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Literal-paraconsistent and literal-paracomplete matrices

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We introduce a family of matrices that define logics in which paraconsistency and/or paracompleteness occurs only at the level of literals, that is, formulas that are propositional letters or their iterated negations. We give a sound and complete axiomatization for the logic defined by the class of all these matrices, we give conditions for the maximality of these logics and we study in detail several relevant examples.

1 Introduction

A classic way of defining a logic is through matrices. In this paper we introduce a class of matrices that generates a large family of deductive systems, or logics, that are paraconsistent and/or paracomplete. We have concentrated on logics for which paraconsistency occurs at the lowest level, that is literals, but at the complex level, that is, formulas that contain a binary connective, the negation behaves classically. Several logics of this kind have appeared in the literature, see for instance [1, 16, 2, 3, 8, 18].

There are two main reasons to study this type of paraconsistency-paracompleteness. The first one is purely methodological: we begin with the simpler case in order to study more complex negations in subsequent work. The second reason is that we think these are the most interesting ones from the point of view of applications to computer science such as databases, artificial intelligence and other areas. Inconsistencies often appear in databases and the most standard way of dealing with them is to devise methods that eliminate them by removing the less reliable information or some other criteria, and turning them into “safe” classically behaved objects. Nevertheless, we think that even though some of these inconsistencies are “strong”, like a person with two different passport numbers, and should be removed from the data base, there are “weak” inconsistencies, such as opinions, credit information, etc., these should remain available in the system because they are all important. An interesting problem is to deal with these inconsistencies using a logic that does not collapse in their presence. Many authors are now working in this problem but we will not pursue it since it is not our main interest, only a motivation. See for instance [9, 2, 3, 8, 18].

In Section 2 we introduce the family of logics defined by a special kind of matrices. In Section 3, we give a sound axiomatization for the logic defined by the class of all such matrices. The completeness of that axiomatization is proven in Section 4. In Section 5 we consider reduced matrices through the study of what we call the negation structure of the matrix, or intuitively, the behavior of the negation function $\sim$. In Section 6 we give some examples including a detailed study of the logics defined by all possible three element reduced matrices and an interesting four element matrix. Finally, in Section 7 our main result gives conditions for the maximality of systems defined by a single matrix.

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2 The logics

2.1 Language

Let $\mathcal{F}m$ be the set of formulas built in the usual recursive way from a denumerable set $\mathrm{Var} = \{p_1, p_2, \ldots\}$ of propositional variables and the connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$.

The *literals* of $\mathcal{F}m$ is the set Lit of all formulas of the form $\neg^k p$, where $\neg^0 p = p$ and $\neg^{k+1} p = \neg(\neg^k p)$, for $p \in \mathrm{Var}$. Formulas that contain a binary connective will be called *complex*.

We will use letters p, q, r etc. as metalinguistic variables for propositional variables, capital letters P, Q, R etc. as variables for complex formulas and Greek letters α, β, γ etc. as variables for general formulas.

2.2 Matrix logics

An *$\mathcal{L}$-matrix* is a pair $\mathcal{A} = \langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and F is a subset of the universe A of $\boldsymbol{A}$, the elements of F are called *the designated elements of $\mathcal{A}$*.

A *valuation into a matrix* $\mathcal{A}$ is a function $v : \mathrm{Var} \longrightarrow A$. Such a valuation v can be extended recursively to $\bar{v} : \mathcal{F}m \longrightarrow A$ in the usual way. We will also write α^v for $\bar{v}(\alpha)$.

Given a matrix $\mathcal{A}$, we define the relation $\vDash_{\mathcal{A}}$ between a set Γ of formulas and a formula φ as follows: $\Gamma \vDash_{\mathcal{A}} \varphi$ if and only if for any valuation v, if $\psi^v \in F$ for all $\psi \in \Gamma$, then $\varphi^v \in F$.

