

# AN ANNOTATED LOGIC DEFINED BY A MATRIX

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ABSTRACT. Of special interest in abstract algebraic logic currently is the problem of extending the general theory of algebraization to logical systems that fail to be structural. The annotated logics  $P_{\mathbf{L}}$  were introduced in the late eighties as a logical framework to deal with deductive databases that contain inconsistent, conflicting or contradictory information. Like many paraconsistent logics they are non-structural, and for this reason they do not have an algebraic semantics in the usual sense. In this paper a structural and algebraizable annotated logic  $P_{\mathcal{M}(\mathbf{L})}$  is constructed that simulates the deductive process of  $P_{\mathbf{L}}$  in a natural way. Unlike  $P_{\mathbf{L}}$ ,  $P_{\mathcal{M}(\mathbf{L})}$  is semantically defined by a single matrix  $\mathcal{M}(\mathbf{L})$ . The matrix is simpler than other matrix semantics of similar kind that have appeared in the literature. If  $\mathcal{M}(\mathbf{L})$  is finite, a sound and complete axiomatization of  $P_{\mathcal{M}(\mathbf{L})}$  is given and this is used to prove that the equivalent algebraic semantics of  $P_{\mathcal{M}(\mathbf{L})}$  is the quasivariety generated by the Leibniz reduction of the underlying  $\mathbf{M}(\mathbf{L})$  of  $\mathcal{M}(\mathbf{L})$ , and to find an equational axiomatization for this quasivariety.

## 1. INTRODUCTION

Abstract algebraic logic can be viewed as the study of the process of associating with a logical system an algebra or class of algebras whose equational deductive process faithfully simulates the deductive process of the logical system. This is called the *algebraization* of the logical system. At this time a satisfactory theory of algebraization is available only for so-called structural logics. A logic is *structural* in this sense if the set of theorems and more generally the consequence relations of the system are closed under simultaneous substitution of compound formulas for atomic subformulas. One of the more interesting problems of abstract algebraic logic is that of extending the general theory of algebraization to logical systems that fail to be structural. The first-order predicate logic in its usual formulation is the prime example of a non-structural logic. Many paraconsistent logics, in particular the annotated logics we consider here also fail to be structural. The standard way of algebraizing first-order logic is to first reformulate it as a structural logic to which the general theory of algebraization can be applied. In this paper we take this approach to algebraize a large class of annotated logics that have been considered in the the literature.

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Artificial intelligence, computer generated proofs, and databases have brought about an increased interest in logic programming based on non-classical logics. A database is said to be *deductive* if, besides the set of facts that make up a database, there is also a set of *rules* from which one can infer new facts that were not originally part of the database. The inference mechanism of a deductive database is normally supported by a logic programming system. A *distributed* database is one whose basic set of facts may be collected from many different sources and thus it may contain inconsistent (conflicting) information, that is, the database may contain both a fact and its logical negation. Assume that from one source we get the information that  $p$  is true while some other source tells us that  $\neg p$  is true. Now consider a statement  $q$ , unrelated to  $p$ . If the database is supported by classical, two-valued logic, then since  $p$  and  $\neg p$  are both true we can deduce that  $q$  is true. But, from the limited amount of information given by  $p$  and  $\neg p$ , we do not want to be able to draw a conclusion about  $q$ . Our goal is to have a database with the capability of containing inconsistent information in a non-destructive fashion. So in the logic that underlies a formalization of reasoning about distributive databases, evidence of the truth of both  $p$  and  $\neg p$  cannot be construed as evidence that an unrelated formula is true. As we know, this is a characteristic feature of *paraconsistency*. A logic with this property called *annotated logic* and designed specifically for reasoning about distributive databases was introduced by Subrahmanian [13]. In fact he defined an annotated logic  $P_{\mathbf{L}}$  for each complete lattice  $\mathbf{L}$ . The elements of the lattice  $\mathbf{L}$  are called *annotation constants*.

Much work has been done on annotated logics, both on its theoretical aspects and in application to problems in artificial intelligence. Blair and Subrahmanian [2] showed that the  $P_{\mathbf{L}}$  are paraconsistent and applied them as the basis of a programming language for reasoning about databases that contain inconsistent information. A complete study of annotated logics, from both the model theoretical and proof theoretical points of view has been done by Abe [1] and da Costa, Subrahmanian and Vago [5]. They show that almost all the basic results of classical model theory can be adapted to these systems. In this paper we follow the axiomatization of  $P_{\mathbf{L}}$  given in [5] and refer to this paper for the properties of  $P_{\mathbf{L}}$  that we use. Kifer and Lozinkii in [9] introduce a first-order annotated logic (APC) with a sound and complete proof procedure, and apply it to several non-monotonic reasoning arguments.

Like most, but not all, paraconsistent logics  $P_{\mathbf{L}}$  fails to be structural. The source of this difficulty is the dichotomy between the formulas for which negation acts classically, and the formulas for which it acts in a non-classical, paraconsistent fashion; the latter class is not closed under substitution. This implies that the consequence relation of  $P_{\mathbf{L}}$  cannot be simulated in standard equational logic in the same way, for example, that the consequence relation of the classical propositional logic can be simulated in the equational logic of the two-element Boolean algebra. In abstract algebraic logic the various ways that a particular class of algebras is attached to a logical system has been studied for the purpose of abstracting a general process for algebraizing logic. There are various proposals for capturing formally the notion of an algebraizable logic. All these notions of algebraizability can be characterized in terms of the existence of a bisimulation in some form between the consequence relation of the logical system and the equational consequence relation of an specific class of algebras called its *equivalent algebraic semantics*. But the structurality

of the logic is a necessary condition in all cases. We get around the problem in the case of  $P_{\mathbf{L}}$  by constructing a (logical) matrix  $\mathcal{M}(\mathbf{L})$  whose associated logic  $P_{\mathcal{M}(\mathbf{L})}$ , which is necessarily structural, naturally simulates  $P_{\mathbf{L}}$ . We then show that, if  $\mathbf{L}$  is finite,  $P_{\mathcal{M}(\mathbf{L})}$  is finitely algebraizable (in the sense of [4]) and its equivalent algebraic semantics is the quasivariety generated by a special quotient  $\mathbf{M}^*(\mathbf{L})$  of the underlying algebra of the matrix  $\mathcal{M}(\mathbf{L})$ . The simulation of  $P_{\mathbf{L}}$  by  $P_{\mathcal{M}(\mathbf{L})}$  is then composed with the bisimulation between  $P_{\mathcal{M}(\mathbf{L})}$  and the equational logic of  $\mathbf{M}^*(\mathbf{L})$  that arises naturally from finite algebraization.

**1.1. Connections with previous work and outline of paper.** Lewin, Mikenberg, and Schwarze [10] introduce for each  $P_{\mathbf{L}}$  a new, structural, annotated system  $SP_{\mathbf{L}}$  that is closely related to  $P_{\mathbf{L}}$  in the sense that it simulates its deductive process in a natural way. They prove  $SP_{\mathbf{L}}$  is finitely algebraizable in the sense of [4] just in case the lattice  $\mathbf{L}$  is finite. In [11, 12] they consider another finitely algebraizable annotated logical system  $\mathcal{SAL}_{\mathbf{L}}$ , called *structural annotated logic based on  $\mathbf{L}$* , that is an axiomatic extension of  $SP_{\mathbf{L}}$ . They investigate its equivalent algebraic semantics and a matrix semantics for  $\mathcal{SAL}_{\mathbf{L}}$ . This matrix semantics is closely related to the one proposed in this paper.

There are some advantages of the systems  $P_{\mathcal{M}(\mathbf{L})}$  over the closely related logics and matrices introduced in [10, 11, 12].  $P_{\mathcal{M}(\mathbf{L})}$  is defined by a single matrix, which is much simpler than the ones that arise through the algebraization process in [11, 12]. The axiomatizations of  $SP_{\mathbf{L}}$  (or of  $\mathcal{SAL}_{\mathbf{L}}$ ) and  $P_{\mathcal{M}(\mathbf{L})}$  are equally complex, but the simulation of the original system  $P_{\mathbf{L}}$  by the logic  $P_{\mathcal{M}(\mathbf{L})}$  is simpler than the one proposed in [10].

In Section 2 we summarize the main facts about the annotated systems  $P_{\mathbf{L}}$  and its semantics. In Section 3 we define the matrices  $\mathcal{M}(\mathbf{L})$  and the logics  $P_{\mathcal{M}(\mathbf{L})}$ . We show that  $P_{\mathcal{M}(\mathbf{L})}$  are algebraizable and that it naturally simulates the deductive properties of  $P_{\mathbf{L}}$ . In the last section we give an axiomatization of  $P_{\mathcal{M}(\mathbf{L})}$  and prove that it is sound and complete. Finally, we make use of this axiomatization prove that, in the case the lattice  $\mathbf{L}$  is finite, the equivalent algebraic semantics of  $P_{\mathcal{M}(\mathbf{L})}$  is the quasivariety generated by a quotient of the underlying algebra of the matrix  $\mathcal{M}(\mathbf{L})$ .

## 2. THE LOGICS $P_{\mathbf{L}}$

This section will introduce the annotated logics  $P_{\mathbf{L}}$ . For proofs of theorems in this section please refer to [5].

**2.1. Logics  $P_{\mathbf{L}}$ .** By an *annotation lattice* we understand an algebra  $\mathbf{L} = \langle L, \vee, \wedge, \sim \rangle$ , where  $\langle L, \vee, \wedge \rangle$  is a complete lattice with largest element  $\top$  and least element  $\perp$ , and  $\sim : L \rightarrow L$  is an arbitrary unary function.

Let  $\mathbf{L}$  be a fixed annotation lattice. The language of  $P_{\mathbf{L}}$  contains the connectives  $\vee, \wedge, \rightarrow, \neg$ , a set  $\text{Va}$  of propositional variables  $p, q, r, \dots$ , annotation constants  $\lambda, \mu, \kappa, \rho, \dots$ , one for each element of the annotation lattice  $\mathbf{L}$ , and various punctuation and delimiting symbols. If  $p$  is a propositional variable and  $\lambda$  is an annotation constant, then the pair  $\langle p, \lambda \rangle$ , which we denote by  $p_{\lambda}$ , is an *annotated variable*. The set  $\text{Fm}_{\mathbf{L}}$  of formulas of  $P_{\mathbf{L}}$  is recursively generated as usual, from the set of annotated variables by the connectives.

The formulas  $\neg \neg \dots \neg p_{\mu}$ , where there are  $k$  negation symbols,  $k \geq 0$ , will be denoted by  $\neg^k p_{\mu}$ , and will be called *hyperliterals*. A formula that is not a hyperliteral will be called a

*complex formula.* Formulas are represented by upper case Latin letters from the beginning of the alphabet,  $A, B, C, \dots$ , and complex formulas by  $F, G, \dots$ .

**2.2. Axioms and Rules.** The axioms and inference rules of  $P_{\mathbf{L}}$  are as follows. For all  $A, B, C \in \text{Fm}_{\mathbf{L}}$ .

- ( $\rightarrow_1$ )  $A \rightarrow (B \rightarrow A)$ ,
- ( $\rightarrow_2$ )  $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ ,
- ( $\rightarrow_3$ )  $((A \rightarrow B) \rightarrow A) \rightarrow A$ ,
- ( $\rightarrow_4$ )  $\frac{A, A \rightarrow B}{B}$
- ( $\wedge_1$ )  $(A \wedge B) \rightarrow A$ ,
- ( $\wedge_2$ )  $(A \wedge B) \rightarrow B$ ,
- ( $\wedge_3$ )  $A \rightarrow (B \rightarrow (A \wedge B))$ ,
- ( $\vee_1$ )  $A \rightarrow (A \vee B)$ ,
- ( $\vee_2$ )  $B \rightarrow (A \vee B)$ ,
- ( $\vee_3$ )  $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$ ,

For all complex formulas  $F, G$  and every  $A \in \text{Fm}_{\mathbf{L}}$ ,

- ( $\neg_1$ )  $(F \rightarrow G) \rightarrow ((F \rightarrow \neg G) \rightarrow \neg F)$ ,
- ( $\neg_2$ )  $F \rightarrow (\neg F \rightarrow A)$ ,
- ( $\neg_3$ )  $F \vee \neg F$ ,
- ( $L_1$ )  $p_{\perp}$ ,
- ( $L_2$ )  $\neg^k p_{\mu} \leftrightarrow \neg^{k-1} p_{\sim \mu}$ , for  $k \geq 1$ ,
- ( $L_3$ )  $p_{\mu} \rightarrow p_{\lambda}$ , for  $\mu \geq \lambda$ ,
- ( $L_4$ ) if  $A \rightarrow p_{\mu_j}$  for each  $j \in J \neq \emptyset$ , then  $A \rightarrow p_{\mu}$ , where  $\mu = \bigvee_{j \in J} \mu_j$ .

**Theorem 2.1** ([5]).  $\Sigma, A \vdash_{P_{\mathbf{L}}} B$  iff  $\Sigma \vdash_{P_{\mathbf{L}}} A \rightarrow B$ .

**2.3. Semantics for  $P_{\mathbf{L}}$ .** An *interpretation*  $I$  for the logic  $P_{\mathbf{L}}$  is a function  $I: \text{Va} \rightarrow L$ . For each interpretation we define a *valuation*  $v_I: \text{Fm}_{\mathbf{L}} \rightarrow \{0, 1\}$  as follows.

- (i) If  $p \in \text{Va}$ , then  $v_I(p_{\mu}) = 1$  iff  $I(p) \geq \mu$ .
- (ii) (a)  $v_I(\neg^k p_{\mu}) = v_I(\neg^{k-1} p_{\sim \mu})$ , for  $k \geq 1$ ,  
(b) If  $A$  is complex, then  $v_I(\neg A) = 1$  iff  $v_I(A) = 0$ .
- (iii) If  $A, B \in \text{Fm}_{\mathbf{L}}$ , then  
(a)  $v_I(A \rightarrow B) = 1$  iff  $v_I(A) = 0$  or  $v_I(B) = 1$ ,  
(b)  $v_I(A \vee B) = 1$  iff  $v_I(A) = 1$  or  $v_I(B) = 1$ ,  
(c)  $v_I(A \wedge B) = 1$  iff  $v_I(A) = 1$  and  $v_I(B) = 1$ .

The concepts of satisfaction, semantical consequence, etc, are defined in the usual way.

**Definition 2.2.** Let  $I$  be an interpretation.

- (i)  $I$  is  $\neg$ -inconsistent if and only if there exists  $p \in \mathcal{P}$  and  $\mu \in L$  such that  $v_I(p_{\mu}) = v_I(\neg p_{\mu}) = 1$ .
- (ii)  $I$  is non-trivial if and only if there exists  $p \in \mathcal{P}$  and  $\mu \in L$  such that  $v_I(p_{\mu}) = 0$ .
- (iii)  $I$  is paraconsistent if and only if  $I$  is both  $\neg$ -inconsistent and non-trivial.

Notice that if  $I(p) = \top$ , then  $v_I(p_\lambda) = v_I(\neg p_\lambda) = 1$  and thus  $P_{\mathbf{L}}$  always has a  $\neg$ -inconsistent interpretation. If  $|L| \geq 2$ , let  $I(p) = \kappa \neq \top$ . Then  $v_I(p_\top) = 0$ , and thus there exists a non-trivial,  $\neg$ -inconsistent, that is, a paraconsistent interpretation

We have the following soundness theorem and the weak completeness theorem.

**Theorem 2.3** ([5]). *Assume  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathbf{L}}$ .*

- (i) *If  $\Sigma \vdash_{P_{\mathbf{L}}} A$  then  $\Sigma \vDash_{P_{\mathbf{L}}} A$ .*
- (ii) *For  $\Sigma$  finite, if  $\Sigma \vDash_{P_{\mathbf{L}}} A$  then  $\Sigma \vdash_{P_{\mathbf{L}}} A$ .*

**Theorem 2.4.** *For any  $\mathbf{L}$ ,  $P_{\mathbf{L}}$  is paraconsistent in the sense that there is a  $\varphi \in \text{Fm}_{\mathbf{L}}$  such that  $\vdash_{P_{\mathbf{L}}} \varphi$  and  $\vdash_{P_{\mathbf{L}}} \neg\varphi$  but it is not the case that  $\vdash_{P_{\mathbf{L}}} \psi$  for all  $\psi \in P_{\mathbf{L}}$ .*

**Theorem 2.5** ([10]). *If  $|L| \geq 2$ ,  $P_{\mathbf{L}}$  is not structural in the sense that there are  $\varphi, \psi \in \text{Fm}_{\mathbf{L}}$  and annotated variable  $p_\lambda$  such that  $\vdash_{P_{\mathbf{L}}} \varphi$  and  $\not\vdash_{P_{\mathbf{L}}} \varphi(p_\lambda/\psi)$ .*

### 3. THE LOGICS $P_{\mathcal{M}(\mathbf{L})}$

As indicated in the introduction  $P_{\mathbf{L}}$  is not algebraizable, in particular there is no bisimulation in the usual sense between the consequence relation  $\vdash_{P_{\mathbf{L}}}$  of  $P_{\mathbf{L}}$  and equational consequence relation  $\vDash_{\mathbf{K}}$  of any class of algebras  $\mathbf{K}$ . However we show that  $P_{\mathbf{L}}$  is algebraizable in a weaker sense by constructing an algebra  $\mathbf{M}^*(\mathbf{L})$  and a natural simulation of  $\vdash_{P_{\mathbf{L}}}$  by  $\vDash_{\mathbf{M}^*(\mathbf{L})}$ . This is done in three stages. In this section construct a matrix  $\mathcal{M}(\mathbf{L}) = \langle \mathbf{M}(\mathbf{L}), D \rangle$  and show that the deductive system  $P_{\mathcal{M}(\mathbf{L})}$  defined by  $\mathcal{M}(\mathbf{L})$  is algebraizable (finitely algebraizable if  $\mathbf{L}$  is finite) and that  $\vdash_{P_{\mathbf{L}}}$  is simulated by  $\vdash_{P_{\mathcal{M}(\mathbf{L})}}$ ; unlike  $P_{\mathbf{L}}$ ,  $P_{\mathcal{M}(\mathbf{L})}$  is semantically defined and structural. Later, in Section 4, we will show that its equivalent quasivariety is generated by a quotient  $\mathbf{M}^*(\mathbf{L})$  of  $\mathbf{M}(\mathbf{L})$ , the underlying algebra of  $\mathcal{M}(\mathbf{L})$ . The desired simulation of  $\vdash_{P_{\mathbf{L}}}$  by  $\vDash_{\mathbf{M}^*(\mathbf{L})}$  is then obtained by composing the simulation of  $\vdash_{P_{\mathbf{L}}}$  by  $\vdash_{P_{\mathcal{M}(\mathbf{L})}}$  with the standard simulation of  $\vdash_{P_{\mathcal{M}(\mathbf{L})}}$  by  $\vDash_{\mathbf{M}(\mathbf{L})}$  that comes from the algebraization.

