra.

One must observe that if θ is a Boolean–congruence on the Boolean algebra $\mathcal{B}$, then it is also an absolute $\mathbf{SAL}_{\tau}$ –congruence on $\mathcal{B}$ considered as a $\mathcal{SAL}_{\tau}$–algebra. Moreover, if we define $\hat{\theta} = \theta \cup \Delta_A$, where Δ_A is the identity on the universe A of $\boldsymbol{A}$, then $\hat{\theta}$ is an absolute $\mathbf{SAL}_{\tau}$ –congruence on $\boldsymbol{A}$.

Now suppose that the Boolean part $\mathcal{B} \neq \mathbf{2}$. Then there are two non–trivial Boolean–congruences θ_1 and θ_2 such that $\theta_1 \cap \theta_2 = \Delta_{\mathcal{B}}$. But then $\hat{\theta}_1 \cap \hat{\theta}_2 = \Delta_A$, so $\boldsymbol{A}$ is not subdirectly irreducible, a contradiction, so $\mathcal{B} = \mathbf{2}$.

Suppose $\mathcal{B} = \mathbf{2}$. We will prove that the $\mathbf{SAL}_{\tau}$ –congruence $\Theta = B \times B \cup \Delta_A$ is a monolith for the lattice of congruences of $\boldsymbol{A}$.

Let θ be any non-trivial $\mathbf{SAL}_{\tau}$ –congruence and let $a\,\theta\,b$, with $a \neq b$.

If both $a, b \in \mathcal{B}$, then $\Theta \subseteq \theta$.

If $a \in \mathcal{B}$ and $b \in \mathcal{H}$, since $\mathbf{1} = a^{\circ}\,\theta\,b^{\circ} = \mathbf{0}$, again $\Theta \subseteq \theta$.

If both $a, b \in \mathcal{H}$ and we assume that $a \to b = \mathbf{0}$, then $\mathbf{1} = a \to a\ \theta\ a \to b = \mathbf{0}$ and thus $\Theta \subseteq \theta$.

A similar argument shows that if we assume that either $f_{\lambda}a \leftrightarrow f_{\lambda}b = \mathbf{0}$, for some $\lambda \in \mathcal{T}$ or that $\neg^k a \leftrightarrow \neg^k b = \mathbf{0}$, for some $k \in \omega$, then $\Theta \subseteq \theta$.

So the only case left is when for all $\lambda \in \mathcal{T}$, $f_{\lambda}a \leftrightarrow f_{\lambda}b = \mathbf{1}$ and for all $k \in \omega$, $\neg^k a \leftrightarrow \neg^k b = \mathbf{1}$. Since for $a, b \in \mathcal{H}$ we also have $a^{\circ} \leftrightarrow b^{\circ} = \mathbf{1}$, we may conclude that $a\Delta b = \mathbf{1}$, so by axiom (4), we have that $a = b$, a contradiction, so this case is not possible and Θ is a monolith for the lattice of congruences of $\boldsymbol{A}$. $\square$

Let $\boldsymbol{A} \in \mathbf{SAL}_{\tau}$ have universe $A = \mathcal{B} \cup \mathcal{H}$. Given any element $a \in A$, $a \wedge a \in \mathcal{B}$. As a matter of fact, even though $\mathcal{B}$ could be larger, at least it contains the Boolean algebra generated by $\{a \wedge a : a \in \mathcal{H}\}$. In this sense, the Boolean part of $\boldsymbol{A}$ cannot be "too small" with respect to $\mathcal{H}$. The implications of this fact can be most easily seen in algebras whose Boolean part is $\mathbf{2}$.

Theorem 13. *Let $\boldsymbol{A}$ be a $\mathcal{SAL}_\tau$ –algebra whose Boolean part is* $\mathbf{2}$*. Then* $\boldsymbol{A}$ *is isomorphic to a subalgebra of a reduced $\mathcal{SAL}_\tau$ –algebra* $\mathbf{2}_{\langle\kappa_1,\ldots,\kappa_m\rangle}$.