For a class $\mathbb{M}$ of matrices we define the relation $\vDash_{\mathbb{M}}$ as follows: $\Gamma \vDash_{\mathbb{M}} \varphi$ if and only if $\Gamma \vDash_{\mathcal{A}} \varphi$, for all $\mathcal{A} \in \mathbb{M}$.

Given a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$, in the usual sense (see for instance [6]), a matrix $\mathcal{A}$ is a *matrix model of $\mathcal{S}$* if $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Gamma \vDash_{\mathcal{A}} \varphi$, in which case F is called an *$\mathcal{S}$-filter*. We observe that if T is an $\mathcal{S}$-theory, then $\langle \mathcal{F}m, T\rangle$ is a matrix model of $\mathcal{S}$. These matrices are called *Lindenbaum matrices for $\mathcal{S}$* but these are not the kind of matrices that we will study here.

Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ be a deductive system. A class $\mathbb{M}$ of matrices will be called a *matrix semantics of $\mathcal{S}$* if for all $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}m$, $\Gamma \vdash_{\mathcal{S}} \varphi$ if and only if $\Gamma \vDash_{\mathbb{M}} \varphi$.

On the other hand, it is easy to check that for any class $\mathbb{M}$ of matrices, the relation $\vDash_{\mathbb{M}}$ given above defines a consequence relation.

2.3 Literal-paraconsistent-paracomplete matrices

Let A be a set such that $\{0, 1\} \subseteq A$ and $F \subseteq A$ such that $1 \in F$ and $0 \notin F$. Let $\sim : A \longrightarrow A$ be a function such that ${\sim}1 = 0$ and ${\sim}0 = 1$. We define *the literal-paraconsistent-paracomplete matrix* (or *LPP-matrix*) $\langle \boldsymbol{A}, F, \sim\rangle$ with the following operations:

$$a \vee b = \begin{cases} 1 & \text{if } a \in F \text{ or } b \in F, \\ 0 & \text{otherwise}, \end{cases}$$

$$a \wedge b = \begin{cases} 1 & \text{if } a \in F \text{ and } b \in F, \\ 0 & \text{otherwise}, \end{cases}$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \notin F \text{ or } b \in F, \\ 0 & \text{otherwise}. \end{cases}$$

We observe that the LPP-matrix $\langle \boldsymbol{A}, F, \sim\rangle$ is just a matrix in the usual sense of the previous paragraph, where the algebra over the universe A has three binary operations defined as above and one unary operation $\sim$, which we want to distinguish because of its relevance in the definition.

Observe, too, that the name "literal-paraconsistent-paracomplete" is slightly misleading since not all these matrices are both paraconsistent and paracomplete. An LPP-matrix is paraconsistent if for some $a \in A$ both $a \in F$ and ${\sim}a \in F$; it is paracomplete if both $a \notin F$ and ${\sim}a \notin F$. As a matter of fact, the matrix $\langle \{1, 0\}, \{1\}, \sim\rangle$, where ${\sim}0 = 1$ and ${\sim}1 = 0$, which defines classical logic (to which we will refer to as CPC), is neither paraconsistent nor paracomplete. In a sense that will be made clear in the next section, this is the only such LPP-matrix. As an example, the matrices $I^n P^k$ introduced in [11] are LPP-matrices.

3 Axiomatization

We will begin our study by defining a sound and complete deductive system for the logic defined by the class of all LPP-matrices $\langle \boldsymbol{A}, F, \sim\rangle$, with no conditions on A, F or $\sim$. This system will be called *literal-paraconsistent-paracomplete logic*.

3.1 Axiomatization of literal-paraconsistent-paracomplete logic

Define the deductive system LPPL with modus ponens (MP) as its only rule and the following axioms:

(A1) $\alpha \to (\beta \to \alpha)$.
(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.
(A3) $(\alpha \wedge \beta) \to \alpha$.
(A4) $(\alpha \wedge \beta) \to \beta$.
(A5) $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$.
(A6) $\alpha \to (\alpha \vee \beta)$.
(A7) $\beta \to (\alpha \vee \beta)$.
(A8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$.
(A9) Axiom for negation: $(\neg P \to \neg Q) \to (Q \to P)$, where P and Q are complex formulas.