**3.1. The Matrix.** The language of  $P_{\mathcal{M}(\mathbf{L})}$  contains all the connectives of  $P_{\mathbf{L}}$ , that is the Boolean connectives  $\rightarrow, \wedge, \vee$  together with paraconsistent negation  $\neg$ . In addition it contains a unary connective  $\#$  and a unary connective  $f_\lambda$  for each  $\lambda \in L$ . Because  $P_{\mathcal{M}(\mathbf{L})}$  is structural, no distinction is made between hyperliterals and complex formulas in the metalanguage. Rather, this is done, virtually, in the objective language by means of the connective  $\#$ . Thus, for any formula  $\varphi$ ,  $\#\varphi$  may be thought of as the ‘‘Boolean content’’ of  $\varphi$ ; hopefully this will become clear when we define the semantics of  $P_{\mathcal{M}(\mathbf{L})}$  shortly. Since  $P_{\mathcal{M}(\mathbf{L})}$  is a structural logic, the propositional variables used to construct its language are not annotated. The formula  $f_\lambda p$  plays essentially the same role for  $P_{\mathcal{M}(\mathbf{L})}$  that the annotated variable  $p_\lambda$  plays for  $P_{\mathbf{L}}$ . The set of propositional variables is again denoted by  $\text{Va}$  and they are again represented by the letters  $p, q, r, \dots$ . Formulas are constructed recursively from the propositional variables in the usual way. They are represented by lower case Greek letters  $\varphi, \psi, \vartheta, \dots$ . The set of formulas is denoted by  $\text{Fm}_{\mathcal{M}(\mathbf{L})}$ .

$P_{\mathcal{M}(\mathbf{L})}$  is defined by a particular matrix  $\mathcal{M}(\mathbf{L})$ , which is finite just in case  $\mathbf{L}$  is finite. Let

$$M(\mathbf{L}) = L \times (L \setminus \{\top\}) \cup \{0, 1, \mathbb{T}\},$$

where  $\mathbb{T}$  is a special element. Let  $\mathbf{M}(\mathbf{L})$  be the algebra

$$\mathbf{M}(\mathbf{L}) = \langle M(\mathbf{L}), \rightarrow^{\mathbf{M}(\mathbf{L})}, \wedge^{\mathbf{M}(\mathbf{L})}, \vee^{\mathbf{M}(\mathbf{L})}, \neg^{\mathbf{M}(\mathbf{L})}, \#^{\mathbf{M}(\mathbf{L})}, f_\kappa^{\mathbf{M}(\mathbf{L})} \rangle_{\kappa \in L}.$$

The operations of  $\mathbf{M}(\mathbf{L})$  are defined as follows. Let

$$D = \{ \langle \lambda, \kappa \rangle \in L \times (L \setminus \{\top\}) : \lambda \leq \kappa \} \cup \{1, \mathbb{T}\},$$

where  $\top$  is the largest element of  $\mathbf{L}$ . It is helpful to think of  $\mathbb{T}$  as representing the elements of the set  $L \times \{\top\}$ . Let  $\mathcal{M}(\mathbf{L})$  be the matrix

$$\mathcal{M}(\mathbf{L}) = \langle \mathbf{M}(\mathbf{L}), D \rangle.$$

In the sequel write  $\cdot^{\mathbf{M}(\mathbf{L})}$ ,  $+^{\mathbf{M}(\mathbf{L})}$ , and  $-^{\mathbf{M}(\mathbf{L})}$  respectively for  $\wedge^{\mathbf{M}(\mathbf{L})}$ ,  $\vee^{\mathbf{M}(\mathbf{L})}$ , and  $\neg^{\mathbf{M}(\mathbf{L})}$ . We also sometimes omit explicit reference to  $\mathbf{L}$  when it is fixed and write  $\mathcal{M}$ ,  $\mathbf{M}$ , and  $M$  respectively for  $\mathcal{M}(\mathbf{L})$ ,  $\mathbf{M}(\mathbf{L})$ , and  $M(\mathbf{L})$ ; this is done systematically in superscripts.

For any  $a \in M$ ,  $\#^{\mathbf{M}} a = 1$  if  $a \in D$  and  $\#^{\mathbf{M}} a = 0$  otherwise, i.e.,

$$\#^{\mathbf{M}} a = \begin{cases} 1 & \text{if } a = \langle \lambda, \kappa \rangle \text{ with } \lambda \leq \kappa, \\ 1 & \text{if } a \in \{0, \mathbb{T}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\#^{\mathbf{M}} 0 = 0$  and  $\#^{\mathbf{M}} 1 = 1$ .

Let  $\mathbf{2} = \langle \{0, 1\}, \rightarrow^{\mathbf{2}}, \cdot^{\mathbf{2}}, +^{\mathbf{2}}, -^{\mathbf{2}} \rangle$  be the two-element Boolean algebra with the standard operations. For all  $a \in M$ ,

$$-^{\mathbf{M}} a = \begin{cases} \langle \sim \lambda, \kappa \rangle & \text{if } a = \langle \lambda, \kappa \rangle \in L \times (L \setminus \{\top\}), \\ -^{\mathbf{2}} a & \text{if } a \in \{0, 1\}, \\ \mathbb{T} & \text{if } a = \mathbb{T}. \end{cases}$$

Note that  $-^{\mathbf{M}} 0 = 1$  and  $-^{\mathbf{M}} 1 = 0$ .

For all  $a \in M$ ,

$$f_{\lambda}^{\mathbf{M}} a = \begin{cases} \langle \lambda, \kappa \rangle & \text{if } a = \langle \mu, \kappa \rangle \in L \times (L \setminus \{\top\}) \\ \mathbb{T} & \text{otherwise.} \end{cases}$$

For all  $a, b \in M$ ,

$$\begin{aligned} a \rightarrow^{\mathbf{M}} b &= \#^{\mathbf{M}} a \rightarrow^{\mathbf{2}} \#^{\mathbf{M}} b \\ a \cdot^{\mathbf{M}} b &= \#^{\mathbf{M}} a \cdot^{\mathbf{2}} \#^{\mathbf{M}} b \\ a +^{\mathbf{M}} b &= \#^{\mathbf{M}} a +^{\mathbf{2}} \#^{\mathbf{M}} b. \end{aligned}$$

Note that, for each  $\star \in \{\rightarrow, \cdot, +\}$ ,  $\#^{\mathbf{M}}(a \star^{\mathbf{M}} b) = \#^{\mathbf{M}} a \star^{\mathbf{2}} \#^{\mathbf{M}} b$ . Thus  $\#^{\mathbf{M}}$  is a homomorphism from  $\langle M, \rightarrow^{\mathbf{M}}, \cdot^{\mathbf{M}}, +^{\mathbf{M}} \rangle$  onto  $\mathbf{2}$ . It does not preserve negation however. For example, if  $\langle \lambda, \kappa \rangle \in L \times (L \setminus \{\top\})$ , and if both  $\lambda \leq \kappa$  and  $\sim \lambda \leq \kappa$ , then  $\#^{\mathbf{M}} -^{\mathbf{M}} \langle \lambda, \kappa \rangle = \#^{\mathbf{M}} \langle \sim \lambda, \kappa \rangle = 1$ , but  $-^{\mathbf{2}} \#^{\mathbf{M}} \langle \lambda, \kappa \rangle = -^{\mathbf{2}} 1 = 0$ .

**Lemma 3.1.** *Let  $a, b \in M$ .*

- (i)  $a \rightarrow^{\mathbf{M}} b \in D$  iff  $a \notin D$  or  $b \in D$ .
- (ii)  $a \cdot^{\mathbf{M}} b \in D$  iff  $a \in D$  and  $b \in D$ .
- (iii)  $a +^{\mathbf{M}} b \in D$  iff  $a \in D$  or  $b \in D$ .

*Proof.* (i).  $a \rightarrow^{\mathbf{M}} b \in D$  iff  $\#^{\mathbf{M}} a \rightarrow^{\mathbf{2}} \#^{\mathbf{M}} b = 1$  iff  $\#^{\mathbf{M}} a = 0$  or  $\#^{\mathbf{M}} b = 1$  iff  $a \notin D$  or  $b \in D$ .

(ii) and (iii) are proved similarly.  $\square$

We take  $P_{\mathcal{M}(\mathbf{L})}$  to be  $\langle \mathbf{Fm}_{\mathcal{M}(\mathbf{L})}, \vDash_{\mathcal{M}(\mathbf{L})} \rangle$ , the logic determined semantically by the matrix  $\mathcal{M}(\mathbf{L}) = \langle \mathbf{M}(\mathbf{L}), D \rangle$ . This is defined as follows. For any *assignment*  $u: \text{Va} \longrightarrow M$  of elements of  $\mathbf{M}(\mathbf{L})$  to the variables, let  $u^*: \mathbf{Fm}_{\mathcal{M}(\mathbf{L})} \longrightarrow \mathbf{M}(\mathbf{L})$  be the unique extension of  $u$  to a homomorphism from the formula algebra  $\mathbf{Fm}_{\mathcal{M}(\mathbf{L})} = \langle \mathbf{Fm}_{\mathcal{M}(\mathbf{L})}, \rightarrow, \wedge, \vee, \neg, \#, \mathbf{f}_\kappa \rangle_{\kappa \in L}$  into  $\mathbf{M}(\mathbf{L})$ . Then, for all  $\varphi_0, \dots, \varphi_{n-1}, \psi \in \mathbf{Fm}_{\mathcal{M}(\mathbf{L})}$ ,

$$\varphi_0, \dots, \varphi_{n-1} \vDash_{\mathcal{M}(\mathbf{L})} \psi \quad \text{iff} \quad u^* \varphi_0, \dots, u^* \varphi_{n-1} \in D \text{ implies } u^* \psi \in D \text{ for every } u: \text{Va} \longrightarrow M(\mathbf{L}).$$

We note that  $\vdash_{P_{\mathcal{M}(\mathbf{L})}} = \vDash_{\mathcal{M}(\mathbf{L})}$ ; since it is semantically defined, it is structural. We also note that  $P_{\mathcal{M}(\mathbf{L})}$  is paraconsistent: the set  $\Delta = \{\mathbf{f}_\lambda p, \neg \mathbf{f}_\lambda p\} \subseteq \mathbf{Fm}_{\mathbf{L}}$  is non-trivial and  $\neg$ -inconsistent.

**3.2. The simulation of of  $\vdash_{P_{\mathbf{L}}}$  by  $\vdash_{P_{\mathcal{M}(\mathbf{L})}}$ .** The translation of the language of  $P_{\mathbf{L}}$  into the language of  $P_{\mathcal{M}(\mathbf{L})}$  is defined as follows.

**Definition 3.2** ([10]). Let  $A$  be in  $\mathbf{Fm}_{\mathbf{L}}$ . The translation of  $t(A) \in \mathbf{Fm}_{\mathcal{M}(\mathbf{L})}$  of  $A$  is defined by recursion on the structure of  $A$  as follows.

- (i) If  $A = p_\lambda$ , then  $t(A) = \mathbf{f}_\lambda p$ .
- (ii) If  $A = \neg B$ , then  $t(A) = \neg t(B)$ .
- (iii) If  $A = B \star C$ , for  $\star \in \{\rightarrow, \wedge, \vee\}$ , then  $t(A) = t(B) \star t(C)$ .

**Lemma 3.3.** *Let  $I: \text{Va} \longrightarrow L$  be an interpretation. Then there exists a function  $u: \text{Va} \longrightarrow M$  such that, for every formula  $A \in \mathbf{Fm}_{\mathbf{L}}$ ,  $v_I(A) = 1$  iff  $u^*(t(A)) \in D$ .*

*Proof.* For every  $p \in \text{Va}$  define

$$u(p) = \begin{cases} \langle \perp, I(p) \rangle & \text{if } I(p) \neq \top, \\ \top & \text{if } I(p) = \top. \end{cases}$$

The proof is by induction on the structure of the annotated formula  $A$ .

Base step: If  $A = p_\kappa$  then  $v_I(A) = v_I(p_\kappa) = 1$  iff  $\kappa \leq I(p)$ .

$$\begin{aligned} u^*(t(A)) &= u^*(t(p_\kappa)) \\ &= u^*(\mathbf{f}_\kappa p) \\ &= \mathbf{f}_\kappa^{\mathbf{M}}(u(p)) \\ &= \begin{cases} \mathbf{f}_\kappa^{\mathbf{M}}(\langle \perp, I(p) \rangle) & \text{if } I(p) \neq \top \\ \mathbf{f}_\kappa^{\mathbf{M}}(\top) & \text{if } I(p) = \top \end{cases} \\ &= \begin{cases} \langle \kappa, I(p) \rangle & \text{if } I(p) \neq \top \\ \top & \text{if } I(p) = \top \end{cases}. \end{aligned}$$

So  $u^*(t(A)) \in D$  iff  $\kappa \leq I(p)$  iff  $v_I(A) = 1$ .

If  $A = \neg^j p_\kappa$  then  $v_I(A) = v_I(\neg^j p_\kappa) = v_I(p_{\sim^j \kappa}) = 1$  iff  $\sim^j \kappa \leq I(p)$ .

$$\begin{aligned}
u^*(t(A)) &= u^*(t(\neg^j p_\kappa)) \\
&= u^*(\neg^j f_\kappa p) \\
&= (\neg^M)^j f_\kappa^M(u(p)) \\
&= \begin{cases} (\neg^M)^j f_\kappa^M(\langle \perp, I(p) \rangle) & \text{if } I(p) \neq \top \\ (\neg^M)^j f_\kappa^M(\top) & \text{if } I(p) = \top \end{cases} \\
&= \begin{cases} (\neg^M)^j \langle \kappa, I(p) \rangle & \text{if } I(p) \neq \top \\ (\neg^M)^j \top & \text{if } I(p) = \top \end{cases} \\
&= \begin{cases} \langle \sim^j \kappa, I(p) \rangle & \text{if } I(p) \neq \top \\ \top & \text{if } I(p) = \top \end{cases}.
\end{aligned}$$

So  $u^*(t(A)) \in D$  iff  $\sim^j \kappa \leq I(p)$  iff  $v_I(A) = 1$ .

Induction step:  $A$  a complex formula.

Let  $A = B \vee C$ ,  $B$  and  $C$  annotated formulas, then

$$\begin{aligned}
v_I(A) = v_I(B \vee C) = 1 &\text{ iff } v_I(B) = 1 \text{ or } v_I(C) = 1 \\
&\text{ iff } u^*(t(B)) \in D \text{ or } u^*(t(C)) \in D \text{ by induction hyp.} \\
&\text{ iff } [u^*(t(B)) +^M u^*(t(C))] \in D \text{ by Lemma 3.1} \\
&\text{ iff } u^*(t(B) \vee t(C)) \in D \\
&\text{ iff } u^*(t(B \vee C)) \in D \\
&\text{ iff } u^*(t(A)) \in D.
\end{aligned}$$

Similarly we prove that for  $A = B \wedge C$  and  $A = B \rightarrow C$ ,  $v_I(A) = 1$  iff  $u^*(t(A)) \in D$ .

Claim:  $u^*(t(C)) \in \{0, 1\}$  for every complex formula  $C \in \text{Fm}_{\mathbf{L}}$ .

We prove the claim by induction on the structure of  $C$ . Note that  $C$  is complex if and only if  $C = B \star E$  with  $\star \in \{\rightarrow, \wedge, \vee\}$  or  $C = \neg B$  with  $B$  complex.

Let  $C = B \star E$  with  $\star \in \{\rightarrow, \wedge, \vee\}$ , then

$$u^*(t(C)) = u^*(t(B \star E)) = u^*(t(B)) \star^M u^*(t(E)) = \#^M u^*(t(B)) \star^2 \#^M u^*(t(E)) \in \{0, 1\}.$$

Let  $C = \neg B$  for  $B$  complex, then

$$u^*(t(C)) = u^*(t(\neg B)) = u^*(\neg t(B)) = -^2 u^*(t(B)) \in \{0, 1\}$$

since by the induction hypothesis  $u^*(t(B)) \in \{0, 1\}$ .

This ends the proof of the Claim.

