

# P1 ALGEBRAS

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ABSTRACT. In [3] the authors proved that the deductive system **P1** introduced by Sette in [6] is algebraizable. In this paper we study the main features of the class of algebras thus obtained. The main results are a complete description of the free algebras in  $n$  generators and that this is not a congruence modular quasi-variety.

## INTRODUCTION

In [1] Blok & Pigozzi develop a theory of algebraizability of deductive systems that generalizes the usual Lindenbaum-Tarski process. Besides giving criteria to determine if a system is algebraizable or not, they provide means to obtain the algebraic counterpart of the system.

In [3] the authors used these techniques and the general framework of algebraizability, to prove that the paraconsistent logic **P1** introduced by Sette in [6] is algebraizable. This is interesting because the best known paraconsistent logics, da Costa's  $C_n, n = 1, 2, \dots$ , systems (see [2]), are not algebraizable. This fact was proved by Mortensen in [5]; for a shorter proof using Blok & Pigozzi's theory see [4]. In this paper we will give a characterization of these algebras, we will describe its free algebras and study some aspects of its lattices of congruences.

Following [1] theorem 2.17 and [6] the axioms that define **P1**-algebras are equivalent to the following.

**Definition 1.** A **P1**- algebra is an algebra  $\mathfrak{A} = \langle A; \rightarrow, ', \mathbf{1} \rangle$  where  $\rightarrow$  is a binary operation,  $'$  is a unary operation and  $\mathbf{1}$  is a constant that verify the identities:

- A1.-  $x \rightarrow (y \rightarrow x) = \mathbf{1}$ ,
- A2.-  $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = \mathbf{1}$ ,
- A3.-  $(x' \rightarrow y') \rightarrow ((x' \rightarrow y'') \rightarrow x) = \mathbf{1}$ ,
- A4.-  $(x \rightarrow x'')' \rightarrow x = \mathbf{1}$ ,
- A5.-  $(x \rightarrow y) \rightarrow (x \rightarrow y)'' = \mathbf{1}$ ,

and the quasi-identities:

- Q1.- If  $x \rightarrow y = \mathbf{1}$  and  $(x \rightarrow x) \rightarrow x = \mathbf{1}$ ,

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then  $(y \rightarrow y) \rightarrow y = \mathbf{1}$ .

Q2.- If  $x \rightarrow y = \mathbf{1}$ ,  $y \rightarrow x = \mathbf{1}$ ,  $x' \rightarrow y' = \mathbf{1}$  and  $y' \rightarrow x' = \mathbf{1}$ ,  
then  $x = y$ .

This is a rather unnatural set of axioms but we will not have to deal too much with it because Corollary (2.2) in [3] states that the class  $\mathfrak{P1}$  of all  $\mathbf{P1}$ -algebras is the quasivariety generated by the three element algebra

$$\mathfrak{A} = \langle \{0, a, 1\}; \rightarrow, ', 1 \rangle \quad ,$$

where  $\langle \{0, 1\}; \rightarrow, ' \rangle$  is the two element Boolean algebra and  $a$  behaves as follows:

$\rightarrow$	0	a	1
0	1	1	1
a	0	1	1
1	0	1	1

	$'$
0	1
a	1
1	0

This is a proper quasivariety i.e. it is not a variety as the following proposition shows.

**Theorem 1.** Let  $\mathfrak{B} = \langle \{b, c\}; \rightarrow, ', b \rangle$  with operations defined by

$\rightarrow$	b	c
b	b	b
c	b	b

	$'$
b	b
c	b

Then  $\mathfrak{B}$  is a homomorphic image of  $\mathfrak{A}$  but is not a  $\mathbf{P1}$ -algebra.

*Proof.*

$$\begin{aligned} \text{Let } f : \mathfrak{A} &\rightarrow \mathfrak{B} \\ 0 &\mapsto b \\ a &\mapsto c \\ 1 &\mapsto b \end{aligned}$$

Then  $f$  is a homomorphism but  $\mathfrak{B}$  fails to verify axiom  $Q_2$ . □

Next we observe that since the class  $\mathfrak{P1}$  is the quasivariety generated by  $\mathfrak{A}$ ,  $\mathfrak{P1} = ISPP_u\{\mathfrak{A}\}$ , where  $I, S, P, P_u$  are the usual class operators of isomorphisms, subalgebras, direct products and ultra-products respectively. Since we have a single finite algebra, any ultra-product is isomorphic to either  $\mathfrak{A}$  or  $\mathbf{2}$ , the two element Boolean algebra, or the trivial algebra, thus  $\mathfrak{P1} = ISP\{\mathfrak{A}\}$ .