We observe that this system was proposed by Puga and da Costa in [16] as an axiomatization of an imaginary logic of Vasiliev. The matrix proposed in that paper (denoted $\mathbb{M}$) is not an LPP-matrix.

Theorem 3.1 (Deduction Theorem) *The following holds in* LPPL*: If* $\Gamma, \varphi \vdash_{\mathsf{LPPL}} \psi$*, then* $\Gamma \vdash_{\mathsf{LPPL}} \varphi \to \psi$.

Theorem 3.2 *Let* $\varphi(p_1, p_2, \dots, p_n)$ *be a classical tautology. Then:*

1. $\vdash_{\mathsf{LPPL}} \varphi(A_1, A_2, \dots, A_n)$*, for complex formulas* $A_1, A_2, \dots, A_n$.

2. *If* φ *does not contain negations, then* $\vdash_{\mathsf{LPPL}} \varphi(\alpha_1, \alpha_2, \dots, \alpha_n)$*, for formulas* $\alpha_1, \alpha_2, \dots, \alpha_n$ *(which may include negations).*

Proof. For part 1., observe that when restricted to complex formulas, the axioms (A1) – (A9) plus (MP) are one of the most standard axiomatizations of classical propositional calculus. A similar remark will take care of 2. □

4 Completeness

Definition 4.1

1. A set Σ of formulas is *non-trivial* if there exists a formula that is not deducible from Σ.

2. A set Σ of formulas is *satisfiable* if there exists a valuation v into an LPP-matrix $\langle \boldsymbol{A}, F, \sim\rangle$ such that $\sigma^v \in F$ for all $\sigma \in \Sigma$.

3. A non-trivial set Σ of formulas is *maximal* if for any $\theta \notin \Sigma$, $\Sigma \cup \{\theta\}$ is trivial.

4. A *deductive filter* is a set $\Gamma \subseteq \mathcal{F}m$ that contains all axioms and is closed under (MP).

Lemma 4.2 *If* Σ *is a non-trivial maximal set of formulas, it is a deductive filter.*

Proof. Let us assume that θ is either an instance of an axiom or obtained by (MP) from τ and $\tau \to \theta \in \Sigma$. In both cases $\Sigma \vdash_{\mathsf{LPPL}} \theta$. If $\theta \notin \Sigma$, then by maximality, $\Sigma \cup \{\theta\}$ is trivial, so for any formula α, $\Sigma \cup \{\theta\} \vdash_{\mathsf{LPPL}} \alpha$, and by the Deduction Theorem, $\Sigma \vdash_{\mathsf{LPPL}} \theta \to \alpha$, so by (MP), $\Sigma \vdash_{\mathsf{LPPL}} \alpha$, contradicting the non-triviality of Σ. Thus $\theta \in \Sigma$, so Σ is closed under (MP) and contains all the axioms of LPPL, that is, it is a deductive filter. □

Theorem 4.3 *If Σ is a non-trivial set of formulas, then it is satisfiable.*

Proof. The proof follows the guidelines of the corresponding proof for CPC. We let $\varphi_1, \dots, \varphi_n, \dots$ be an enumeration of the formulas of the language. Then we extend Σ to a non-trivial maximal set of formulas recursively as follows:

$$\Sigma_0 = \Sigma, \quad \Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\varphi_n\} & \text{if it is non-trivial,} \\ \Sigma_n & \text{otherwise.} \end{cases}$$

By definition, each Σ_n is non-trivial, so we define

$$\bar{\Sigma} = \bigcup\{\Sigma_n \,:\, n \in \mathbb{N}\}.$$

Obviously $\bar{\Sigma}$ is non-trivial, for if not, $\bar{\Sigma} \vdash_{\mathsf{LPPL}} A \wedge \neg A$, for some complex formula A and since proofs are of finite length, one of the Σ_n would be non-trivial. On the other hand $\bar{\Sigma}$ is maximal by construction. So by Lemma 4.2, $\bar{\Sigma}$ is closed under deductions and contains all the axioms. Also, $\bar{\Sigma}$ verifies the following property: If A is a complex formula, then $A \in \bar{\Sigma}$ if and only if $\neg A \notin \bar{\Sigma}$.