If  $A = \neg C$  for  $C$  complex then

$$\begin{aligned}
v_I(A) = 1 & \text{ iff } v_I(C) = 0 \\
& \text{ iff } u^*(t(C)) \notin D && \text{by the ind. hyp.} \\
& \text{ iff } u^*(t(C)) = 0 && \text{by the Claim} \\
& \text{ iff } \neg^M u^*(t(C)) = 1 \\
& \text{ iff } u^*(\neg t(C)) = 1 \\
& \text{ iff } u^*(t(\neg C)) = 1 \\
& \text{ iff } u^*(t(\neg C)) \in D && \text{by the Claim} \\
& \text{ iff } u^*(t(A)) \in D.
\end{aligned}$$

□

**Lemma 3.4.** *Let  $u: \text{Va} \longrightarrow M$ . Then there exists an interpretation  $I: \text{Va} \longrightarrow L$  such that, for every formula  $A \in \text{Fm}_L$ ,  $v_I(A) = 1$  iff  $u^*(t(A)) \in D$ .*

*Proof.* Define

$$I(p) = \begin{cases} \lambda & \text{if } u(p) = \langle \kappa, \lambda \rangle, \\ \top & \text{if } u(p) \in \{0, 1, \mathbb{T}\} \end{cases}$$

The proof is by induction on the structure of the annotated formula  $A$ .

Suppose  $A = p_\rho$ . There are two cases.

Case (1):  $u(p) = \langle \kappa, \lambda \rangle$ . Then  $v_I(A) = 1$  if and only if  $\rho \leq I(p) = \lambda$ , so

$$u^*(t(A)) = u^*(t(p_\rho)) = u^*(f_\rho p) = f_\rho^M(u(p)) = f_\rho^M(\langle \kappa, \lambda \rangle) = \langle \rho, \lambda \rangle,$$

and thus

$$u^*(t(A)) \in D \text{ iff } \rho \leq \lambda = I(p) \text{ iff } v_I(A) = 1.$$

Case (2):  $u(p) \in \{0, 1, \mathbb{T}\}$ . Then

$$u^*(t(A)) = u^*(t(p_\rho)) = u^*(f_\rho p) = f_\rho^M(u(p)) = \mathbb{T},$$

and thus

$$u^*(t(A)) \in D \text{ iff } I(p) = \top \geq \rho \text{ iff } v_I(A) = 1.$$

For  $A = \neg^j p_\rho$  there are also two cases.

Case (1):  $u(p) = \langle \kappa, \lambda \rangle$ . Then

$$\begin{aligned}
u^*(t(A)) &= u^*(t(\neg^j p_\rho)) \\
&= u^*(\neg^j f_\rho(p)) \\
&= (\neg^M)^j f_\rho^M(u(p)) \\
&= (\neg^M)^j f_\rho^M(\langle \kappa, \lambda \rangle) \\
&= (\neg^M)^j \langle \rho, \lambda \rangle \\
&= \langle \sim^j \rho, \lambda \rangle
\end{aligned}$$

so

$$u^*(t(A)) \in D \text{ iff } \sim^j \rho \leq \lambda = I(p) \text{ iff } v_I(p_{\sim^j \rho}) = 1 \text{ iff } v_I(A) = 1.$$

Case (2):  $u(p) \in \{0, 1, \mathbb{T}\}$ . Then

$$u^*(t(A)) = (-^M)^j u^*(f_\rho p) = (-^M)^j f_\rho^M(u(p)) = (-^M)^j \mathbb{T} = \mathbb{T},$$

so

$$u^*(t(A)) \in D \text{ iff } I(p) = \mathbb{T} \geq \sim^j \rho \text{ iff } v_I(A) = 1.$$

The induction step is the same as in Lemma 3.3.  $\square$

**Theorem 3.5.** *Let  $A_1, \dots, A_n, B$  be annotated formulas in  $\text{Fm}_L$ . Then*

$$A_1, \dots, A_n \vDash_{P_L} B \text{ iff } t(A_1), \dots, t(A_n) \vDash_{P_{\mathcal{M}(L)}} t(B).$$

*Proof.*  $\implies$ . Let  $A_1, \dots, A_n \vDash_{P_L} B$ .

Assume  $u^*(t(A_1)), \dots, u^*(t(A_n)) \in D$  for some  $u: \text{Va} \longrightarrow M$ . Then by Lemma 3.4 there exists an interpretation  $I$  such that  $v_I(A_1) = \dots = v_I(A_n) = 1$ , so by hypothesis  $v_I(B) = 1$ . Again by Lemma 3.4 this gives  $u^*(t(B)) \in D$ . Therefore  $t(A_1), \dots, t(A_n) \vDash_{P_{\mathcal{M}(L)}} t(B)$ .

$\impliedby$ . Let  $t(A_1), \dots, t(A_n) \vDash_{P_{\mathcal{M}(L)}} t(B)$ .

Given an interpretation  $I: \text{Va} \longrightarrow L$  such that  $v_I(A_1) = \dots = v_I(A_n) = 1$ , by Lemma 3.3 there exists a function  $u: \text{Va} \longrightarrow M$  such that  $u^*(t(A_1)), \dots, u^*(t(A_n)) \in D$ . So  $u^*(t(B)) \in D$  by hypothesis. Hence, again by Lemma 3.3,  $v_I(B) = 1$ . Therefore  $A_1, \dots, A_n \vDash_{P_L} B$ .  $\square$

**Theorem 3.6.** *For  $A_1, \dots, A_n, B$  annotated formulas in  $\text{Fm}_L$  the following are equivalent:*

- (i)  $A_1, \dots, A_n \vdash_{P_L} B$
- (ii)  $A_1, \dots, A_n \vDash_{P_L} B$
- (iii)  $t(A_1), \dots, t(A_n) \vdash_{P_{\mathcal{M}(L)}} t(B)$

*Proof.* (i)  $\iff$  (ii) is Theorem 2.3.

(ii)  $\iff$  (iii) is Theorem 3.5.  $\square$

**3.3. The algebraizability of  $P_{\mathcal{M}(L)}$ .** By the *Leibniz congruence of  $D$* , in symbols  $\Omega_M D$ , we mean the largest congruence  $\Phi$  on  $\mathbf{M}(L)$  that is compatible with  $D$  in the sense that  $a \equiv b \pmod{\Phi}$  and  $a \in D$  implies  $b \in D$ . It is well known that  $a \equiv b \pmod{\Omega_M D}$  iff, for every  $\varphi(p, q_0, \dots, q_{n-1}) \in \text{Fm}_{\mathcal{M}(L)}$  and every sequence  $c_0, \dots, c_{n-1}$  of elements of  $\mathbf{M}$ ,

$$\varphi^{\mathbf{M}}(a, c_0, \dots, c_{n-1}) \in D \text{ iff } \varphi^{\mathbf{M}}(b, c_0, \dots, c_{n-1}) \in D.$$

Let  $\kappa, \lambda \in L$ . By the  $\kappa$ -signature of  $\lambda$  we mean the set  $\text{sig}_\kappa \lambda = \{j < \omega : \sim^j \leq \kappa\}$ .

**Lemma 3.7.** *For all  $\kappa, \lambda, \lambda' \in L$ ,  $\langle \lambda, \kappa \rangle \equiv \langle \lambda', \kappa \rangle \pmod{\Omega_M D}$  iff  $\lambda$  and  $\lambda'$  have the same  $\kappa$ -signature.*

*Proof.* Suppose  $\lambda$  and  $\lambda'$  have the same  $\kappa$ -signature. Let  $\varphi(p, q_0, \dots, q_{n-1})$  be any formula, and let  $\bar{a} = a_0, \dots, a_{n-1} \in M$ . We prove the following result by induction on the structure of  $\varphi$ .

- (1) Either  $\varphi^{\mathbf{M}}(\langle \lambda, \kappa \rangle, \bar{a}) = \varphi^{\mathbf{M}}(\langle \lambda', \kappa \rangle, \bar{a})$ , or  $\varphi^{\mathbf{M}}(\langle \lambda, \kappa \rangle, \bar{a}) = \langle \sim^i \lambda, \kappa \rangle$  and  $\varphi^{\mathbf{M}}(\langle \lambda', \kappa \rangle, \bar{a}) = \langle \sim^i \lambda', \kappa \rangle$  for some  $i < \omega$ .

This is obvious if  $\varphi(p, \bar{q}) \in \{p, q_0, \dots, q_{n-1}\}$ . Suppose  $\varphi(p, \bar{q})$  is of the form  $\Box \psi(p, \bar{q})$  with  $\Box \in \{\neg, \#, f_\mu\}_{\mu \in L}$ . If  $\psi^{\mathbf{M}}(\langle \lambda, \kappa \rangle, \bar{a}) = \psi^{\mathbf{M}}(\langle \lambda', \kappa \rangle, \bar{a})$ , then obviously  $\varphi^{\mathbf{M}}(\langle \lambda, \kappa \rangle, \bar{a}) = \varphi^{\mathbf{M}}(\langle \lambda', \kappa \rangle, \bar{a})$ . So we assume that  $\psi^{\mathbf{M}}(\langle \lambda, \kappa \rangle, \bar{a}) = \langle \sim^i \lambda, \kappa \rangle$  and  $\psi^{\mathbf{M}}(\langle \lambda', \kappa \rangle, \bar{a}) = \langle \sim^i \lambda', \kappa \rangle$ . If  $\Box = \neg$ , then  $\psi^{\mathbf{M}}(\langle \lambda, \kappa \rangle, \bar{a}) = \langle \sim^{i+1} \lambda, \kappa \rangle$  and  $\psi^{\mathbf{M}}(\langle \lambda', \kappa \rangle, \bar{a}) = \langle \sim^{i+1} \lambda', \kappa \rangle$ .

If  $\square = \#$ , then  $\psi^M(\langle \lambda, \kappa \rangle, \bar{a}) = \#^M \langle \sim^i \lambda, \kappa \rangle = \#^M \langle \sim^i \lambda', \kappa \rangle = \psi^M(\langle \lambda', \kappa \rangle, \bar{a})$ , since  $\sim^i \lambda \leq \kappa$  iff  $\sim^i \lambda' \leq \kappa$  by the assumption that  $\lambda$  and  $\lambda'$  have the same  $\kappa$ -signature.

If  $\square = f_\mu$ , then  $\psi^M(\langle \lambda, \kappa \rangle, \bar{a}) = f_\mu^M \langle \sim^i \lambda, \kappa \rangle = \langle \mu, \kappa \rangle = f_\mu^M \langle \sim^i \lambda', \kappa \rangle = \psi^M(\langle \lambda', \kappa \rangle, \bar{a})$ .

We now assume  $\varphi(p, \bar{q}) = \psi(p, \bar{q}) \star \vartheta(p, \bar{q})$  for  $\star \in \{\rightarrow, \wedge, \vee\}$ . By the induction hypothesis and the premiss that  $\lambda$  and  $\lambda'$  have the same  $\kappa$ -signature we have that  $\#^M \psi^M(\langle \lambda, \kappa \rangle, \bar{a}) = \#^M \psi^M(\langle \lambda', \kappa \rangle, \bar{a})$  and  $\#^M \vartheta^M(\langle \lambda, \kappa \rangle, \bar{a}) = \#^M \vartheta^M(\langle \lambda', \kappa \rangle, \bar{a})$ . Thus

$$\begin{aligned} \psi^M(\langle \lambda, \kappa \rangle, \bar{a}) &= \#^M \psi(\langle \lambda, \kappa \rangle, \bar{a}) \star^2 \#^M \vartheta(\langle \lambda, \kappa \rangle, \bar{a}) \\ &= \#^M \psi(\langle \lambda', \kappa \rangle, \bar{a}) \star^2 \#^M \vartheta(\langle \lambda', \kappa \rangle, \bar{a}) \\ &= \psi^M(\langle \lambda', \kappa \rangle, \bar{a}). \end{aligned}$$

So (1) holds. It follows immediately that  $\varphi^M(\langle \lambda, \kappa \rangle, \bar{a}) \in D$  iff  $\varphi^M(\langle \lambda', \kappa \rangle, \bar{a}) \in D$  and hence that  $\langle \lambda, \kappa \rangle \equiv \langle \lambda', \kappa \rangle \pmod{\Omega_M D}$ .

If  $\lambda$  and  $\lambda'$  do not have the same  $\kappa$ -signature, then for some  $i < \omega$ ,  $\sim^i \lambda \leq \kappa$  and  $\sim^i \lambda' \not\leq \kappa$ , or vice-versa. In this case take  $\varphi(p) = \neg^i p$ . Then either  $\varphi^M(\langle \lambda, \kappa \rangle) \in D$  and  $\varphi^M(\langle \lambda', \kappa \rangle) \notin D$  or vice-versa. Thus  $\langle \lambda, \kappa \rangle \not\equiv \langle \lambda', \kappa \rangle \pmod{\Omega_M D}$ .  $\square$

As an easy consequence of this result we have that

$$\Omega_M D = \{ \langle \langle \lambda, \kappa \rangle, \langle \lambda', \kappa \rangle \rangle : \lambda, \lambda', \kappa \in L, \text{sig}_\kappa \lambda = \text{sig}_\kappa \lambda' \} \cup \text{Id}_M;$$

to see this we observe that  $1, \mathbb{T} \not\equiv 0 \pmod{\Omega_M D}$  since  $1, \mathbb{T} \in D$  and  $0 \notin D$ ;  $1 \not\equiv \mathbb{T} \pmod{\Omega_M D}$  since  $-^M \mathbb{T} \in D$  while  $-^M 1 \notin D$ ; for all  $\kappa, \lambda \in L$  with  $\kappa \neq \mathbb{T}$ , we have  $0, 1, \mathbb{T} \not\equiv \langle \lambda, \kappa \rangle \pmod{\Omega_M D}$  because  $f_{\mathbb{T}} 0 = f_{\mathbb{T}} 1 = f_{\mathbb{T}} \mathbb{T} = \mathbb{T} \in D$  and  $f_{\mathbb{T}} \langle \lambda, \kappa \rangle = \langle \mathbb{T}, \kappa \rangle \notin D$ .

Let  $M^*(L) = M(L)/\Omega_M D$ ,  $D^* = D/\Omega_M D$ , and

$$\mathcal{M}^*(L) = \langle M^*(L), D^* \rangle.$$

$\mathcal{M}^*(L)$  is the *reduced form* of  $\mathcal{M}(L)$ . In the sequel we write  $[\langle \lambda, \kappa \rangle]$  for the  $(\Omega_M)$ -equivalence class of  $\langle \lambda, \kappa \rangle$ ; we denote the equivalence classes of  $0, 1, \mathbb{T}$  by the same symbols however because they are singletons.

Let  $\text{or}_L$  be the smallest natural number  $i$  such that for every  $\kappa \in L$  there exists a  $j < i$  such that  $\sim^i \kappa = \sim^j \kappa$ .  $\text{or}_L$  is called the  $\sim$ -order of  $L$ . Let  $\text{cf}_L$  be the smallest natural number  $i$  such that, for every  $\kappa \in L$ ,  $\sim^{\text{or}_L} \kappa = \sim^i \kappa$ .  $\text{cf}_L$  is called the *cycle-free length* of  $L$ . Then  $\sim^{\text{or}_L} \kappa = \sim^{\text{cf}_L} \kappa$  for every  $\kappa \in L$ , and clearly  $\text{cf}_L < \text{or}_L$ .

We define some special terms that will be used in the axiomatization. For any  $\kappa \in L$  and any formula  $\varphi$  let

$$S_\kappa \varphi = f_\kappa \varphi \wedge \bigwedge_{\lambda \not\leq \kappa} \neg \# f_\lambda \varphi.$$

Note that  $S_{\mathbb{T}} \varphi = f_{\mathbb{T}} \varphi$ . For all  $\langle \lambda, \kappa \rangle \in L \times (L \setminus \{\mathbb{T}\})$  let

$$T_{\langle \lambda, \kappa \rangle} \varphi = \bigwedge_{\substack{j < \text{or}_L \\ \sim^j \lambda \leq \kappa}} \neg^j \varphi \wedge \bigwedge_{\substack{j < \text{or}_L \\ \sim^j \lambda \not\leq \kappa}} \neg \# \neg^j \varphi.$$

**Lemma 3.8.** *For all  $a, b \in M$ ,*

- (i)  $a \in D$  iff  $\#^M a \in D$ ,
- (ii)  $a \notin D$  iff  $-^M \#^M a \in D$ .

(iii)  $a \rightarrow^M b \in D$  iff  $a \notin D$  or  $b \in D$ ,

For every finite system  $a_0, \dots, a_{n-1}$  of elements of  $M$ ,

(iv)  $\prod_{i < n}^M a_i \in D$  iff  $a_i \in D$  for every  $i < n$ .

(v)  $\sum_{i < n}^M a_i \in D$  iff  $a_i \in D$  for some  $i < n$ .

*Proof.* (i).  $a \in D$  implies  $\#^M a = 1 \in D$ ;  $a \notin D$  implies  $\#^M a = 0 \notin D$ .

(ii).  $a \notin D$  implies  $-^M \#^M a = -^M 0 = 1 \in D$ ;  $a \in D$  implies  $-^M \#^M a = -^M 1 = 0 \notin D$ .

(iii).  $a \rightarrow^M b \in D$  iff  $\#^M a \rightarrow^2 \#^M b = 1$  iff  $\#^M a = 0$  or  $\#^M b = 1$  iff  $a \notin D$  or  $b \in D$ .

(iv).  $\prod_{i < n}^M a_i \in D$  iff  $\prod_{i < n}^2 \#^M a_i = 1$  iff  $\#^M a_i = 1$  for every  $i < n$  iff  $a_i \in D$  for every  $i < n$ .

(v). The proof is similar to that of (iv).  $\square$

**Lemma 3.9.** For any  $a \in M$ ,

(i)  $f_{\top}^M a \in D$  iff  $a \in \{0, 1, \top\}$ ,

(ii) if  $\kappa \neq \top$ ,  $S_{\kappa}^M a \in D$  iff  $a = \langle \mu, \kappa \rangle$  for some  $\mu \in L$ .