So, up to isomorphism,  $\mathbf{P1}$ -algebras are subalgebras of powers of  $\mathfrak{A}$ , thus one can think of their elements as tuples or, more precisely, as functions  $b : I \rightarrow \mathfrak{A}$ , for some set of indices  $I$ .

We will say that an element  $b$  in a  $\mathbf{P1}$ -algebra  $\mathfrak{B} \leq \mathfrak{A}^I$  is *Boolean* iff  $b_i \in \{0, 1\}$ , for all  $i \in I$ . Otherwise we say that  $b$  is *non-standard*.

One can easily check that  $b$  is Boolean iff  $(b \rightarrow b) \rightarrow b = b$ , for if  $b$  is Boolean, the identity holds and if for some  $i \in I, b_i = a$ , then  $(b_i \rightarrow b_i) \rightarrow b_i = (a \rightarrow a) \rightarrow a = 1 \rightarrow a = 1 \neq b_i$ .

Thus **P1**-algebras consist of two parts, a Boolean algebra **B**, the set of all Boolean elements, and a set  $S$ , the set of all non-standard elements.

Notice also that any non atomic term evaluated in a **P1**-algebra is Boolean.

### FREE ALGEBRAS

**Theorem 2.** *The free **P1**-algebra generated by a single generator  $x$  is the algebra depicted in diagram 1.*

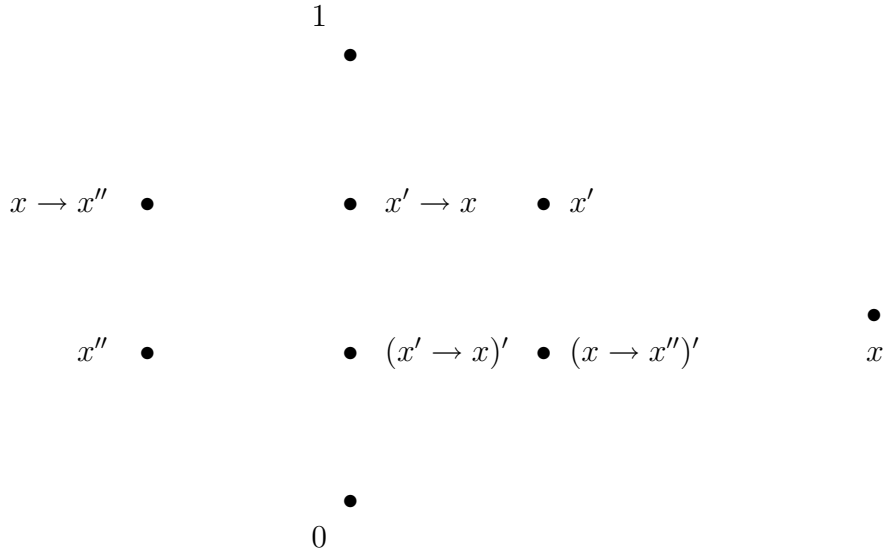


Diagram 1

*Proof.* Take any element  $x$  in a **P1**-algebra  $\mathfrak{B}$ . Since  $\mathfrak{B} \leq \mathfrak{A}^I$ , for some set  $I$ , and  $\mathfrak{A}^I \cong \mathfrak{A}^J \times \mathfrak{A}^K$ , where  $J = \{i \in I : x_i \neq a\}$  and  $K = \{i \in I : x_i = a\}$ ,  $x$  can be regarded as an ordered pair  $(\mathbf{b}, \mathbf{a})$ , where  $\mathbf{b}$  is a sequence of 0's and 1's that behaves as an element in a Boolean algebra and  $\mathbf{a}$  is a sequence of  $a$ 's that behaves as  $a$ .