In order to prove this from right to left, assume $A \notin \bar{\Sigma}$. Then $\bar{\Sigma} \cup \{A\}$ is trivial, so $\bar{\Sigma} \cup \{A\} \vdash_{\mathsf{LPPL}} \neg A$ and thus $\bar{\Sigma} \vdash_{\mathsf{LPPL}} (A \rightarrow \neg A)$. But $\vdash_{\mathsf{LPPL}} (A \rightarrow \neg A) \rightarrow \neg A$, since it is a substitution of complex formulas in a classical tautology, so by (MP), $\bar{\Sigma} \vdash_{\mathsf{LPPL}} \neg A$ and by Lemma 4.2, $\neg A \in \bar{\Sigma}$.

A similar argument gives the proof in the other direction.

In order to finish the proof of the theorem, we define the matrix

$$\mathcal{M} = \langle \{0,1\} \cup \mathrm{Lit}, F, \sim \rangle,$$

where $F = \{\neg^k p \in \mathrm{Lit} \,:\, \neg^k p \in \bar{\Sigma}\} \cup \{1\}$ and $\sim$: Lit $\longrightarrow$ Lit is defined by $\sim\alpha = \neg\alpha$. Now define the valuation $v(p) = p$. We have $\alpha^v \in F$ if and only if $\alpha \in \bar{\Sigma}$. The proof is by induction on α.

i) If $\alpha \in \mathrm{Lit}$, then the condition holds by definition.

ii) If $\alpha = \neg\beta$, there are two cases.

a) If $\beta \in \mathrm{Lit}$, then $\alpha \in \mathrm{Lit}$ and we are in case i).

b) If $\beta \notin \mathrm{Lit}$, then both α and β are complex formulas and thus, using the induction hypothesis, the following are equivalent: $\alpha^v \in F$, $\alpha^v = 1$, $\beta^v = 0$, $\beta \notin \bar{\Sigma}$, $\alpha \in \bar{\Sigma}$.

iii) If $\alpha = (\beta \rightarrow \gamma)$, then using the induction hypothesis, the following are equivalent: $\alpha^v \in F$, $(\beta \rightarrow \gamma)^v = 1$, $\beta^v \notin F$ or $\gamma^v \in F$, $\beta \notin \bar{\Sigma}$ or $\gamma \in \bar{\Sigma}$. For proving that the last is equivalent to $(\beta \rightarrow \gamma) \in \bar{\Sigma}$, we have two cases.

If $\beta \notin \bar{\Sigma}$, then by construction $\bar{\Sigma} \cup \{\beta\}$ is trivial. So in particular, $\bar{\Sigma} \cup \{\beta\} \vdash \gamma$, or $\bar{\Sigma} \vdash (\beta \rightarrow \gamma)$. Assume that $(\beta \rightarrow \gamma) \notin \bar{\Sigma}$. Then since this is a complex formula, $\neg(\beta \rightarrow \gamma) \in \bar{\Sigma}$. This clearly contradicts the fact that $\bar{\Sigma}$ is not trivial.

If $\gamma \in \bar{\Sigma}$, then $(\beta \rightarrow \gamma) \in \bar{\Sigma}$ by (A1) and (MP). So in any case, $(\beta \rightarrow \gamma) \in \bar{\Sigma}$.