*Proof.* (i). If  $a \notin \{0, 1, \top\}$ , i.e.,  $a = \langle \mu, \kappa \rangle \in L \times (L \setminus \{\top\})$ , then  $f_{\top}^M a = \langle \top, \kappa \rangle \notin D$ . If  $a \in \{0, 1, \top\}$ , then  $f_{\top}^M a = \top \in D$ .

(ii). Assume  $\kappa \neq \top$ . Then

$$\begin{aligned} S_{\kappa}^M a \in D &\text{ iff } f_{\kappa}^M a \in D \text{ and } -^M \#^M f_{\lambda}^M a \in D \text{ for all } \lambda \not\leq \kappa \\ &\text{ iff } f_{\kappa}^M a \in D \text{ and } f_{\lambda}^M a \notin D \text{ for all } \lambda \not\leq \kappa. \end{aligned}$$

If  $a = \langle \mu, \nu \rangle \in L \times (L \setminus \{\top\})$ , then for all  $\lambda \not\leq \kappa$ ,

$$\begin{aligned} S_{\kappa}^M a \in D &\text{ iff } \langle \kappa, \nu \rangle \in D \text{ and } \langle \kappa, \nu \rangle \notin D \text{ for all } \lambda \not\leq \kappa \\ &\text{ iff } \kappa \leq \nu \text{ and } \lambda \not\leq \nu \text{ for all } \lambda \not\leq \kappa \\ &\text{ iff } \nu = \kappa \text{ by Lem. 3.7.} \end{aligned}$$

If  $a \in \{0, 1, \top\}$ , then  $f_{\top}^M a \in D$  by (i). Since  $\top \not\leq \kappa$ , have have  $S_{\kappa}^M a \notin D$ .  $\square$

**Lemma 3.10.** Let  $a \in M$  and  $\langle \lambda, \kappa \rangle \in L \times (L \setminus \{\top\})$ . Then  $S_{\kappa}^M a, T_{\langle \lambda, \kappa \rangle} a \in D$  iff  $a \equiv \langle \lambda, \kappa \rangle \pmod{\Omega_M D}$ .

*Proof.* Assume  $S_{\kappa}^M a \in D$ . Then by Lem. 3.9,  $a = \langle \mu, \kappa \rangle$  for some  $\mu \in L$ . Thus

$$\begin{aligned} T_{\langle \lambda, \kappa \rangle}^M a \in D &\text{ iff } \sim^j \langle \mu, \kappa \rangle \in D \text{ for each } j < \text{or}_L, \sim^j \lambda \leq \kappa \text{ and} \\ &\sim^j \langle \mu, \kappa \rangle \notin D \text{ for each } j < \text{or}_L, \sim^j \lambda \not\leq \kappa \\ &\text{ iff } \langle \sim^j \mu, \kappa \rangle \in D \text{ iff } \sim^j \lambda \leq \kappa, \text{ for all } j < \text{or}_L \\ &\text{ iff } \sim^j \mu \leq \kappa \text{ iff } \sim^j \lambda \leq \kappa, \text{ for all } j < \text{or}_L \\ &\text{ iff } \mu \text{ and } \lambda \text{ have the same } \kappa\text{-signature} \\ &\text{ iff } a \equiv \langle \lambda, \kappa \rangle \pmod{\Omega_M D} \text{ by Lem. 3.7.} \end{aligned}$$

If  $S_{\kappa}^M a \notin D$ , then  $a \in \{0, 1, \top\}$ ; thus  $a \not\equiv \langle \lambda, \kappa \rangle \pmod{\Omega_M D}$  for all  $\langle \lambda, \kappa \rangle \in L \times (L \setminus \{\top\})$ .  $\square$

As usual,  $\varphi \leftrightarrow \psi$  is an abbreviation of  $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ . We also use  $\mathbf{1}$  for an abbreviation of  $\top \rightarrow \top$ . Note that  $\mathbf{1}^M = 1$ .

**Lemma 3.11.** *Assume  $\mathbf{L}$  is finite. For all  $a, b \in M(\mathbf{L})$ ,  $a \equiv b \pmod{\Omega_M D}$  iff the following hold.*

- (i)  $a \leftrightarrow^M b \in D$ ,
- (ii)  $\neg^M a \leftrightarrow^M \neg^M b \in D$ ,
- (iii)  $f_{\top}^M a \leftrightarrow^M f_{\top}^M b \in D$ ,
- (iv)  $S_{\kappa}^M a \leftrightarrow^M S_{\kappa}^M b \in D$  for all  $\kappa \in L \setminus \{\top\}$ ,
- (v)  $T_{\langle \lambda, \kappa \rangle}^M a \leftrightarrow^M T_{\langle \lambda, \kappa \rangle}^M b \in D$  for all  $\langle \lambda, \kappa \rangle \in L \times L \setminus \{\top\}$ .

*Proof.*  $\Leftarrow$ :  $\imath$ From  $f_{\top}^M a \leftrightarrow^M f_{\top}^M b \in D$  we get  $f_{\top}^M a \in D$  iff  $f_{\top}^M b \in D$  by Lem. 3.8. Thus by Lem. 3.9 either  $a, b \in \{0, 1, \top\}$  or  $a, b \in L \times L \setminus \{\top\}$ . Assume first of all that  $a, b \in \{0, 1, \top\}$ .  $\imath$ From  $a \leftrightarrow^M b \in D$  get that either  $a = b = 0$  or  $a, b \in \{1, \top\}$ , and in the latter case we can conclude from  $\neg^M a \leftrightarrow \neg^M b \in D$  that either  $a = b = 1$  or  $a = b = \top$ . Now suppose  $a, b \in L \times L \setminus \{\top\}$ .  $\imath$ From (iv) and (v) we have by Lems. 3.9 and 3.10 that  $a = b = [\langle \lambda, \kappa \rangle]$  for some  $\langle \lambda, \kappa \rangle \in L \times L \setminus \top$ .

$\Rightarrow$ : The proof is similar and is omitted.  $\square$

$\imath$ From the lemma we get that, if  $\mathbf{L}$  is finite,

$$\Delta(x, y) = \{x \leftrightarrow y, \neg x \leftrightarrow y, f_{\top} x \leftrightarrow f_{\top} y\} \cup \{S_{\kappa} x \leftrightarrow S_{\kappa} y : \kappa \in L \setminus \{\top\}\} \\ \cup \{T_{\langle \lambda, \kappa \rangle} x \leftrightarrow T_{\langle \lambda, \kappa \rangle} y : \langle \lambda, \kappa \rangle \in L \setminus \{\top\}\}.$$

is an *finite equivalence system* for  $P_{\mathcal{M}(\mathbf{L})}$  in the sense of [4]. Furthermore, from the fact that  $\#^M a = 1$  iff  $a \in D$ , we have that

$$\delta(x) \approx \eta(x) = \# x \approx \mathbf{1}$$

is a *defining equation* for  $P_{\mathcal{M}(\mathbf{L})}$ , also in the sense of [4]. Hence  $P_{\mathcal{M}(\mathbf{L})}$  is *algebraizable* in the sense of [4]; in the current standard terminology  $P_{\mathcal{M}(\mathbf{L})}$  is said to be *finitely algebraizable*.

It follows that for a unique quasivariety  $\mathbf{Q}$  there is a bisimulation in the usual sense between  $\vdash_{P_{\mathcal{M}(\mathbf{L})}}$  and the consequence relation  $\vDash_{\mathbf{Q}}$  of the equational logic of  $\mathbf{Q}$ . Moreover, the simulation  $\vdash_{P_{\mathcal{M}(\mathbf{L})}}$  by  $\vDash_{\mathbf{Q}}$  is given by the defining equation  $\# x \approx \mathbf{1}$ , that is, for all  $\varphi_0, \dots, \varphi_{n-1}, \psi \in \mathbf{Fm}_{\mathcal{M}(\mathbf{L})}$ ,

$$(2) \quad \varphi_0, \dots, \varphi_{n-1} \vdash_{P_{\mathcal{M}(\mathbf{L})}} \psi \quad \text{iff} \quad \# \varphi_0 \approx \mathbf{1}, \dots, \# \varphi_{n-1} \approx \mathbf{1} \vDash_{\mathbf{Q}} \# \psi \approx \mathbf{1}.$$

We will show in Section 4.1, as a consequence of the the axiomatization of  $P_{\mathcal{M}(\mathbf{L})}$  given in the next section, that  $\mathbf{Q}$  is the quasivariety generated by  $\mathbf{M}^*(\mathbf{L})$ .

If  $\mathbf{L}$  is not finite, then  $S_{\kappa} x \leftrightarrow S_{\kappa} y$  and  $T_{\langle \lambda, \kappa \rangle} x \leftrightarrow T_{\langle \lambda, \kappa \rangle} y$  are infinitary formulas. They can however be replaced respectively by the infinite sets of finitary formulas

$$\{f_{\kappa} x \leftrightarrow f_{\kappa} y\} \cup \{\neg \# f_{\lambda} x \leftrightarrow \neg \# f_{\lambda} y : \lambda \not\leq \kappa\}, \\ \{\neg^j x \leftrightarrow \neg^j y : \sim^j \lambda \leq \kappa, j < \omega\} \cup \{\neg \# \neg^j x \leftrightarrow \neg \# \neg^j y : \sim^j \lambda \not\leq \kappa, j < \omega\}.$$

$\Delta(x, y)$  with these replacements is a (infinite) equivalence system for  $P_{\mathcal{M}(\mathbf{L})}$ , and  $P_{\mathcal{M}(\mathbf{L})}$  is *algebraizable* in the sense of [7, 8].

4. AN AXIOMATIZATION OF  $P_{\mathcal{M}(\mathbf{L})}$ 

In this section we give a set of axioms and rules for  $P_{\mathcal{M}(\mathbf{L})}$  and prove that this axiomatization is sound and complete.

The axioms for  $P_{\mathcal{M}(\mathbf{L})}$  consist of the following formulas, for all  $\varphi, \psi, \vartheta \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ . In the sequel  $\kappa, \lambda, \mu, \nu$  are assumed to range over all elements of the lattice  $\mathbf{L}$  except where otherwise indicated. For formulas of the form  $\varphi \rightarrow \psi \rightarrow \vartheta$  association is assumed to the right.

*The positive Boolean axioms of  $P_{\mathbf{L}}$ :*

- ( $\rightarrow_0$ )  $\varphi \rightarrow (\psi \rightarrow \varphi)$ .
- ( $\rightarrow_1$ )  $(\varphi \rightarrow (\psi \rightarrow \vartheta)) \rightarrow (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \vartheta)$ .
- ( $\rightarrow_2$ )  $((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$ .
- ( $\wedge_0$ )  $\varphi \wedge \psi \rightarrow \varphi$ .
- ( $\wedge_1$ )  $\varphi \wedge \psi \rightarrow \psi$ .
- ( $\wedge_2$ )  $\varphi \rightarrow \psi \rightarrow (\varphi \wedge \psi)$ .
- ( $\vee_0$ )  $\varphi \rightarrow \varphi \vee \psi$ .
- ( $\vee_1$ )  $\psi \rightarrow \varphi \vee \psi$ .
- ( $\vee_2$ )  $(\varphi \rightarrow \vartheta) \rightarrow (\psi \rightarrow \vartheta) \rightarrow (\varphi \vee \psi \rightarrow \vartheta)$ .

*The axiom of paraconsistent negation:*

- ( $\neg \#$ )  $(\neg \# \varphi \rightarrow \neg \# \psi) \rightarrow (\neg \# \varphi \rightarrow \# \psi) \rightarrow \# \varphi$ .

*The axioms of the lattice  $L$ :*

- ( $L_0$ )  $f_\lambda f_\kappa \varphi \leftrightarrow f_\lambda \varphi$ .
- ( $L_1$ )  $f_\perp \varphi$ , where  $\perp$  is the least element of  $L$ .
- ( $L_2$ )  $f_\kappa \varphi \rightarrow f_\lambda \varphi$ , if  $\lambda \leq \kappa$ .
- ( $L_3$ )  $f_\kappa \varphi \wedge f_\lambda \varphi \rightarrow f_{\kappa+\lambda} \varphi$ , for all  $\kappa, \lambda \in L$ , where  $\kappa + \lambda$  is the join of  $\kappa, \lambda \in L$ .

*The axioms of the annotation lattice  $\mathbf{L}$ :*

- (ann $_0$ )  $\neg^j f_\lambda \varphi \leftrightarrow f_{\sim^j \lambda} \varphi$ , for all  $j < \text{or}_{\mathbf{L}}$ .
- (ann $_1$ )  $f_\lambda \neg \varphi \leftrightarrow f_\lambda \varphi$ .
- (ann $_2$ )  $\neg \# f_\top \varphi \rightarrow (\neg^{\text{or}_{\mathbf{L}}} \varphi \leftrightarrow \neg^{\text{cf}_{\mathbf{L}}} \varphi)$ .
- (ann $_3$ )  $S_\kappa \varphi \rightarrow \bigvee_{\lambda \in L} T_{\langle \lambda, \kappa \rangle} \varphi$ .

*The axioms of Boolean content:*

- (#0)  $\varphi \leftrightarrow \# \varphi$ .
- (#1)  $f_\top \# \varphi$ .
- (#2)  $f_\top \varphi \rightarrow (\neg \# \varphi \rightarrow \neg \varphi)$ .
- (#3)  $f_\top \varphi \rightarrow (\neg \neg \varphi \leftrightarrow \varphi)$ .
- (#4)  $f_\top (\varphi \rightarrow \psi)$ .
- (#5)  $f_\top (\varphi \wedge \psi)$ .
- (#6)  $f_\top (\varphi \vee \psi)$ .
- (#7)  $\neg (\varphi \rightarrow \psi) \rightarrow \neg \# (\varphi \rightarrow \psi)$ .
- (#8)  $\neg (\varphi \wedge \psi) \rightarrow \neg \# (\varphi \wedge \psi)$ .
- (#9)  $\neg (\varphi \vee \psi) \rightarrow \neg \# (\varphi \vee \psi)$ .

Let  $\mathcal{S}(\mathbf{L}) = \langle \text{Fm}_{\mathcal{M}(\mathbf{L})}, \vdash_{\mathcal{S}(\mathbf{L})} \rangle$  be the deductive system defined by the preceding axioms with

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi} \quad \text{modus ponens}$$

as the only rule of inference.  $\vdash_{\mathcal{S}(\mathbf{L})}$  is the consequence relation of  $\mathcal{S}(\mathbf{L})$ . Thus, for any  $\Gamma \subseteq \text{Fm}_{\mathcal{M}(\mathbf{L})}$  and any  $\varphi \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ ,  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \varphi$  iff there is a finite sequence  $\psi_0, \dots, \psi_n$  of formulas such that  $\psi_n = \varphi$  and, for each  $i \leq n$ , either  $\psi_i \in \Gamma$ ,  $\psi_i$  is an axiom, or  $\psi_i$  is a direct consequence of previous formulas in the sequence by modus ponens. When  $\mathbf{L}$  is fixed we write  $\mathcal{S}$  for  $\mathcal{S}(\mathbf{L})$ .

The deduction theorem holds in  $\mathcal{S}$ , i.e., for all  $\Gamma \cup \{\varphi, \psi\} \subseteq \text{Fm}_{\mathcal{M}(\mathbf{L})}$ ,

$$\Gamma, \varphi \vdash_{\mathcal{S}(\mathbf{L})} \psi \quad \text{implies} \quad \Gamma \vdash_{\mathcal{S}(\mathbf{L})} \varphi \rightarrow \psi.$$

This is an immediate consequence of Axs.  $(\rightarrow_0)$  and  $(\rightarrow_1)$  and the fact that modus ponens is the only inference rule.

By a *theory of  $\mathcal{S}(\mathbf{L})$*  we mean any set  $\Theta$  of formulas that is closed under the consequence relation  $\vdash_{\mathcal{S}(\mathbf{L})}$ , i.e.,  $\Gamma \subseteq \Theta$  and  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \varphi$  implies  $\varphi \in \Theta$ .

We first prove the soundness of  $\mathcal{S}(\mathbf{L})$  (with respect to  $\vdash_{\mathcal{P}_{\mathcal{M}(\mathbf{L})}}$ ).

**Lemma 4.1.** *For each of the axioms  $\psi$  of the lattice  $\mathbf{L}$ ,  $u^*\psi \in D$  for every  $u: \text{Va} \rightarrow M$ .*

*Proof.*  $(L_0)$ :  $\psi = \text{f}_\lambda \text{f}_\kappa \varphi \leftrightarrow \text{f}_\lambda \varphi$ .

If  $u^*\varphi \in \{0, 1, \mathbb{T}\}$ , then  $u^*(\text{f}_\lambda \text{f}_\kappa \varphi) = \text{f}_\lambda^M \text{f}_\kappa^M u^*\varphi = \text{f}_\lambda^M \mathbb{T} = \mathbb{T} = \text{f}_\lambda^M u^*\varphi$ . If  $u^*\varphi = \langle \mu, \nu \rangle$ , then  $u^*(\text{f}_\lambda \text{f}_\kappa \varphi) = \text{f}_\lambda^M \text{f}_\kappa^M \langle \mu, \nu \rangle = \text{f}_\lambda^M \langle \kappa, \nu \rangle = \langle \lambda, \nu \rangle = \text{f}_\lambda^M \langle \mu, \nu \rangle$ . So, in both cases,  $u^*(\text{f}_\lambda \text{f}_\kappa \varphi)$  and  $u^*(\text{f}_\lambda \varphi)$  are the same element of  $M$ , say  $a$ . Hence  $u^*\psi = \#^M a \leftrightarrow^2 \#^M a = 1 \in D$ .

$(L_1)$ :  $\psi = \text{f}_\perp \varphi$ .

If  $u^*\varphi \in \{0, 1, \mathbb{T}\}$ , then  $u^*\psi = \mathbb{T} \in D$ . If  $u^*\varphi = \langle \mu, \nu \rangle$ , then  $u^*\psi = \langle \perp, \nu \rangle \in D$ .