The reader can easily check that the algebra generated by  $x = (\mathbf{b}, \mathbf{a})$  is (isomorphic to) the one depicted in the diagram.  $\square$

**Theorem 3.** *The free **P1**-algebra in two generators  $x$  and  $y$  is the algebra*

$$\mathbf{2}^{2^2} \times \mathbf{2}^2 \times \mathbf{2}^2 \times \mathbf{2} \cup \{x, y\}.$$

*Proof.* As in the previous case, if  $x$  and  $y$  are elements in a **P1**-algebra  $\mathfrak{B}$ , if we let

- i)  $I_1 = \{i : x_i \neq a \text{ and } y_i \neq a\}$ ,
- ii)  $I_2 = \{i : x_i \neq a \text{ and } y_i = a\}$ ,
- iii)  $I_3 = \{i : x_i = a \text{ and } y_i \neq a\}$ ,
- iv)  $I_4 = \{i : x_i = y_i = a\}$ ,

$\mathfrak{B}$  is (isomorphic to) a subalgebra of  $\mathfrak{A}^I \cong \mathfrak{A}^{I_1} \times \mathfrak{A}^{I_2} \times \mathfrak{A}^{I_3} \times \mathfrak{A}^{I_4}$ , and we can regard  $x$  and  $y$  (after reordering) as 4-tuples,  $x = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{a}_3, \mathbf{a}_4)$ ,  $y = (\mathbf{c}_1, \mathbf{a}_2, \mathbf{c}_3, \mathbf{a}_4)$ , where  $\mathbf{b}_1, \mathbf{c}_1$  are in  $\mathbf{2}^{I_1}$ ,  $\mathbf{b}_2$  is in  $\mathbf{2}^{I_2}$ ,  $\mathbf{c}_3$  is in  $\mathbf{2}^{I_3}$  and the  $\mathbf{a}_i$ 's are sequences of  $a$ 's of the appropriate length. For any variable  $z$ , let  $z^* = z \rightarrow z''$ . Then  $x^* = (1, 1, 0, 0)$  and  $y^* = (1, 0, 1, 0)$ , so

$$\begin{aligned} x^*y^* &= (1, 0, 0, 0), \\ x^*y^{*'} &= (0, 1, 0, 0), \\ x^{*'}y^* &= (0, 0, 1, 0), \\ x^{*'}y^{*'} &= (0, 0, 0, 1), \end{aligned}$$

are in the algebra generated by  $x$  and  $y$ , and so are

$$\begin{aligned} (\mathbf{b}_1, 0, 0, 0) &= x'' x^*y^*, \\ (\mathbf{c}_1, 0, 0, 0) &= y'' x^*y^*, \\ (0, \mathbf{b}_2, 0, 0) &= x'' x^*y^{*'}, \\ (0, 0, \mathbf{c}_3, 0) &= y'' x^{*'}y^*. \end{aligned}$$

The first two elements will generate the Boolean algebra  $\mathbf{2}^{2^2}$  as a subalgebra of  $\mathfrak{A}^{I_1}$ , the other two elements will generate the Boolean algebra  $\mathbf{2}^2$  as subalgebras of  $\mathfrak{A}^{I_2}$  and  $\mathfrak{A}^{I_3}$ , respectively. Since any complex term evaluated in  $a$  is either 0 or 1, the fourth component is isomorphic to  $\mathbf{2}$ .  $\square$

This result can be generalized in the obvious way.

**Theorem 4.** *The free **P1**-algebra in  $n$  generators  $x_1, \dots, x_n$  is the algebra*

$$\prod_{i=0}^n \binom{n}{i} \mathbf{2}^{2^i} \cup \{x_1, \dots, x_n\}.$$

**Corollary 5.** *The free spectrum of the class of  $\mathfrak{P1}$  is  $\langle 2^{3^n} + n : n \in \omega \rangle$ .*

**Corollary 6.** *If  $\mathfrak{B} = \mathbf{2}^n \cup S$  is a **P1**-algebra, then  $|S| \leq 3^n - 2^n$ .*

## CONGRUENCES

Let  $\mathfrak{B}$  be a **P1**-algebra, and let  $x = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{a}_3, \mathbf{a}_4)$  and  $y = (\mathbf{c}_1, \mathbf{a}_2, \mathbf{c}_3, \mathbf{a}_4)$  be as before. Then the Boolean part of  $\mathfrak{B}$  is (isomorphic to) a sub-Boolean algebra

$$B_1 \times B_2 \times B_3 \times B_4 \leq \mathbf{2}^{I_1} \times \mathbf{2}^{I_2} \times \mathbf{2}^{I_3} \times \mathbf{2}^{I_4}$$

The following theorem describes the principal congruence  $\Theta(x, y)$  generated by  $x$  and  $y$ .