For proving the implication in the other direction, if $(\beta \rightarrow \gamma) \in \bar{\Sigma}$, then either $\beta \notin \bar{\Sigma}$ or else, by (MP), $\gamma \in \bar{\Sigma}$.

iv) If $\alpha = (\beta \wedge \gamma)$, then, using the induction hypothesis, the following are equivalent: $\alpha^v \in F$, $(\beta \wedge \gamma)^v = 1$, $\beta^v \in F$ and $\gamma^v \in F$, and by Lemma 4.2 and axioms (A3), (A4), and (A5), the latter is equivalent to $(\beta \wedge \gamma) \in \bar{\Sigma}$.

v) If $\alpha = (\beta \vee \gamma)$, the proof is similar to the previous case using axioms (A6), (A7), (A8) and the induction hypothesis. ☐

5 Negation structures, the Leibniz congruence and reduced matrices

We begin this section with an important concept in algebraic logic.

Definition 5.1 [6, Definiton 1.4] Let $\boldsymbol{A}$ be an algebra and $F \subseteq A$. We define the binary relation on A:

$$\boldsymbol{\Omega}_{\boldsymbol{A}} F = \{\langle a, b\rangle \,:\, \varphi^{\boldsymbol{A}}(a, \bar{c}) \in F \text{ if and only if } \varphi^{\boldsymbol{A}}(b, \bar{c}) \in F, \text{ for all } \varphi(p, q_1, \dots, q_n) \in \mathcal{F}m \text{ and all } \bar{c} \in A^n\},$$

where $\varphi^{\boldsymbol{A}}$ is the interpretation of the formula $\varphi(p, q_1, \dots, q_n)$ in $\boldsymbol{A}$ replacing the propositional letters $p, q_1, \dots, q_n$ by $a, c_1, \dots, c_n$, as usual.

A congruence Θ on $\boldsymbol{A}$ is *compatible with the subset* F *of* A if for all $a, b \in A$, from $a \in F$ and $\langle a, b\rangle \in \Theta$ it follows that $b \in F$.

Theorem 5.2 [6, Theorem 1.5] *Given an algebra* $\boldsymbol{A}$ *and any* $F \subseteq A$, $\boldsymbol{\Omega_A} F$ *is the largest congruence on* $\boldsymbol{A}$ *compatible with* F.

The congruence $\boldsymbol{\Omega_A}(F)$ is *the Leibniz relation on* $\boldsymbol{A}$ *over* F. The corresponding operator on the power set of A, denoted $\boldsymbol{\Omega_A}$, is called *the Leibniz operator on* A. If $\boldsymbol{A}$ is the formula algebra of $\mathcal{F}m$, the Leibniz operator is simply denoted $\boldsymbol{\Omega}$.

Let $\mathcal{M} = \langle \boldsymbol{A}, F, \sim\rangle$ be an LPP-matrix. *The negation structure of* $\mathcal{M}$ is a function $\mathrm{nstr}_{\mathcal{M}} : A \longrightarrow \{0,1\}^{\mathbb{N}}$ such that $\mathrm{nstr}_{\mathcal{M}}(a)(k) = 1$ if and only if $\sim^k a \in F$.

The negation type of $a \in A$ is the function $\mathrm{nstr}_{\mathcal{M}}(a)$. If $\mathcal{M}$ is finite, then each negation type is eventually periodic, one can think of it as a finite sequence of 0's and 1's.

Lemma 5.3 *Let* v *be a valuation. For any* $a, b \in A$, *define*

$$v(a|b)(p) = \begin{cases} p^v & \text{if } p^v \neq a, \\ b & \text{if } p^v = a. \end{cases}$$

Then if $\mathrm{nstr}_{\mathcal{M}}(a) = \mathrm{nstr}_{\mathcal{M}}(b)$ *and* α *is a formula,* $\alpha^v \in F$ *if and only if* $\alpha^{v(a|b)} \in F$.