$(L_2)$ :  $\psi = \text{f}_\kappa \varphi \rightarrow \text{f}_\lambda \varphi$  with  $\lambda \leq \kappa$ .

If  $u^*\varphi \in \{0, 1, \mathbb{T}\}$ , then  $u^*\text{f}_\lambda \varphi = \mathbb{T} \in D$ . So  $u^*\psi \in D$  by Lem. 3.8(iii). If  $u^*\varphi = \langle \mu, \nu \rangle$ , then  $u^*\text{f}_\kappa \varphi = \langle \kappa, \nu \rangle$  and  $u^*\text{f}_\lambda \varphi = \langle \lambda, \nu \rangle$ . If  $u^*\text{f}_\kappa \varphi \in D$ , then  $\kappa \leq \nu$ . Thus  $\lambda \leq \nu$  and hence  $u^*\text{f}_\lambda \varphi \in D$ . So  $u^*\psi \in D$  by Lem. 3.8(iii) again.

$(L_3)$ :  $\psi = \text{f}_\kappa \varphi \wedge \text{f}_\lambda \varphi \rightarrow \text{f}_{\kappa+\lambda} \varphi$ .

If  $u^*\varphi \in \{0, 1, \mathbb{T}\}$ , then  $u^*\text{f}_{\kappa+\lambda} \varphi = \mathbb{T} \in D$ . So  $u^*\psi \in D$ . If  $u^*\varphi = \langle \mu, \nu \rangle$ , then  $u^*\text{f}_\kappa \varphi = \langle \kappa, \nu \rangle$  and  $u^*\text{f}_\lambda \varphi = \langle \lambda, \nu \rangle$ . If  $u^*\text{f}_\kappa \varphi, u^*\text{f}_\lambda \varphi \in D$ , then  $\kappa, \lambda \leq \nu$ , and hence  $\kappa + \lambda \leq \nu$ , which implies  $u^*\text{f}_{\kappa+\lambda} \varphi \in D$ . So  $u^*\psi \in D$  by Lem. 3.8(iv).  $\square$

**Lemma 4.2.** *For each of the axioms  $\psi$  of the annotation lattice  $\mathbf{L}$ ,  $u^*\psi \in D$  for every  $u: \text{Va} \rightarrow M$ .*

*Proof.*  $(\text{ann}_0)$ :  $\psi = \neg^j \text{f}_\lambda \varphi \leftrightarrow \text{f}_{\sim^j \lambda} \varphi$  with  $j < \text{or}_{\mathbf{L}}$ .

If  $j = 0$ , then  $\neg^j \text{f}_\lambda \varphi = \text{f}_{\sim^j \lambda} \varphi = \text{f}_\lambda \varphi$ . So assume  $j > 0$ .

If  $u^*\varphi \in \{0, 1, \mathbb{T}\}$ , then  $u^*\neg^j \text{f}_\lambda \varphi = u^*\text{f}_{\sim^j \lambda} \varphi = \mathbb{T}$ .

If  $u^*\varphi = \langle \mu, \nu \rangle \in L \times (L \setminus \{\mathbb{T}\})$ , then  $u^*\neg^j \text{f}_\lambda \varphi = -^M j \langle \lambda, \nu \rangle = \langle \sim^j \lambda, \nu \rangle = u^*\text{f}_{\sim^j \lambda} \varphi$ .

$(\text{ann}_1)$ :  $\psi = \text{f}_\lambda \neg \varphi \leftrightarrow \text{f}_\lambda \varphi$ .

If  $u^*\varphi \in \{0, 1, \mathbb{T}\}$ , then  $u^*\neg \text{f}_\lambda \varphi = -^M \text{f}_\lambda^M u^*\varphi = -^M \mathbb{T} = \mathbb{T}^M \mathbb{T} = \text{f}_\lambda^M u^*\varphi = u^*\text{f}_\lambda \varphi$ .

If  $u^*\varphi = \langle \mu, \nu \rangle$ , then  $u^*\text{f}_\lambda \neg \varphi = \text{f}_\lambda^M \langle \sim \mu, \nu \rangle = \langle \lambda, \nu \rangle = u^*\text{f}_\lambda \varphi$ .

(ann<sub>2</sub>):  $u^*\psi = \neg \# \text{f}_\top \varphi \rightarrow (\neg^{\text{orL}} \varphi \leftrightarrow \neg^{\text{cfL}} \varphi)$ .

Assume  $u^* \neg \# \text{f}_\top \varphi \in D$ . Then  $u^* \text{f}_\top \varphi \notin D$  by Lem. 3.8(ii), and hence  $u^*\varphi \in L \times (L \setminus \{\top\})$  by Lem. 3.9(i). Let  $u^*\varphi = \langle \mu, \nu \rangle$ . Then  $u^* \neg^{\text{orL}} \varphi = -^M \text{orL} \langle \mu, \nu \rangle = \langle \sim^{\text{orL}} \mu, \nu \rangle = \langle \sim^{\text{cfL}} \mu, \nu \rangle = u^* \neg^{\text{cfL}} \varphi$ . So  $u^*(\neg^{\text{orL}} \varphi \leftrightarrow \neg^{\text{cfL}} \varphi) \in D$ .

(ann<sub>3</sub>):  $\psi = \text{S}_\kappa \varphi \rightarrow \bigvee_{\lambda \in L} \text{T}_{\langle \lambda, \kappa \rangle} \varphi$ .

If  $u^* \text{S}_\kappa \varphi \in D$ , then  $u^*\varphi = \langle \mu, \kappa \rangle$  for some  $\mu \in L$  by Lem. 3.9. Thus  $u^* \text{T}_{\langle \mu, \kappa \rangle} \varphi \in D$  by Lem. 3.10, and hence  $u^*(\bigvee_{\lambda \in L} \text{T}_{\langle \lambda, \kappa \rangle} \varphi) \in D$ .  $\square$

**Lemma 4.3.** *For each of the axioms  $\psi$  of Boolean content,  $u^*\psi \in D$  for every  $u: \text{Va} \rightarrow M$ .*

*Proof.* (#<sub>0</sub>):  $\psi = \varphi \leftrightarrow \# \varphi$ .

$u^*\psi \in D$  by Lem. 3.8(i).

(#<sub>1</sub>):  $\psi = \text{f}_\top \# \varphi$ .

$u^* \# \varphi \in \{0, 1\}$ . So  $u^*\psi = \text{f}_\top^M u^* \# \varphi = \mathbb{T} \in D$ .

(#<sub>2</sub>):  $\psi = \text{f}_\top \varphi \rightarrow (\neg \# \varphi \rightarrow \neg \varphi)$ .

Assume  $u^* \text{f}_\top \varphi \in D$ . Then  $u^*\varphi \in \{0, 1, \mathbb{T}\}$  by Lem. 3.9(i). If  $u^*\varphi \in \{0, 1\}$ , then  $u^* \neg \# \varphi = -^M \#^M u^*\varphi = -^M u^*\varphi = u^* \neg \varphi$ . If  $u^*\varphi = \mathbb{T}$ , then  $u^* \neg \varphi = -^M \mathbb{T} = \mathbb{T} \in D$ . So in both cases we have  $u^*(\neg \# \varphi \rightarrow \neg \varphi) \in D$ .

(#<sub>3</sub>):  $\psi = \text{f}_\top \varphi \rightarrow (\neg \neg \varphi \leftrightarrow \varphi)$ .

Assume  $u^* \text{f}_\top \varphi \in D$ , i.e.,  $u^*\varphi \in \{0, 1, \mathbb{T}\}$ . Then  $u^* \neg \neg \varphi = -^M -^M u^*\varphi = u^*\varphi$ .

(#<sub>4</sub>)–(#<sub>6</sub>):  $\psi = \text{f}_\top(\varphi \star \vartheta)$  with  $\star \in \{\rightarrow, \wedge, \vee\}$ .

$u^*(\varphi \star \vartheta) = \#^M u^*\varphi \star^2 \#^M u^*\vartheta \in \{0, 1\}$ . So  $u^*\psi = \text{f}_\top^M u^*(\varphi \star \vartheta) = \mathbb{T} \in D$ .

(#<sub>7</sub>)–(#<sub>9</sub>):  $\psi = \neg(\varphi \star \vartheta) \rightarrow \neg \#(\varphi \star \vartheta)$ , with  $\star \in \{\rightarrow, \wedge, \vee\}$ .

Then  $u^*(\varphi \star \vartheta) \in \{0, 1\}$  and hence  $u^*(\varphi \star \vartheta) = u^* \#(\varphi \star \vartheta)$ .  $\square$

**Theorem 4.4** (soundness of  $\mathcal{S}(\mathbf{L})$ ). *For all  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{M}(\mathbf{L})}$ ,*

$$\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \varphi \quad \text{implies} \quad \Gamma \vdash_{\text{P}_{\mathcal{M}(\mathbf{L})}} \varphi.$$

*Proof.* Assume  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \varphi$  and let  $u: \text{Va} \rightarrow M$  such that  $u^*\psi \in D$  for each  $\psi \in \Gamma$ . Let  $\vartheta_0, \dots, \vartheta_n$  with  $\vartheta_n = \vartheta$  be an  $\mathcal{S}(\mathbf{L})$ -derivation of  $\varphi$  from  $\Gamma$ . Consider any  $i \leq n$ . If  $\vartheta_i \in \Gamma$ , then  $u^*\vartheta_i \in D$  by assumption. If  $\vartheta_i$  is an axiom of  $\mathcal{S}(\mathbf{L})$ , then  $u^*\vartheta_i \in D$  by Lems. 4.1–4.3. If  $\vartheta_j = \vartheta_k \rightarrow \vartheta_i$  for some  $j, k < i$ , then  $u^*\vartheta_k \rightarrow^M u^*\vartheta_i, u^*\vartheta_k \in D$  by the induction hypothesis, and hence  $u^*\vartheta_i \in D$  by Lem. 3.1(i). So  $u^*\vartheta \in D$  and thus  $\Gamma \vdash_{\text{P}_{\mathcal{M}(\mathbf{L})}} \varphi$ .  $\square$

We now turn to the proof of the completeness of  $\mathcal{S}(\mathbf{L})$ .

**Lemma 4.5.** (i) *Let  $\varphi(p_0, \dots, p_{n-1})$  be a positive Boolean tautology, i.e., a Boolean tautology that contains only the connectives  $\rightarrow, \wedge$ , and  $\vee$ . Then  $\vdash_{\mathcal{S}(\mathbf{L})} \varphi(\psi_0, \dots, \psi_{n-1})$  for all  $\psi_0, \dots, \psi_{n-1} \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ .*

(ii) *Let  $\varphi(p_0, \dots, p_{n-1})$  be any Boolean tautology (possibly containing the connective  $\neg$ ). Then  $\vdash_{\mathcal{S}(\mathbf{L})} \varphi(\# \psi_0, \dots, \# \psi_{n-1})$  for all  $\psi_0, \dots, \psi_{n-1} \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ .*

*Proof.* It is well known that axioms ( $\rightarrow_0$ )–( $\vee_2$ ) axiomatize all positive Boolean tautologies. When  $(\neg \varphi \rightarrow \neg \psi) \rightarrow (\neg \varphi \rightarrow \psi) \rightarrow \varphi$  is adjoined, we get all Boolean tautologies.  $\square$

**Lemma 4.6.** *For any theory  $\Theta$  of  $\mathcal{S}(\mathbf{L})$ ,  $\varphi \star \psi \leftrightarrow \# \varphi \star \# \psi \in \Theta$ .*

*Proof.*  $(\varphi \leftrightarrow \# \varphi) \rightarrow (\psi \leftrightarrow \# \psi) \rightarrow (\varphi \star \psi \leftrightarrow \# \varphi \star \# \psi)$  is a positive tautology and hence is in  $\Theta$  by Lem. 4.5. And by Ax.  $(\#_0)$   $\varphi \leftrightarrow \# \varphi$  and  $\psi \leftrightarrow \# \psi$  are both in  $\Theta$ . Thus  $\varphi \star \psi \leftrightarrow \# \varphi \star \# \psi \in \Theta$  by modus ponens.  $\square$

**Lemma 4.7.** *Let  $\Theta$  be a theory of  $\mathcal{S}(\mathbf{L})$ . Then for all  $\varphi, \psi \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$*

(i)  $\varphi \in \Theta$  iff  $\# \varphi \in \Theta$ .

*For every finite system  $\varphi_0, \dots, \varphi_{n-1}$  of elements of  $\Theta$ ,*

(ii)  $\bigwedge_{i < n} \varphi_i \in \Theta$  iff  $\varphi_j \in \Theta$  for all  $j < n$ .

*Proof.* (i): by Ax.  $(\#_0)$ .

(ii). By Lem. 4.5(i),  $\bigwedge_{i < n} \varphi_i \rightarrow \varphi_j \in \Theta$  for each  $j < n$ . So  $\bigwedge_{i < n} \varphi_i$  implies  $\varphi_j \in \Theta$  for each  $j < n$ .  $\varphi_0 \rightarrow \dots \rightarrow \varphi_{n-1} \rightarrow \bigwedge_{i < n} \varphi_i \in \Theta$ , again by Lem. 4.5(i). So  $\varphi_j \in \Theta$  for all  $j < n$  implies  $\bigwedge_{i < n} \varphi_i \in \Theta$ .  $\square$

**Definition 4.8.** A subset  $\Gamma$  of formulas is *Boolean inconsistent* if there exists a  $\varphi \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$  such that  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \varphi$  and  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \neg \# \varphi$ . If no such  $\varphi$  exists,  $\Gamma$  is *Boolean consistent*.

**Lemma 4.9.**  *$\Gamma$  is Boolean inconsistent iff  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \psi$  for all  $\psi \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ .*

*Proof.* Suppose  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \varphi$  and  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \neg \# \varphi$  for some  $\varphi \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ . Then  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \# \varphi$  by Lem. 4.7(i) and  $\vdash_{\mathcal{S}(\mathbf{L})} \# \varphi \rightarrow (\neg \# \varphi \rightarrow \# \psi)$  for every  $\psi \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$  by Lem. 4.5(ii). So  $\vdash_{\mathcal{S}(\mathbf{L})} \# \psi$  by modus ponens, and hence  $\vdash_{\mathcal{S}(\mathbf{L})} \psi$  by Lem. 4.7(i). This proves the implication from left to right. The reverse implication is obvious.  $\square$

**Lemma 4.10.** *For all  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{M}(\mathbf{L})}$ ,  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \varphi$  iff  $\Gamma \cup \{\neg \# \varphi\}$  is Boolean inconsistent.*

*Proof.* The implication from left to right is obvious. Suppose  $\Gamma \cup \{\neg \# \varphi\}$  is Boolean inconsistent. Then  $\Gamma, \neg \# \varphi \vdash_{\mathcal{S}(\mathbf{L})} \# \varphi$  and hence  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \neg \# \varphi \rightarrow \# \varphi$  by the deduction theorem. We also have  $\vdash_{\mathcal{S}(\mathbf{L})} \neg \# \varphi \rightarrow \neg \# \varphi$  by Lem. 4.5(i) and  $\vdash_{\mathcal{S}} (\neg \# \varphi \rightarrow \neg \# \varphi) \rightarrow (\neg \# \varphi \rightarrow \# \varphi) \rightarrow \# \varphi$  by Ax.  $(\neg \#)$ . By two applications of modus ponens we get  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \# \varphi$  and hence finally  $\Gamma \vdash_{\mathcal{S}(\mathbf{L})} \varphi$  by Lem. 4.7(i).  $\square$

Every maximal Boolean consistent set of formulas is a theory of  $\mathcal{S}(\mathbf{L})$ .

**Lemma 4.11.** *Let  $\Theta$  be a maximal Boolean consistent set of formulas. For all  $\varphi, \psi \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ ,*

(i)  $\varphi \notin \Theta$  iff  $\neg \# \varphi \in \Theta$ .

(ii)  $\varphi \rightarrow \psi \in \Theta$  iff  $\varphi \notin \Theta$  or  $\psi \in \Theta$ .

*For every finite system  $\varphi_0, \dots, \varphi_{n-1}$  of formulas,*

(iii)  $\bigvee_{i < n} \varphi_i \in \Theta$  iff  $\varphi_j \in \Theta$  for some  $j < n$ .

*Proof.* (i). Suppose  $\neg \# \varphi \notin \Theta$ . Then  $\Theta \cup \{\neg \# \varphi\}$  is inconsistent by the maximality of  $\Theta$ . Thus  $\Theta \vdash_{\mathcal{S}(\mathbf{L})} \varphi$  by Lem. 4.10, and hence  $\varphi \in \Theta$  since  $\Theta$  is a theory. This proves the implication from left to right.

For the reverse implication, if  $\neg \# \varphi \in \Theta$ , then  $\varphi \notin \Theta$  because  $\Theta$  is consistent.

(ii). If  $\varphi \rightarrow \psi \in \Theta$  and  $\varphi \in \Theta$ , then  $\psi \in \Theta$  by modus ponens. So  $\varphi \rightarrow \psi \in \Theta$  implies  $\varphi \notin \Theta$  or  $\psi \in \Theta$ .  $\psi \rightarrow (\varphi \rightarrow \psi) \in \Theta$  because it is a positive tautology. So, if  $\psi \in \Theta$ , then  $\varphi \rightarrow \psi \in \Theta$ . If  $\psi \notin \Theta$ , then  $\neg \# \varphi \in \Theta$ .  $\neg \# \varphi \rightarrow (\# \varphi \rightarrow \# \psi) \in \Theta$  by Lem. 4.5(ii). So  $\# \varphi \rightarrow \# \psi \in \Theta$ , and hence  $\varphi \rightarrow \psi \in \Theta$  by Lem. 4.6.