**Theorem 7.** *With the notations above,*

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \Theta(x, y) (\beta_1, \beta_2, \beta_3, \beta_4)$$

*iff*

$$x = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \text{ and } y = (\beta_1, \beta_2, \beta_3, \beta_4)$$

or

- i)  $(\alpha_1, \beta_1)$  is in the principal congruence generated by  $\mathbf{b}_1$  and  $\mathbf{c}_1$  in  $B_1$ ,
- ii)  $\alpha_2, \beta_2$  are elements of  $B_2$ ,
- iii)  $\alpha_3, \beta_3$  are elements in  $B_3$  and
- iv)  $\alpha_4, \beta_4$  are elements in  $B_4$  and  $\alpha_4 = \beta_4$ .

*Proof.*

- $\supseteq$ ) The set  $C$  described is a congruence that contains the pair  $(x, y)$ .
- $\subseteq$ ) Let  $\theta$  be any congruence such that  $x\theta y$ . We will prove that  $C \subseteq \theta$ .

So let  $((\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4)) \in C$ . By i),  $(\alpha_1, \beta_1) \in \Theta_{B_1}(\mathbf{b}_1, \mathbf{c}_1)$ , the principal congruence generated by  $\mathbf{b}_1$  and  $\mathbf{c}_1$  in  $B_1$ . Recall that  $\Theta_{B_1}(\mathbf{b}_1, \mathbf{c}_1) = \bigcup \theta_n$ , where

$$\begin{aligned} \theta_0 &= \{(\mathbf{b}_1, \mathbf{c}_1), (\mathbf{c}_1, \mathbf{b}_1)\} \cup \{(u, u) : u \in B_1\}, \\ \theta_{n+1} &= \theta_n \cup \theta_n \circ \theta_n \cup \{(u_1 \rightarrow v_1, u_2 \rightarrow v_2) : (u_1, u_2), (v_1, v_2) \in \theta_n\} \\ &\quad \cup \{(u', v') : (u, v) \in \theta_n\} \end{aligned}$$

It follows by induction that  $(\alpha_1, 0, 0, 0) \theta (\beta_1, 0, 0, 0)$ . In the first step of the induction we use the fact that  $(\mathbf{b}_1, 0, 0, 0) = x''x^*y^* \theta y''x^*y^* = (\mathbf{c}_1, 0, 0, 0)$ .

Next we observe that since  $(0, 1, 0, 0) = x^*y^{*'} \theta y^*x^*y^{*'} = (0, 0, 0, 0)$ , we have  $(0, \alpha_2, 0, 0) \theta (0, 0, 0, 0) \theta (0, \beta_2, 0, 0)$ .

Similarly, we get that  $(0, 0, \alpha_3, 0) \theta (0, 0, \beta_3, 0)$ .

Finally, since  $\alpha_4 = \beta_4$ , we have  $(0, 0, 0, \alpha_4) \theta (0, 0, 0, \beta_4)$ .

But  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\alpha_1, 0, 0, 0) \vee (0, \alpha_2, 0, 0) \vee (0, 0, \alpha_3, 0) \vee (0, 0, 0, \alpha_4)$  and

$(\beta_1, \beta_2, \beta_3, \beta_4) = (\beta_1, 0, 0, 0) \vee (0, \beta_2, 0, 0) \vee (0, 0, \beta_3, 0) \vee (0, 0, 0, \beta_4)$ , where  $\vee$  is the Boolean join, thus  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \theta (\beta_1, \beta_2, \beta_3, \beta_4)$ .



## SUBDIRECTLY IRREDUCIBLES

**Theorem 9.** *The only subdirectly irreducible  $\mathbf{P1}$ -algebras are  $\mathbf{2}$  and  $\mathfrak{A}$ .*

*Proof.* Observe that if  $\theta$  is a congruence of the Boolean part of a  $\mathbf{P1}$ -algebra, then  $\hat{\theta} = \theta \cup \Delta$ , is a  $\mathbf{P1}$ -congruence. So the Boolean part of a subdirectly irreducible  $\mathbf{P1}$ -algebra has to be  $\mathbf{2}$ . There are only two such algebras, namely,  $\mathbf{2}$  and  $\mathfrak{A}$  and it is easy to check that they are both subdirectly irreducible.  $\square$

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