Theorem 5.4 $\langle a, b\rangle \in \boldsymbol{\Omega_A} F$ *if and only if* $\mathrm{nstr}_{\mathcal{M}}(a) = \mathrm{nstr}_{\mathcal{M}}(b)$.

Proof. We know that $\langle a, b\rangle \in \boldsymbol{\Omega_A} F$ if and only if for all $\varphi(p, q_1, \dots, q_n) \in \mathcal{F}m$ and all $\bar{c} \in A^n$,

$$\varphi^{\boldsymbol{A}}(a, \bar{c}) \in F \quad \text{if and only if} \quad \varphi^{\boldsymbol{A}}(b, \bar{c}) \in F.$$

This is equivalent to

$$(\star) \qquad \varphi^v \in F \quad \text{if and only if} \quad \varphi^{v(a|b)} \in F,$$

for any valuation v such that $v(p) = a$.

If $(\star)$ holds, then for $\varphi = \neg^k p$, we have $\mathrm{nstr}_{\mathcal{M}}(a) = \mathrm{nstr}_{\mathcal{M}}(b)$.

On the other hand, if $\mathrm{nstr}_{\mathcal{M}}(a) = \mathrm{nstr}_{\mathcal{M}}(b)$, by Lemma 5.3, $(\star)$ holds. □

We say the matrix $\mathcal{M}|_{\boldsymbol{\Omega_A} F} = \langle \boldsymbol{A}|_{\boldsymbol{\Omega_A} F}, F|_{\boldsymbol{\Omega_A} F}, \sim\rangle$ is a *reduced matrix*. Notice that in a reduced matrix there is one single element for each negation type present in $\mathcal{M} = \langle \boldsymbol{A}, F, \sim\rangle$.

It is a well known fact (see [10]) that $\mathcal{M}$ and $\mathcal{M}|_{\boldsymbol{\Omega_A} F}$ define the same deductive system, so if the matrices $\mathcal{M}$ and $\mathcal{M}'$ are such that both contain elements with the same negation types, they have the same reduction, so they give rise to the same deductive system. It is enough then to study reduced matrices.

6 Some examples

We will illustrate the logics that arise in this process by studying the logics associated with some three element matrices which have appeared previously in the literature under different names and formalizations. We also study a four element LPP-matrix which we think is important because it is the smallest matrix that is both paraconsistent and paracomplete. For each of them we will give an axiomatization and prove that they are complete using the well known method of Kalmár. As far as we know, the first use of this method in the context of paraconsistency appeared in [18]. By the way that paper discusses precisely the system in Subsection 6.1.4.

6.1 Three element matrices

In this section we will study matrices $\langle\{0, \frac{1}{2}, 1\}, F, \sim\rangle$. There are three possible functions $\sim$, namely,

$$\text{either} \quad \sim_1 \tfrac{1}{2} = \tfrac{1}{2} \quad \text{or} \quad \sim_2 \tfrac{1}{2} = 1 \quad \text{or} \quad \sim_3 \tfrac{1}{2} = 0.$$

Likewise, there are two possible filters, namely, $F_1 = \{1\}$ and $F_2 = \{1, \frac{1}{2}\}$. Of the six combinations only four of them are reduced:

$$\mathcal{M}^3_{1,1} = \langle\{0, \tfrac{1}{2}, 1\}, F_1, \sim_1\rangle, \quad \mathcal{M}^3_{1,3} = \langle\{0, \tfrac{1}{2}, 1\}, F_1, \sim_3\rangle,$$
$$\mathcal{M}^3_{2,1} = \langle\{0, \tfrac{1}{2}, 1\}, F_2, \sim_1\rangle, \quad \mathcal{M}^3_{2,2} = \langle\{0, \tfrac{1}{2}, 1\}, F_2, \sim_2\rangle.$$

Throughout this and the following sections for any formula α, we will let $\alpha^\circ = \neg(\alpha \wedge \neg\alpha)$ and $\alpha^\bullet = \alpha \vee \neg\alpha$. Also, in order to ease the notation, we will simply write $\vdash$ instead of the corresponding $\vdash_{\mathcal{M}}$.