(iii).  $\varphi_j \rightarrow \bigvee_{i < n} \varphi_i \in \Theta$  for each  $j < n$  by Lem. 4.5(i). Thus  $\varphi_i \in \Theta$  for some  $j < n$  implies  $\bigvee_{i < n} \varphi_i \in \Theta$ . Suppose  $\varphi_j \notin \Theta$  for every  $j < n$ . Then  $\neg \# \varphi_j \in \Theta$  for each  $j < n$ , and hence  $\bigwedge_{i < n} \neg \# \varphi_i \in \Theta$  by (ii).  $(\bigwedge_{i < n} \neg \# \varphi_i) \rightarrow (\neg \# \bigvee_{i < n} \varphi_i) \in \Theta$  by Lem. 4.5(ii). So  $\neg \# \bigvee_{i < n} \varphi_i \in \Theta$  and hence  $\bigvee_{i < n} \varphi_i \notin \Theta$ .  $\square$

In the sequel  $\Theta$  is assumed to be a maximal Boolean consistent set of formulas.

**Lemma 4.12.** *For any  $\varphi \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ ,  $S_\kappa \varphi \in \Theta$  for exactly one  $\kappa \in L$ .*

*Proof.* Let  $K = \{\lambda \in L : f_\lambda \varphi \in \Theta\}$ .  $\perp \in K$  by Ax. (L<sub>1</sub>). If  $\mu \leq \lambda \in K$ , then  $\mu \in K$  by Ax. (L<sub>2</sub>).  $\lambda, \mu \in K$  implies  $\lambda + \mu \in K$  by Ax. (L<sub>3</sub>). Thus  $K$  is an ideal of  $L$ , and since  $L$  is a finite lattice,  $K$  is a principal ideal. Let  $\kappa = \sum K$ . Then, for all  $\lambda \not\leq \kappa$ ,  $f_\lambda \varphi \notin \Theta$  and hence  $\neg \# f_\lambda \varphi \in \Theta$  by Lem. 4.11(i). So  $S_\kappa \varphi = f_\kappa \varphi \wedge \bigwedge_{\lambda \not\leq \kappa} \neg \# f_\lambda \varphi \in \Theta$ . Suppose  $\lambda \neq \kappa$ . If  $\lambda \not\leq \kappa$ , then  $\lambda \notin K$  and hence  $f_\lambda \varphi \notin \Theta$ . So  $S_\lambda \varphi \notin \Theta$ . If  $\kappa \not\leq \lambda$ , then  $\neg \# f_\kappa \varphi$  is a conjunct of  $S_\lambda \varphi$ , and hence  $S_\lambda \varphi \notin \Theta$  because  $f_\kappa \varphi \in \Theta$ .  $\square$

The following lemma is an immediate consequence of the preceding proof.

**Lemma 4.13.** *If  $S_\kappa \varphi \in \Theta$ , then for every  $\mu \in L$ ,  $f_\mu \varphi \in \Theta$  iff  $\mu \leq \kappa$ .*  $\square$

**Lemma 4.14.** *If  $S_\kappa \varphi \in \Theta$  with  $\kappa \neq \top$ , then  $T_{\langle \lambda, \kappa \rangle} \varphi \in \Theta$  for at least one  $\lambda \in L$ . And, for every  $\mu \in L$ ,  $T_{\langle \mu, \kappa \rangle} \varphi \in \Theta$  iff  $\mu$  has the same  $\kappa$ -signature as  $\lambda$ .*

*Proof.* Assume  $S_\kappa \varphi \in \Theta$ . By Ax. (ann<sub>3</sub>) and Lem. 4.11(iii),  $T_{\langle \lambda, \kappa \rangle} \varphi \in \Theta$  for at least one  $\lambda \in L$ . Suppose  $\mu$  and  $\lambda$  have the same  $\kappa$ -signature, i.e.,  $\sim^i \mu \leq \kappa$  iff  $\sim^i \lambda \leq \kappa$ , for every  $i < \omega$ . Then  $T_{\langle \mu, \kappa \rangle} \varphi$  and  $T_{\langle \lambda, \kappa \rangle} \varphi$  have exactly the same conjuncts and hence are identical. Suppose now that  $\lambda$  and  $\mu$  do not have the same  $\kappa$ -signature. Then, for some  $j < \omega$ ,  $\sim^j \lambda \leq \kappa$  and  $\sim^j \mu \not\leq \kappa$ , or  $\sim^j \lambda \not\leq \kappa$  and  $\sim^j \mu \leq \kappa$ . If  $T_{\langle \mu, \kappa \rangle} \varphi \in \Theta$ , then in both cases  $\neg^j \varphi, \neg \# \neg^j \varphi \in \Theta$ , contradicting the Boolean consistency of  $\Theta$ . Thus  $T_{\langle \mu, \kappa \rangle} \varphi \notin \Theta$ .  $\square$

**Lemma 4.15.** *Assume  $S_\kappa \varphi$  and  $T_{\langle \lambda, \kappa \rangle} \vartheta \in \Theta$  with  $\kappa \neq \top$ . Then  $\varphi \in \Theta$  iff  $\lambda \leq \kappa$ .*

*Proof.*  $T_{\langle \lambda, \kappa \rangle} \varphi \in \Theta$  implies  $\neg^j \varphi \in \Theta$  iff  $\sim^j \lambda \leq \kappa$ . Taking  $j = 0$  we get  $\varphi \in \Theta$  iff  $\lambda \leq \kappa$ .  $\square$

**Lemma 4.16.** *If  $S_\top \varphi (= f_\top \varphi) \in \Theta$ , then either  $\varphi \in \Theta$  or  $\neg \varphi \in \Theta$ .*

*Proof.* By Lem. 4.11(i), either  $\varphi \in \Theta$  or  $\neg \# \varphi \in \Theta$ . If  $f_\top \varphi \in \Theta$ , then by Ax. (#<sub>2</sub>),  $\neg \# \varphi \in \Theta$  implies  $\neg \varphi \in \Theta$ .  $\square$

**Lemma 4.17.** *For every  $\varphi \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ , exactly one of the following conditions holds.*

- (i) *There exist  $\lambda, \kappa \in L$  with  $\kappa \neq \top$  such that  $S_\kappa \varphi, T_{\langle \lambda, \kappa \rangle} \varphi \in \Theta$  with  $\lambda \leq \kappa$  and  $\varphi \in \Theta$ .*
- (ii) *There exist  $\lambda, \kappa \in L$  with  $\kappa \neq \top$  such that  $S_\kappa \varphi, T_{\langle \lambda, \kappa \rangle} \varphi \in \Theta$  with  $\lambda \not\leq \kappa$  and  $\varphi \notin \Theta$ .*
- (iii)  *$S_\top \varphi (= f_\top \varphi), \varphi, \neg \varphi \in \Theta$ .*
- (iv)  *$S_\top \varphi, \varphi \in \Theta$  and  $\neg \varphi \notin \Theta$ .*
- (v)  *$S_\top \varphi \in \Theta, \varphi \notin \Theta$ , and  $\neg \varphi \in \Theta$ .*

*Proof.* By Lem. 4.12 either  $S_\kappa \varphi \in \Theta$  with  $\kappa \neq \top$  or  $S_\top \varphi (= f_\top \varphi) \in \Theta$ . In the first case, by Lem. 4.14,  $T_{\langle \lambda, \kappa \rangle} \varphi \in \Theta$  for some  $\lambda \in L$ . Either  $\lambda \leq \kappa$  or  $\lambda \not\leq \kappa$ . In the first case  $\varphi \in \Theta$  and in the second  $\varphi \notin \Theta$  by Lem. 4.15. This gives (i) or (ii).

Assume now that  $f_\top \varphi \in \Theta$ . Then by Lem. 4.16 either  $\varphi \in \Theta$  or  $\neg \varphi \in \Theta$ . This gives (iii), (iv), or (v).

Thus as least one of (i)–(v) holds. By Lems. 4.12 and 4.14, (i), (ii), and any one of (iii)–(v) are pairwise inconsistent. Clearly (i)–(v) are pairwise inconsistent.  $\square$

**Corollary 4.18.** *If  $S_{\top}\varphi \in \Theta$ , then  $\varphi \in \Theta$  or  $\neg\varphi \in \Theta$ .*  $\square$

**Lemma 4.19.** *For any  $\kappa \in L$ ,  $S_{\kappa}\neg\varphi \in \Theta$  iff  $S_{\kappa}\varphi \in \Theta$ .*

*Proof.* By Ax. (ano<sub>1</sub>) we have  $f_{\kappa}\neg\varphi \in \Theta$  iff  $f_{\kappa}\varphi \in \Theta$ . Also for all  $\lambda \not\leq \kappa$ ,  $f_{\lambda}\neg\varphi \in \Theta$  iff  $f_{\lambda}\varphi \in \Theta$ ; equivalently, by Lem. 4.11(i),  $\neg\#\neg f_{\lambda}\varphi \in \Theta$  iff  $\neg\#f_{\lambda}\varphi \in \Theta$ . Thus  $f_{\kappa}\neg\varphi \wedge \bigvee_{\lambda \not\leq \kappa} \neg\#f_{\lambda}\neg\varphi \in \Theta$  iff  $f_{\kappa}\varphi \wedge \bigvee_{\lambda \not\leq \kappa} \neg\#f_{\lambda}\varphi \in \Theta$ , i.e.,  $S_{\kappa}\neg\varphi \in \Theta$  iff  $S_{\kappa}\varphi \in \Theta$ .  $\square$

**Lemma 4.20.** *Assume  $\kappa \neq \top$ .  $S_{\kappa}\neg\varphi, T_{\langle\lambda, \kappa\rangle}\neg\varphi \in \Theta$  iff  $\exists \mu \in L, \text{sig}_{\kappa} \sim \mu = \text{sig}_{\kappa} \lambda (S_{\kappa}\varphi, T_{\langle\mu, \kappa\rangle}\varphi \in \Theta)$ .*

*Proof.*  $\Leftarrow$ . Assume  $S_{\kappa}\varphi, T_{\langle\mu, \kappa\rangle}\varphi \in \Theta$  with  $\text{sig}_{\kappa} \sim \mu = \text{sig}_{\kappa} \lambda$ . Then  $S_{\kappa}\neg\varphi \in \Theta$  by Lem. 4.19.

$$\begin{aligned} T_{\langle\sim\mu, \kappa\rangle}\neg\varphi &= \bigwedge_{\substack{j < \text{or}_L \\ \sim^j \sim \mu \leq \kappa}} \neg^j \neg\varphi \wedge \bigwedge_{\substack{j < \text{or}_L \\ \sim^j \sim \mu \not\leq \kappa}} \neg\#\neg^j \neg\varphi \\ &= \bigwedge_{\substack{j < \text{or}_L \\ \sim^{j+1} \mu \leq \kappa}} \neg^{j+1} \varphi \wedge \bigwedge_{\substack{j < \text{or}_L \\ \sim^{j+1} \mu \not\leq \kappa}} \neg\#\neg^{j+1} \varphi \\ &= \bigwedge_{\substack{1 \leq j \leq \text{or}_L \\ \sim^j \mu \leq \kappa}} \neg^j \varphi \wedge \bigwedge_{\substack{1 \leq j \leq \text{or}_L \\ \sim^j \mu \not\leq \kappa}} \neg\#\neg^j \varphi \\ &= \left( \bigwedge_{\substack{1 \leq j < \text{or}_L \\ \sim^j \mu \leq \kappa}} \neg^j \varphi \wedge \bigwedge_{\substack{1 \leq j < \text{or}_L \\ \sim^j \mu \not\leq \kappa}} \neg\#\neg^j \varphi \right) \wedge \psi, \end{aligned}$$

where

$$\psi = \begin{cases} \neg^{\text{or}_L} \varphi & \text{if } \sim^{\text{or}_L} \mu \leq \kappa, \\ \neg\#\neg^{\text{or}_L} \varphi & \text{if } \sim^{\text{or}_L} \mu \not\leq \kappa. \end{cases}$$

Recall that  $\sim^{\text{or}_L} \mu = \sim^{\text{cf}_L} \mu$  and that  $\text{cf}_L < \text{or}_L$ . Also, by Ax. (ann<sub>2</sub>),

$$(3) \quad \neg^{\text{or}_L} \varphi \in \Theta \quad \text{iff} \quad \neg^{\text{cf}_L} \varphi \in \Theta,$$

since  $\neg\#f_{\top}\varphi \in \Theta$  because  $S_{\kappa}\varphi \in \Theta$  and  $\top \not\leq \kappa$  by assumption.

If  $\sim^{\text{cf}_L} \mu \leq \kappa$ , then  $\psi = \neg^{\text{or}_L} \varphi$  and  $\neg^{\text{cf}_L} \varphi$  is a conjunct of  $T_{\langle\mu, \kappa\rangle}\varphi$ . Thus  $\neg^{\text{cf}_L} \varphi \in \Theta$  and hence also  $\psi \in \Theta$  by (3). If  $\sim^{\text{cf}_L} \mu \not\leq \kappa$ , then  $\psi = \neg\#\neg^{\text{or}_L} \varphi$  and  $\neg\#\neg^{\text{cf}_L} \varphi$  is a conjunct of  $T_{\langle\mu, \kappa\rangle}\varphi$ . Thus  $\neg\#\neg^{\text{cf}_L} \varphi \in \Theta$  and hence  $\neg^{\text{cf}_L} \varphi \notin \Theta$ . So  $\neg^{\text{or}_L} \varphi \notin \Theta$  by (3), and hence  $\psi = \neg\#\neg^{\text{or}_L} \varphi \in \Theta$ . We conclude that  $T_{\langle\sim\mu, \kappa\rangle}\neg\varphi \in \Theta$ . Finally, we get  $T_{\langle\lambda, \kappa\rangle}\neg\varphi \in \Theta$  by Lem. 4.14.

$\Rightarrow$ . Assume  $S_{\kappa}\neg\varphi, T_{\langle\lambda, \kappa\rangle}\neg\varphi \in \Theta$ . Then  $S_{\kappa}\varphi \in \Theta$  by Lem. 4.19. Thus, by Lem. 4.14,  $T_{\langle\mu, \kappa\rangle}\varphi \in \Theta$  for at least one  $\mu \in L$ . By the first part of the proof  $T_{\langle\mu, \kappa\rangle}\neg\varphi \in \Theta$ , and by Lem. 4.14 again  $\sim\mu$  and  $\lambda$  have the same signature.  $\square$

**Lemma 4.21.** (i)  $S_{\top}\neg\varphi, \neg\varphi, \neg\neg\varphi \in \Theta$  iff  $S_{\top}\varphi, \varphi, \neg\varphi \in \Theta$ .

- (ii)  $S_{\top}\neg\varphi, \neg\varphi \in \Theta$  and  $\neg\neg\varphi \notin \Theta$  iff  $S_{\top}\varphi \in \Theta, \varphi \notin \Theta,$  and  $\neg\varphi \in \Theta.$
- (iii)  $S_{\top}\neg\varphi \in \Theta, \neg\varphi \notin \Theta,$  and  $\neg\neg\varphi \in \Theta$  iff  $S_{\top}\varphi, \varphi \in \Theta$  and  $\neg\varphi \notin \Theta.$

*Proof.* By Lem. 4.19,  $S_{\top}\neg\varphi, \neg\varphi \in \Theta$  iff  $S_{\top}\varphi, \neg\varphi \in \Theta.$  Thus (i) and (ii) follow from Ax. (#3). Again by Lem. 4.19,  $S_{\top}\neg\varphi \in \Theta$  and  $\neg\varphi \notin \Theta$  iff  $S_{\top}\varphi \in \Theta$  and  $\neg\varphi \notin \Theta.$  Thus (iii) also follows from Ax. (#3).  $\square$

**Lemma 4.22.** *For all  $\kappa, \mu \in L,$   $S_{\kappa}f_{\mu}\varphi \in \Theta$  iff  $S_{\kappa}\varphi \in \Theta.$*

*Proof.*  $S_{\kappa}f_{\mu}\varphi = f_{\kappa}f_{\mu}\varphi \wedge \bigwedge_{\lambda \not\leq \kappa} \neg\#f_{\lambda}f_{\mu}\varphi.$  By Ax. ( $L_0$ ) we have  $f_{\kappa}f_{\mu}\varphi \in \Theta$  iff  $f_{\kappa}\varphi \in \Theta$ . By the same axiom,  $f_{\lambda}f_{\mu}\varphi \notin \Theta$  iff  $f_{\lambda}\varphi \notin \Theta,$  i.e.,  $\neg\#f_{\lambda}f_{\mu}\varphi \in \Theta$  iff  $\neg\#f_{\lambda}\varphi \in \Theta,$  for every  $\lambda \in L.$  So  $S_{\kappa}f_{\mu}\varphi \in \Theta$  iff  $f_{\kappa}\varphi \wedge \bigwedge_{\lambda \not\leq \kappa} \neg\#f_{\lambda}\varphi \in \Theta,$  i.e., iff  $S_{\kappa}\varphi \in \Theta.$   $\square$

**Lemma 4.23.** *For every  $\mu \in L,$  if  $S_{\kappa}f_{\mu}\varphi \in \Theta$  and  $\kappa \neq \top,$  then  $T_{\langle\mu,\kappa\rangle}f_{\mu}\varphi \in \Theta.$*

*Proof.* Assume  $S_{\kappa}f_{\mu}\varphi \in \Theta$  and  $\kappa \neq \top.$  Then  $S_{\kappa}\varphi \in \Theta$  by Lem. 4.22, and by Ax. (ann<sub>0</sub>) and Lem. 4.13

$$\sim^j \mu \leq \kappa \quad \text{iff} \quad f_{\sim^j \mu}\varphi \in \Theta \quad \text{iff} \quad \neg^j f_{\mu}\varphi \in \Theta.$$

Thus  $\sim^j \mu \not\leq \kappa$  iff  $\neg^j f_{\mu}\varphi \notin \Theta$  iff  $\neg\#\neg^j f_{\mu}\varphi \in \Theta$  by Lem 4.11(i). Thus

$$T_{\langle\mu,\kappa\rangle}f_{\mu}\varphi = \bigwedge_{\substack{j < \text{or } L \\ \sim^j \mu \leq \kappa}} \neg^j f_{\mu}\varphi \wedge \bigwedge_{\substack{j < \text{or } L \\ \sim^j \mu \not\leq \kappa}} \neg\#\neg^j f_{\mu}\varphi \in \Theta.$$

$\square$

As an immediate consequence of these last two lemmas we have.