Proof. Theorem 10 states that $\mathbf{SAL}_\tau = \mathbb{SP}(\{\mathbf{2}_\tau\})$, so the Boolean elements of $\boldsymbol{A}$ may always be thought of as "tuples" of zeros and ones and the hyperliteral elements of $\boldsymbol{A}$ as tuples of zeros, ones and elements of $\mathcal{T}$.

Let $\boldsymbol{A}$ be a subalgebra of $\Pi_{i\in I}\mathbf{2}_\tau$ whose Boolean part $\mathcal{B}$ is $\mathbf{2}$.

If for some $i,\, j \in I$, $a(i) \in \mathcal{T}_\kappa$ for some κ and $a(j) \in \{\mathbf{0},\mathbf{1}\}$, then $a^\circ(i) = 0$ and $a^\circ(j) = 1$. This contradicts the fact that $a^\circ \in \mathcal{B} = \{\mathbf{0},\mathbf{1}\}$. So either for all $i \in I$ $a(i) \in \{\mathbf{0},\mathbf{1}\}$, or for all $i \in I$ $a(i) \in \mathcal{T}_\kappa$ for some κ.

Assume that for $i,\, j \in I$, $a(i) \in \mathcal{T}_{\kappa_i}$ and $a(j) \in \mathcal{T}_{\kappa_j}$, with $\kappa_j \neq \kappa_i$, say $\kappa_j \nleq \kappa_i$. Then $f_{\kappa_j}a(i) \notin \mathcal{I}^*_{\kappa_i}$ but $f_{\kappa_j}a(j) \in \mathcal{I}^*_{\kappa_j}$, and thus

$$f_{\kappa_j}a(i) \wedge f_{\kappa_j}a(i) = 0 \ \text{ and } \ f_{\kappa_j}a(j) \wedge f_{\kappa_j}a(j) = 1,$$

and thus $f_{\kappa_j}a \wedge f_{\kappa_j}a \notin \mathcal{B}$, which is a contradiction. So for any $a \in \mathcal{H}$, for all $i \in I$, $a(i) \in \mathcal{T}_\kappa$ for a unique κ. This means that $\mathcal{H}$ is contained in a union of powers of $\mathcal{T}_\kappa$'s.

Assume now that for $i,\, j \in I$, $a(i) \neq a(j)$. Then for some $m < t + s$, $\sim^m a(i) \in \mathcal{I}^*_\kappa$, but $\sim^m a(j) \notin \mathcal{I}^*_\kappa$, so

$$\sim^m a(i) \wedge \sim^m a(i) = 1 \ \text{ and } \ \sim^m a(j) \wedge \sim^m a(j) = 0,$$

and thus $\sim^m a \wedge \sim^m a \notin \mathcal{B}$, which again is a contradiction. So if we let

$$diag(\mathcal{T}^I_\kappa) = \{a : \text{ for all } \ i,\, j \in I,\ a(i) = a(j)\},$$

then $\boldsymbol{A}$ is a subalgebra of

$$\boldsymbol{B} = \mathbf{2} \cup \bigcup_{\kappa\in F} diag(\mathcal{T}^I_\kappa),$$

where $F \subseteq \mathcal{T}$.

To finish our proof, taking $F = \{\kappa_1,\ldots,\kappa_m\}$, we simply have to point out that fixing one $i \in I$, the mapping

$$\begin{array}{rcccll} f & : & \boldsymbol{B} & \longrightarrow & \mathbf{2}_{\langle\kappa_1,\ldots,\kappa_m\rangle} & \\ & & \mathbf{0} & \longmapsto & \mathbf{0} & \\ & & \mathbf{1} & \longmapsto & \mathbf{1} & \\ & & a & \longmapsto & a(i)\,, & \quad \text{for } \ a \in \mathcal{H}, \end{array}$$

is a monomorphism. □

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