6.1.1 $\mathcal{M}^3_{1,1} = \langle\{0, \frac{1}{2}, 1\}, F_1, \sim_1\rangle$

The following is an axiomatization for the logic $\mathcal{S}_{1,1}$ defined by this matrix. This system appears in [14] under the name I^1_2.

The axioms are:

($\mathrm{A}_{1,1}$.1) The axioms of LPPL.

($\mathrm{A}_{1,1}$.2) $\alpha^\bullet \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$.

($\mathrm{A}_{1,1}$.3) $\beta \leftrightarrow \neg\neg\beta$.

Modus Ponens is the only rule of inference.

We observe that by Theorem 3.2, for any complex formula A, $A^\bullet$ holds in $\mathcal{S}_{1,1}$. We will use this without further mention.

Lemma 6.1 *The following hold in system* $\mathcal{S}_{1,1}$*:*

1. $\neg(\beta \vee \neg\beta) \vdash \neg(\neg^k\beta \vee \neg^{k+1}\beta)$.
2. $\beta, \neg\beta \vdash \gamma$.
3. $\beta, \neg(\beta \wedge \beta) \vdash \gamma$.
4. $\beta, \neg\gamma \vdash \neg(\beta \to \gamma)$.
5. $\beta, \neg(\gamma \wedge \gamma) \vdash \neg(\beta \to \gamma)$.
6. $\neg(\beta \vee \neg\beta) \vdash \neg(\beta \wedge \beta)$.

Proof. The proofs are straightforward using the axioms of LPPL, Theorems 3.1 and 3.2. Axiom ($\mathrm{A}_{1,1}$.3) is used in the proof of 1., axiom ($\mathrm{A}_{1,1}$.2) is used in the proof of 2. □

Lemma 6.2 *Let α be a formula and let $p_1, \ldots, p_n$ be the propositional variables that appear in α. For any valuation v define*

$$\alpha' = \begin{cases} \alpha & \text{if } v(\alpha) = 1, \\ \neg(\alpha \wedge \alpha) & \text{if } v(\alpha) = \frac{1}{2}, \\ \neg\alpha & \text{if } v(\alpha) = 0, \end{cases} \qquad \overline{\alpha} = \begin{cases} \alpha^\bullet & \text{if } v(\alpha) = 1, \\ \neg(\alpha \vee \neg\alpha) & \text{if } v(\alpha) = \frac{1}{2}, \\ \alpha^\bullet & \text{if } v(\alpha) = 0. \end{cases}$$

Then

$$\overline{p_1}, \ldots, \overline{p_n}, p'_1, \ldots, p'_n \vdash \alpha'.$$

Proof. The proof is by induction on the complexity of α.

Case I: $\alpha = p$. Then the lemma is true since $\alpha' = p'$ and $p' \vdash p'$.

Case II: $\alpha = \neg\beta$. Then both α and β share the same variables $p_1, \ldots, p_n$.

If $\beta^v = 1$, then $\alpha^v = 0$, so $\alpha' = \neg\alpha = \neg\neg\beta$ and $\beta' = \beta$. But then, by induction hypothesis,

$$\overline{p_1}, \ldots, \overline{p_n}, p'_1, \ldots, p'_n \vdash \beta,$$

and by axiom ($\mathrm{A}_{1,1}$.3), $\overline{p_1}, \ldots, \overline{p_n}, p'_1, \ldots, p'_n \vdash \alpha'$.

If $\beta^v = 0$, then $\alpha^v = 1$, so $\beta' = \neg\beta = \alpha = \alpha'$, and the lemma follows by induction hypothesis.