**Corollary 4.24.** *If  $\kappa \neq \top,$  then  $S_{\kappa}f_{\mu}\varphi, T_{\langle\mu,\kappa\rangle}f_{\mu}\varphi \in \Theta$  iff  $S_{\kappa}\varphi \in \Theta.$*   $\square$

**Lemma 4.25.** *For every  $\mu \in L,$   $S_{\top}f_{\mu}\varphi, f_{\mu}\varphi, \neg f_{\mu}\varphi \in \Theta$  iff  $S_{\top}\varphi \in \Theta.$*

*Proof.* The implication from left to right follows immediately from Lem. 4.22. For the reverse implication assume  $S_{\top}\varphi \in \Theta.$  Then  $S_{\top}f_{\mu}\varphi \in \Theta$  by Lem. 4.22 again, and  $f_{\mu}\varphi \in \Theta$  for every  $\mu \in L$  by Lem. 4.13. In particular,  $f_{\sim^j \mu}\varphi \in \Theta$  for every  $\mu \in L.$  Thus  $\neg f_{\mu}\varphi \in \Theta$  by Ax. (ann<sub>0</sub>) with  $j = 1.$   $\square$

**Lemma 4.26.** (i)  $S_{\top}\#\varphi \in \Theta.$

(ii)  $\#\varphi \in \Theta$  iff  $\neg\#\varphi \notin \Theta.$

*Proof.*  $S_{\top}\#\varphi \in \Theta$  by Ax. (#1).  $\neg\#\varphi \in \Theta$  iff  $\varphi \notin \Theta$  (by Lem. 4.11(i)) iff  $\#\varphi \notin \Theta$  (by Lem. 4.7(i)).  $\square$

**Lemma 4.27.** *Let  $\star \in \{\rightarrow, \wedge, \vee\}.$*

(i)  $S_{\top}(\varphi \star \psi) \in \Theta.$

(ii)  $\varphi \star \psi \in \Theta$  iff  $\neg(\varphi \star \psi) \notin \Theta.$

*Proof.* (i): Immediate by Axs. (#4)–(#6). (ii). By Lem. 4.7(i),  $\varphi \star \psi \in \Theta$  iff  $\#(\varphi \star \psi) \in \Theta.$  And by Axs. (#2), (#7)–(#9) and part (i),  $\neg(\varphi \star \psi) \in \Theta$  iff  $\#\neg(\varphi \star \psi) \in \Theta.$  The equivalences of (ii) now follow directly from Lem. 4.26(ii).  $\square$

We now apply the previous lemmas to prove the main lemma used in the proof of the completeness of  $\mathcal{S}(L).$

**Lemma 4.28.** *Let  $\Theta$  be a maximal Boolean consistent set of formulas. Assume  $u: \text{Va} \rightarrow M^*$  such that, for each  $p \in \text{Va}$ ,*

$$u p = \begin{cases} [\langle \lambda, \kappa \rangle] & \text{if } S_\kappa p, T_{\langle \lambda, \kappa \rangle} p \in \Theta, \\ \mathbb{T} & \text{if } S_\top p, p, \neg p \in \Theta, \\ 1 & \text{if } S_\top p, p \in \Theta, \neg p \notin \Theta, \\ 0 & \text{if } S_\top p \in \Theta, p \notin \Theta, p \in \Theta. \end{cases}$$

Then, for every  $\varphi \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ ,

- (i)  $u^* \varphi = [\langle \lambda, \kappa \rangle]$ , with  $\kappa \neq \top$ , iff  $S_\kappa \varphi, T_{\langle \lambda, \kappa \rangle} \varphi \in \Theta$ ,
- (ii)  $u^* \varphi = \mathbb{T}$  iff  $S_\top \varphi, \varphi, \neg \varphi \in \Theta$ ,
- (iii)  $u^* \varphi = 1$  iff  $S_\top \varphi, \varphi \in \Theta, \neg \varphi \notin \Theta$ ,
- (iv)  $u^* \varphi = 0$  iff  $S_\top \varphi \in \Theta, \varphi \notin \Theta, \neg \varphi \in \Theta$ .

*Proof.* The proof is by induction on the structure of  $\varphi$ . The conclusions (i)–(iv) hold by hypothesis if  $\varphi$  is a propositional variable.

$$\varphi = \neg \psi.$$

- (i)  $u^* \varphi = [\langle \lambda, \kappa \rangle]$  iff  $\exists \mu \in L, \text{sig}_k \sim \mu = \text{sig}_k \lambda (u^* \psi = \langle \mu, \kappa \rangle)$   
iff  $\exists \mu \in L, \text{sig}_k \sim \mu = \text{sig}_k \lambda (S_\kappa \psi, T_{\langle \mu, \kappa \rangle} \psi \in \Theta)$ , by ind. hyp.  
iff  $S_\kappa \neg \psi, T_{\langle \lambda, \kappa \rangle} \neg \psi \in \Theta$ , by Lem. 4.20  
iff  $S_\kappa \varphi, T_{\langle \lambda, \kappa \rangle} \varphi \in \Theta$ .

- (ii)  $u^* \varphi = \mathbb{T}$  iff  $u^* \psi = \mathbb{T}$   
iff  $S_\top \psi, \psi, \neg \psi \in \Theta$ , by ind. hyp.  
iff  $S_\top \neg \psi, \neg \psi, \neg \neg \psi \in \Theta$ , by Lem. 4.21(i)  
iff  $S_\top \varphi, \varphi, \neg \varphi \in \Theta$ .

- (iii)  $u^* \varphi = 1$  iff  $u^* \psi = 0$   
iff  $S_\top \psi \in \Theta, \psi \notin \Theta$ , and  $\neg \psi \notin \Theta$ , by ind. hyp.  
iff  $S_\top \neg \psi, \neg \psi \in \Theta$ , and  $\neg \neg \psi \notin \Theta$ , by Lem. 4.21(ii)  
iff  $S_\top \varphi, \varphi \in \Theta$  and  $\neg \varphi \notin \Theta$ .

- (iv)  $u^* \varphi = 0$  iff  $u^* \psi = 1$   
iff  $S_\top \psi, \psi \in \Theta$  and  $\neg \psi \notin \Theta$ , by ind. hyp.  
iff  $S_\top \neg \psi \in \Theta, \neg \psi \notin \Theta$ , and  $\neg \neg \psi \in \Theta$ , by Lem. 4.21(iii)  
iff  $S_\top \varphi \in \Theta, \varphi \notin \Theta$ , and  $\neg \varphi \in \Theta$ .

$$\varphi = f_\mu \psi.$$

- (i)  $u^* \varphi = [\langle \lambda, \kappa \rangle]$  with  $\kappa \neq \top$
- iff  $\text{sig}_\kappa \lambda = \text{sig}_\kappa \mu$  and  $\exists \nu \in L (u^* \psi = [\langle \nu, \kappa \rangle])$ , by Lem. 3.7
  - iff  $\text{sig}_\kappa \lambda = \text{sig}_\kappa \mu$  and  $\exists \nu \in L (S_\kappa \psi, T_{\langle \nu, \kappa \rangle} \psi \in \Theta)$ , by ind. hyp.
  - iff  $\text{sig}_\kappa \lambda = \text{sig}_\kappa \mu$  and  $S_\kappa \psi \in \Theta$ , by Lem. 4.14
  - iff  $\text{sig}_\kappa \lambda = \text{sig}_\kappa \mu$  and  $S_\kappa f_\mu \psi, T_{\langle \mu, \kappa \rangle} f_\mu \psi \in \Theta$ , by Lem. 4.24
  - iff  $S_\kappa f_\mu \psi, T_{\langle \lambda, \kappa \rangle} f_\mu \psi \in \Theta$ , by Lem. 3.7.
- (ii)  $u^* \varphi = \top$  iff  $u^* \psi \in \{0, 1, \top\}$
- iff  $S_\top \psi \in \Theta$ , by ind. hyp. and Cor. 4.18
  - iff  $S_\top f_\mu \psi, f_\mu \psi, \neg f_\mu \psi \in \Theta$ , by Lem. 4.25
  - iff  $S_\top \varphi, \varphi, \neg \varphi \in \Theta$ .

$u^* \varphi$  never equals 0 or 1, and by Lem. 4.25  $S_\top \varphi \in \Theta$  implies  $\varphi, \neg \varphi \in \Theta$ . So the equivalences (iii) and (iv) hold vacuously in this case.

For  $\star \in \{\rightarrow, \wedge, \vee\}$  we have  $u^*(\varphi \star \psi) \neq [\langle \lambda, \kappa \rangle]$  for all  $\langle \lambda, \kappa \rangle \in L \times (L \setminus \{\top\})$ , and  $S_\kappa(\varphi \star \psi) \notin \Theta$  for all  $\kappa \neq \top$  by Axs. (#4)–(#6) and Lem. 4.12. Thus the equivalence (i) holds vacuously.

Again for  $\star \in \{\rightarrow, \wedge, \vee\}$  we have  $u^*(\varphi \star \psi) \neq \top$  and  $\neg(\varphi \star \psi) \in \Theta$  iff  $\varphi \star \psi \notin \Theta$  by Axs. (#2), (#4)–(#9). Thus the equivalence (ii) also vacuously holds.

In order to verify the equivalences (iii) and (iv) we first show that, for every formula  $\psi$  such that the equivalences (i)–(iv) hold,

$$(2) \quad u^* \# \psi = 1 \quad \text{iff} \quad \psi \in \Theta \quad \text{and} \quad u^* \# \psi = 0 \quad \text{iff} \quad \psi \notin \Theta.$$

$$\begin{aligned} u^* \# \psi = 1 & \quad \text{iff} \quad \exists \kappa < \top \exists \lambda \leq \kappa (u^* \psi = \langle \lambda, \kappa \rangle) \text{ or } u^* \psi = \top \text{ or } u^* \psi = 1 \\ & \quad \text{iff} \quad \exists \kappa < \top \exists \lambda \leq \kappa (S_\kappa \psi, T_{\langle \lambda, \kappa \rangle} \psi \in \Theta) \text{ or } S_\top \psi, \psi, \neg \psi \in \Theta \text{ or } S_\top \psi, \psi \in \Theta, \neg \psi \notin \Theta \\ & \quad \text{by hypothesis} \\ & \quad \text{iff} \quad \exists \kappa < \top (S_\kappa \psi, \psi \in \Theta) \text{ or } S_\top \psi, \psi, \neg \psi \in \Theta \text{ or } S_\top \psi, \psi \in \Theta, \neg \psi \notin \Theta \\ & \quad \text{by Lem. 4.15} \\ & \quad \text{iff} \quad \psi \in \Theta. \end{aligned}$$

This gives the first part of (2); The second part follows from the first and the fact that  $u^* \# \psi \in \{0, 1\}$ .

We now return to the verification of (iii) and (iv) in the cases  $\varphi = \psi \star \varphi$  with  $\star \in \{\rightarrow, \wedge, \vee\}$ .

$$\varphi = \psi \rightarrow \vartheta.$$

- (iii)  $u^*\varphi = 1$  iff  $u^*\#\psi = 0$  or  $u^*\#\vartheta = 1$   
iff  $\psi \notin \Theta$  and  $\vartheta \in \Theta$ , by (2) and the ind. hyp.  
iff  $\psi \rightarrow \vartheta \in \Theta$ , by Lem. 4.11(ii)  
iff  $S_{\top}(\psi \rightarrow \vartheta), \psi \rightarrow \vartheta \in \Theta, \neg(\psi \rightarrow \vartheta) \notin \Theta$ , by Lem. 4.27  
iff  $S_{\top}\varphi, \varphi \in \Theta, \neg\varphi \notin \Theta$ .

(iv)

- $u^*\varphi = 0$  iff  $u^*\#\psi = 1$  and  $u^*\#\vartheta = 0$   
iff  $\psi \in \Theta$  and  $\vartheta \notin \Theta$ , by (2) and the ind. hyp.  
iff  $\psi \rightarrow \vartheta \notin \Theta$ , by Lem. 4.11(ii)  
iff  $S_{\top}(\psi \rightarrow \vartheta) \in \Theta, \psi \rightarrow \vartheta \notin \Theta, \neg(\psi \rightarrow \vartheta) \in \Theta$ , by Lem. 4.27  
iff  $S_{\top}\varphi \in \Theta, \varphi \notin \Theta, \neg\varphi \in \Theta$ .

$$\varphi = \psi \wedge \vartheta.$$

- (iii)  $u^*\varphi = 1$  iff  $u^*\#\psi = 1$  and  $u^*\#\vartheta = 1$   
iff  $\psi \in \Theta$  and  $\vartheta \in \Theta$ , by (2) and the ind. hyp.  
iff  $\psi \wedge \vartheta \in \Theta$ , by Lem. 4.7(ii)  
iff  $S_{\top}(\psi \wedge \vartheta), \psi \wedge \vartheta \in \Theta, \neg(\psi \wedge \vartheta) \notin \Theta$ , by Lem. 4.27  
iff  $S_{\top}\varphi, \varphi \in \Theta, \neg\varphi \notin \Theta$ .

- (iv)  $u^*\varphi = 0$  iff  $u^*\#\psi = 0$  or  $u^*\#\vartheta = 0$   
iff  $\psi \notin \Theta$  or  $\vartheta \notin \Theta$ , by (2) and the ind. hyp.  
iff  $\psi \wedge \vartheta \notin \Theta$ , by Lem. 4.7(ii)  
iff  $S_{\top}(\psi \wedge \vartheta) \in \Theta, \psi \wedge \vartheta \notin \Theta, \neg(\psi \wedge \vartheta) \in \Theta$ , by Lem. 4.27  
iff  $S_{\top}\varphi \in \Theta, \varphi \notin \Theta, \neg\varphi \in \Theta$ .

$$\varphi = \psi \vee \vartheta.$$

- (iii)  $u^*\varphi = 1$  iff  $u^*\#\psi = 1$  or  $u^*\#\vartheta = 1$   
iff  $\psi \in \Theta$  or  $\vartheta \in \Theta$ , by (2) and the ind. hyp.  
iff  $\psi \vee \vartheta \in \Theta$ , by Lem. 4.11(iii)  
iff  $S_{\top}(\psi \vee \vartheta), \psi \vee \vartheta \in \Theta, \neg(\psi \vee \vartheta) \notin \Theta$ , by Lem. 4.27  
iff  $S_{\top}\varphi, \varphi \in \Theta, \neg\varphi \notin \Theta$ .

$$\begin{aligned}
\text{(iv)} \quad u^*\varphi = 0 & \text{ iff } u^*\#\psi = 0 \text{ and } u^*\#\vartheta = 0 \\
& \text{ iff } \psi \notin \Theta \text{ and } \vartheta \notin \Theta, && \text{by (2) and the ind. hyp.} \\
& \text{ iff } \psi \vee \vartheta \notin \Theta, && \text{by Lem. 4.11(iii)} \\
& \text{ iff } S_{\top}(\psi \vee \vartheta) \in \Theta, \psi \vee \vartheta \notin \Theta, \neg(\psi \vee \vartheta) \in \Theta, && \text{by Lem. 4.27} \\
& \text{ iff } S_{\top}\varphi \in \Theta, \varphi \notin \Theta, \neg\varphi \in \Theta.
\end{aligned}$$

$\varphi = \#\psi$ .  
 $u^*\#\psi \in \{0, 1\}$  while  $S_{\top}\#\psi \in \Theta$  and  $\#\psi \in \Theta$  iff  $\neg\#\psi \notin \Theta$  by Lem. 4.26. So the equivalences (i) and (ii) hold vacuously.

$$\begin{aligned}
\text{(iii)} \quad u^*\varphi = 1 & \text{ iff } \psi \in \Theta, && \text{by (2)} \\
& \text{ iff } \#\psi \in \Theta, && \text{by Lem. 4.7(i)} \\
& \text{ iff } S_{\top}\#\psi, \#\psi \in \Theta, \neg\#\psi \notin \Theta, && \text{by Lem. 4.26} \\
& \text{ iff } S_{\top}\varphi, \varphi \in \Theta, \neg\varphi \notin \Theta.
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad u^*\varphi = 0 & \text{ iff } \psi \notin \Theta, && \text{by (2)} \\
& \text{ iff } \#\psi \notin \Theta, && \text{by Lem. 4.7(i)} \\
& \text{ iff } S_{\top}\#\psi, \#\psi \notin \Theta, \neg\#\psi \in \Theta, && \text{by Lem. 4.26} \\
& \text{ iff } S_{\top}\varphi \in \Theta, \varphi \notin \Theta, \neg\varphi \in \Theta.
\end{aligned}$$

□

**Corollary 4.29.** *Let  $\Theta$  be a maximal, Boolean consistent set of formulas. Assume  $u: \text{Va} \rightarrow M^*$  as in the hypothesis of the lemma. Then, for all  $\varphi \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ , we have  $u^*\varphi \in D^*$  iff  $\varphi \in \Theta$ .*

*Proof.*  $\implies$ . Suppose  $u^*\psi \in D^*$ . Then  $u^* = [\langle \lambda, \kappa \rangle]$ , with  $\lambda \leq \kappa < \top$  or  $u^*\varphi = \mathbb{T}$ , or  $u^*\varphi = 1$ . If  $u^*\varphi = [\langle \lambda, \kappa \rangle]$ , with  $\lambda \leq \kappa < \top$ , then  $S_{\kappa}\varphi, T_{\langle \lambda, \kappa \rangle}\varphi \in \Theta$  by the lemma. Thus  $\varphi \in \Theta$  by Lem. 4.15. If  $u^*\varphi = \mathbb{T}$ , then  $S_{\top}\varphi, \varphi, \neg\varphi \in \Theta$ , again by the lemma; in particular,  $\varphi \in \Theta$ . Finally, if  $u^*\varphi = 1$ , then by the lemma  $S_{\top}\varphi, \varphi \in \Theta, \neg\varphi \notin \Theta$ ; in particular  $\varphi \in \Theta$ .

$\impliedby$ . Suppose  $\varphi \in \Theta$ . Then by Lem. 4.17 exactly one of the following conditions holds.

- $S_{\kappa}, T_{\langle \lambda, \kappa \rangle} \in \Theta$  with  $\lambda \leq \kappa < \top$ .
- $S_{\top}\varphi, \varphi, \neg\varphi \in \Theta$ .
- $S_{\top}\varphi, \varphi \in \Theta, \neg\varphi \notin \Theta$ .

Thus by the lemma,  $u^*\varphi = \langle \lambda, \kappa \rangle$ , with  $\lambda \leq \kappa < \top$ ,  $u^*\varphi = \mathbb{T}$ , or  $u^*\varphi = 1$ , i.e.,  $u^*\varphi \in D^*$ . □

**Lemma 4.30.** *Assume  $\Gamma \not\vdash_{\mathcal{S}(\mathbf{L})} \varphi$ . Then there exists a  $u: \text{Va} \rightarrow M^*$  such that  $u^*\psi \in D^*$  for every  $\psi \in \Gamma$  and  $u^*\varphi \notin D^*$ .*

*Proof.*  $\Gamma \cup \{\neg \# \varphi\}$  is Boolean consistent by Lem. 4.10. Let  $\Theta$  be a maximal Boolean consistent set of formulas that includes  $\Gamma \cup \{\neg \# \varphi\}$ ; such a  $\Theta$  exists by Zorn's lemma. Since  $\Theta$  is consistent,  $\varphi \notin \Theta$ . Let  $u: \text{Va} \rightarrow M^*$  be defined as follows for every  $p \in \text{Va}$ .

$$u p = \begin{cases} [(\lambda, \kappa)] & \text{if } S_{\kappa} p, T_{(\lambda, \kappa)} p \in \Theta, \text{ with } \kappa \neq \top, \\ \top & \text{if } S_{\top} p, p, \neg p \in \Theta, \\ 1 & \text{if } S_{\top} p, p \in \Theta \text{ and } \neg p \notin \Theta, \\ 0 & \text{if } S_{\top} p \in \Theta, p \notin \Theta, \text{ and } \neg p \in \Theta. \end{cases}$$

$u^*$  is well defined by Lems. 4.14 and 4.17. By Cor. 4.29 we have that, for all  $\vartheta \in \text{Fm}_{\mathcal{M}(\mathbf{L})}$ ,  $u^* \vartheta \in D^*$  iff  $\vartheta \in \Theta$ . Thus  $u^* \psi \in D^*$  for all  $\psi \in \Gamma$ , and  $u^* \varphi \notin D^*$ .  $\square$

**Theorem 4.31** (Completeness of  $\mathcal{S}(\mathbf{L})$ ). *For any  $\Gamma \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{M}(\mathbf{L})}$ ,*

$$\Gamma \vdash_{\text{P}_{\mathcal{M}(\mathbf{L})}} \varphi \text{ implies } \Gamma \vdash_{\mathcal{S}(\mathbf{L})} \varphi.$$

*Proof.* Assume  $\Gamma \not\vdash_{\mathcal{S}(\mathbf{L})} \varphi$ . By Lem. 4.30  $\Gamma \not\vdash_{\text{P}_{\mathcal{M}(\mathbf{L})}} \varphi$ .  $\square$

**4.1. The equivalent quasivariety of  $\text{P}_{\mathcal{M}(\mathbf{L})}$ .** In this section we show that, under the assumption  $\mathbf{L}$  is finite, the equivalent quasivariety of  $\text{P}_{\mathcal{M}(\mathbf{L})}$  is generated by  $\mathbf{M}^*(\mathbf{L})$  and obtain an axiomatization for it. For the rest of the section we assume  $\mathbf{L}$  is finite, unless explicitly indicated otherwise.

We begin by reviewing some results from abstract algebraic logic that we shall need; in particular, we apply a general result from abstract algebraic logic to obtain an intermediate characterization of the equivalent quasivariety in terms of matrix models of  $\text{P}_{\mathcal{M}(\mathbf{L})}$ .

From the completeness and soundness theorems of the previous section we have that  $\text{P}_{\mathcal{M}(\mathbf{L})}$  and  $\mathcal{S}(\mathbf{L})$  coincide in the sense that  $\vdash_{\text{P}_{\mathcal{M}(\mathbf{L})}} = \vdash_{\mathcal{S}(\mathbf{L})}$ .

Let  $\mathbf{A}$  be an arbitrary algebra (of the same language type as  $\mathbf{M}(\mathbf{L})$ ). A subset  $F$  of the universe  $A$  if  $\mathbf{A}$  is a  $\text{P}_{\mathcal{M}(\mathbf{L})}$ -filter if it is closed under  $\vdash_{\text{P}_{\mathcal{M}(\mathbf{L})}}$  in the sense that, if  $\varphi_0, \dots, \varphi_{n-1} \vdash_{\text{P}_{\mathcal{M}(\mathbf{L})}} \psi$ , then, for every  $h: \mathbf{Fm}_{\mathcal{M}(\mathbf{L})} \rightarrow \mathbf{A}$  such that  $h \varphi_0, \dots, h \varphi_{n-1} \in F$ , we have  $h \psi \in F$ . Equivalently,  $F$  contains the  $h$ -image of every axiom of  $\text{P}_{\mathcal{M}(\mathbf{L})}$  and, if  $h \varphi, h(\varphi \rightarrow \psi) \in F$ , then  $h \psi \in F$ . The *Leibniz congruence of  $F$* , in symbols  $\Omega_{\mathbf{A}} F$ , is the largest congruence on  $\mathbf{A}$  compatible with  $F$  in the sense that, if  $a \in F$  and  $a \equiv b \pmod{\Omega_{\mathbf{A}} F}$ , then  $b \in F$ .

**Lemma 4.32.** *Let  $\mathbf{A}$  and  $\mathbf{B}$  be algebras and  $h: \mathbf{A} \rightarrow \mathbf{B}$  a surjective homomorphism. Let  $F$  be a  $\text{P}_{\mathcal{M}(\mathbf{L})}$ -filter of  $\mathbf{B}$ . Then  $h^{-1}(F)$  is a  $\text{P}_{\mathcal{M}(\mathbf{L})}$ -filter of  $\mathbf{A}$  and  $\Omega_{\mathbf{A}} h^{-1}(F) = h^{-1}(\Omega_{\mathbf{B}} F)$ .*

*Proof.* It is straightforward to check that  $h^{-1}(F)$  is a  $\text{P}_{\mathcal{M}(\mathbf{L})}$ -filter of  $\mathbf{A}$  and that  $h^{-1}(\Omega_{\mathbf{B}} F)$  is a congruence on  $\mathbf{A}$  that is compatible with  $h^{-1}(F)$ . Hence  $h^{-1}(\Omega_{\mathbf{B}} F) \subseteq \Omega_{\mathbf{A}} h^{-1}(F)$ . For the inclusion in the other direction, note that the relation-kernel  $h^{-1}(\text{Id}_{\mathbf{B}})$  of  $h$  is compatible with  $h^{-1}(F)$  and hence is included in  $\Omega_{\mathbf{A}} h^{-1}(F)$ . So  $h(\Omega_{\mathbf{A}} h^{-1}(F))$  is a congruence on  $\mathbf{B}$ , since  $h$  is surjective, and it is easy to check that it is compatible with  $F$ . So  $h(\Omega_{\mathbf{A}} h^{-1}(F)) \subseteq \Omega_{\mathbf{B}}(F)$ , and then  $\Omega_{\mathbf{A}} h^{-1}(F) = h^{-1}h(\Omega_{\mathbf{A}} h^{-1}(F)) \subseteq h^{-1}(\Omega_{\mathbf{B}}(F))$ .  $\square$

**Lemma 4.33.** *Let  $\mathbf{A}$  be an algebra. If a congruence relation  $\alpha$  of  $\mathbf{A}$  is compatible with a  $\text{P}_{\mathcal{M}(\mathbf{L})}$ -filter  $F$  of  $\mathbf{A}$ , then it is compatible with every  $\text{P}_{\mathcal{M}(\mathbf{L})}$ -filter of  $\mathbf{A}$  that includes  $F$ .*

*Proof.* Let  $\alpha$  be a congruence compatible with  $F$  and  $G$  a  $P_{\mathcal{M}(\mathbf{L})}$ -filter that includes  $F$ . Let  $a, b \in A$  and assume  $a \equiv b \pmod{\alpha}$  and  $a \in G$ . Then  $a \rightarrow^{\mathbf{A}} a \equiv a \rightarrow^{\mathbf{A}} b \pmod{\alpha}$  and  $a \rightarrow^{\mathbf{A}} a \in F$ . So  $a \rightarrow^{\mathbf{A}} b \in F \subseteq G$  by compatibility. Together with the assumption  $a \in G$  this implies that  $b \in G$ . So  $\alpha$  is compatible with  $G$ .  $\square$

In the terminology of abstract algebraic logic this lemma says that  $P_{\mathcal{M}(\mathbf{L})}$  is *protoalgebraic*.

A (*matrix*) *model* of  $P_{\mathcal{M}(\mathbf{L})}$  is a pair  $\langle \mathbf{A}, F \rangle$  where  $\mathbf{A}$  is any algebra and  $F$  is a  $P_{\mathcal{M}(\mathbf{L})}$ -filter of  $\mathbf{A}$ . The model  $\langle \mathbf{A}, F \rangle$  is *reduced* if  $\Omega_{\mathbf{A}} F = \text{Id}_A$ , the identity relation on  $A$ . The class of all models of  $P_{\mathcal{M}(\mathbf{L})}$  is denoted by  $\text{Mod } P_{\mathcal{M}(\mathbf{L})}$  and the class of all reduced models by  $\text{Mod}^* P_{\mathcal{M}(\mathbf{L})}$ .  $\text{Alg Mod}^* P_{\mathcal{M}(\mathbf{L})}$  is the class of underlying algebras of all reduced models of  $P_{\mathcal{M}(\mathbf{L})}$ . Thus  $\mathbf{A} \in \text{Alg Mod}^* P_{\mathcal{M}(\mathbf{L})}$  iff there exists a  $P_{\mathcal{M}(\mathbf{L})}$ -filter of  $\mathbf{A}$  such that  $\Omega_{\mathbf{A}} F = \text{Id}_A$ . As a consequence of the finite algebraizability of  $P_{\mathcal{M}(\mathbf{L})}$  we have that  $\text{Alg Mod}^* P_{\mathcal{M}(\mathbf{L})}$  is a quasivariety and in fact the equivalent quasivariety of  $P_{\mathcal{M}(\mathbf{L})}$  ([4, Corollary 5.3]).

For any class  $\mathbf{K}$  of algebras,  $\mathbf{SK}$  and  $\mathbf{PK}$  denote the classes of all algebras isomorphic respectively to a subalgebra of a member of  $\mathbf{K}$  and a direct product of a system of members of  $\mathbf{K}$ .

**Lemma 4.34.** *Assume  $\mathbf{L}$  is a finite annotation lattice. For every maximal, Boolean consistent theory  $\Theta$  of  $P_{\mathcal{M}(\mathbf{L})}$ ,  $\mathbf{Fm}_{\mathcal{M}(\mathbf{L})} / \Omega_{\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}} \Theta \in \mathbf{S}\{M^*(\mathbf{L})\}$ .*

*Proof.* Let  $\Theta$  be a maximal, Boolean consistent set of formulas. Assume  $u: \text{Va} \rightarrow M^*$  as in the hypothesis of the Lem. 4.28. Then, by Cor. 4.29,  $u^*$  is a homomorphism from  $\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}$  into  $M^*(\mathbf{L})$  such that  $u^{*-1}(D^*) = \Theta$ . Then by Lem. 4.32,

$$\Omega_{\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}} \Theta = \Omega_{\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}} u^{*-1}(D^*) = u^{*-1}(\Omega_{M^*(\mathbf{L})} D^*) = u^{*-1}(\text{Id}_{M^*(\mathbf{L})}),$$

the relation kernel of  $u^*$ . Thus  $\mathbf{Fm}_{\mathcal{M}(\mathbf{L})} / \Omega_{\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}}$  is isomorphic to a subalgebra of  $M(\mathbf{L})$ .  $\square$

In the following theorem we prove that  $M^*(\mathbf{L})$  generates the equivalent quasivariety of  $P_{\mathcal{M}(\mathbf{L})}$ .

**Theorem 4.35.** *Assume  $\mathbf{L}$  is a finite annotation lattice. Then*

$$\text{Alg Mod}^* P_{\mathcal{M}(\mathbf{L})} = \mathbf{SP}\{M^*(\mathbf{L})\}.$$

*Proof.* The inclusion from right-to-left is straightforward. To prove the inclusion in the opposite direction consider any  $\mathbf{A} \in \text{Mod}^* P_{\mathcal{M}(\mathbf{L})}$ . Then there is a  $P_{\mathcal{M}(\mathbf{L})}$ -filter  $F$  of  $\mathbf{A}$  such that  $\Omega_{\mathbf{A}} F = \text{Id}_A$ . Let  $\alpha$  be any infinite cardinal such that  $\mathbf{A}$  has a set of generators of cardinality less than or equal to  $\alpha$ , and let  $\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha$  be the algebra of formulas over a set of  $\alpha$  propositional variable symbols. Let  $h: \mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha \rightarrow \mathbf{A}$  be a surjective homomorphism, and let  $\Theta = h^{-1}(F)$ ;  $\Theta$  is a  $P_{\mathcal{M}(\mathbf{L})}$ -theory. Moreover, by Lem. 4.32,  $\Omega_{\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha} \Theta = h^{-1}(\Omega_{\mathbf{A}} F) = h^{-1}(\text{Id}_A)$ , the relation-kernel of  $h$ , and hence  $\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha / \Omega_{\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha} \Theta \cong \mathbf{A}$ . Let  $\{\Phi_i : i \in I\}$  be the set of all maximal Boolean consistent  $P_{\mathcal{M}(\mathbf{L})}$ -theories on  $\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha$  that include  $\Theta$ . Then  $\Theta = \bigcap_{i \in I} \Phi_i$ . We claim that

$$\Omega_{\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha} \bigcap_{i \in I} \Phi_i = \bigcap_{i \in I} \Omega_{\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha} \Phi_i.$$

The inclusion from left to right follows from Lem. 4.33, and the opposite inclusion from the definition of the Leibniz congruence. We conclude from this that  $\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha / \Omega_{\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha} \Theta$ , and hence  $\mathbf{A}$ , is isomorphic to a subdirect product of the algebras  $\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha / \Omega_{\mathbf{Fm}_{\mathcal{M}(\mathbf{L})}^\alpha} \Phi_i$ , for  $i \in I$ . Consequently,  $\mathbf{A}$  is isomorphic to a subdirect product of subalgebras of  $\mathbf{M}^*(\mathbf{L})$  by Lem. 4.34.  $\square$

The axiomatization of  $\mathbf{P}_{\mathcal{M}(\mathbf{L})}$  gives rise automatically to the following axiomatization of the quasivariety generated by  $\mathbf{M}^*(\mathbf{L})$  (see [4, Theorem 2.17] and [6, Theorem 1.13]): the identities

$$\varphi \approx \mathbf{1} \text{ for each axiom } \varphi \text{ of } \mathbf{P}_{\mathcal{M}(\mathbf{L})}$$

and the two quasi-identities

$$\#x \approx \mathbf{1} \ \& \ \#(x \rightarrow y) \Rightarrow \#y \approx \mathbf{1} \quad \text{and} \quad \bigwedge_{\delta(x,y) \in \Delta(x,y)} \# \delta(x,y) \approx \mathbf{1} \Rightarrow x \approx y.$$

The two quasivarieties cannot be replaced by identities, that is to say,  $\mathbf{SP}\{\mathbf{M}^*(\mathbf{L})\}$  is not a variety. To see this observe that the equivalence relation that identifies 0 and 1, and no other distinct pairs of elements of  $\mathbf{M}^*(\mathbf{L})$ , is a congruence whose associated quotient algebra is not a member of  $\mathbf{SP}\{\mathbf{M}^*(\mathbf{L})\}$ , so  $\mathbf{SP}\{\mathbf{M}^*(\mathbf{L})\}$  is not closed under homomorphic images.

Combining Theorems 4.35 and 3.6 with (2) we get the desired simulation of  $\vdash_{\mathbf{P}_{\mathbf{L}}}$  in  $\vDash_{\mathbf{M}^*(\mathbf{L})}$ : for all  $A_0, \dots, A_{n-1}, B \in \mathbf{Fm}_{\mathbf{L}}$ ,

$$A_0, \dots, A_{n-1} \vdash_{\mathbf{P}_{\mathbf{L}}} B \quad \text{iff} \quad \#t(A_0) \approx \mathbf{1}, \dots, \#t(A_{n-1}) \approx \mathbf{1} \vDash_{\mathbf{M}^*(\mathbf{L})} \#t(B) \approx \mathbf{1}.$$

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