If $\beta^v = \frac{1}{2}$, then $\alpha^v = \frac{1}{2}$, so $\alpha' = \neg(\alpha \wedge \alpha)$ and $\beta' = \neg(\beta \wedge \beta)$. Also, this implies $\beta = \neg^k p$ and $\alpha = \neg^{k+1} p$. But in this case by Lemma 6.1, 1., $\neg(p \vee \neg p) \vdash \alpha'$, so the lemma holds.

Case III: $\alpha = \beta \to \gamma$.

If $\beta^v \in \{0, \frac{1}{2}\}$, then $\alpha^v = 1$, so $\alpha' = \alpha = \beta \to \gamma$ and $\beta' = \neg\beta$ or $\beta' = \neg(\beta \wedge \beta)$. Then by induction hypothesis, $\overline{p_1}, \ldots, \overline{p_n}, p'_1, \ldots, p'_n \vdash \beta'$, so by Lemma 6.1, 2. or 3., $\overline{p_1}, \ldots, \overline{p_n}, p'_1, \ldots, p'_n, \beta \vdash \gamma$, and the result follows by the Deduction Theorem.

If $\gamma^v = 1$, then $\alpha^v = 1$, so $\alpha' = \alpha = \beta \to \gamma$ and $\gamma' = \gamma$. Then by induction hypothesis,

$$\overline{p_1}, \ldots, \overline{p_n}, p'_1, \ldots, p'_n \vdash \gamma,$$

so by (A1) and (MP), $\overline{p_1}, \ldots, \overline{p_n}, p'_1, \ldots, p'_n \vdash \beta \to \gamma$.

If $\beta^v = 1$ and $\gamma^v \in \{0, \frac{1}{2}\}$, then $\alpha^v = 0$, so $\alpha' = \neg(\beta \to \gamma)$, $\beta' = \beta$ and $\gamma' = \neg\gamma$ or $\gamma' = \neg(\gamma \wedge \gamma)$. Then by induction hypothesis, $\overline{p_1}, \ldots, \overline{p_n}, p'_1, \ldots, p'_n \vdash \beta$, and $\overline{p_1}, \ldots, \overline{p_n}, p'_1, \ldots, p'_n \vdash \neg\gamma$, so by Lemma 6.1, 4. or 5., the lemma holds.

Case IV: $\alpha = \beta \wedge \gamma$ or $\alpha = \beta \vee \gamma$. These cases are treated similarly. ☐

Theorem 6.3 (Completeness Theorem for $\mathcal{S}_{1,1}$) *Let α be an $\mathcal{M}^3_{1,1}$-tautology. Then in $\mathcal{S}_{1,1}$, $\vdash \alpha$.*

Proof. Since α is a tautology, for any valuation v, $\alpha^v = 1$, so $\alpha' = \alpha$. Let $p_1, \ldots, p_n$ be the propositional variables that appear in α. We choose three valuations v_1, v_2 and v_3 such that $v_1(p_n) = 1$, $v_2(p_n) = 0$ and $v_3(p_n) = \frac{1}{2}$, and they agree on all other variables. This means that the corresponding $\overline{p_i}$'s and p'_i's are the same for $i < n$ and differ for p_n. Let $\Sigma = \{\overline{p_1}, \ldots, \overline{p_{n-1}}, p'_1, \ldots, p'_{n-1}\}$. Then by Lemma 6.2, we have

$$\Sigma, p_n^\bullet, p_n \vdash \alpha, \quad \Sigma, p_n^\bullet, \neg p_n \vdash \alpha, \quad \Sigma, \neg(p_n \vee \neg p_n), \neg(p_n \wedge p_n) \vdash \alpha.$$

From the first two, $\Sigma, p_n^\bullet \vdash p_n \to \alpha$ and $\Sigma, p_n^\bullet \vdash \neg p_n \to \alpha$ (by Theorem 3.1), $\Sigma, p_n^\bullet \vdash (p_n \vee \neg p_n) \to \alpha$ (by (A8) and (MP)), $\Sigma, p_n^\bullet \vdash \alpha$ (by (MP)).

From the third one we can eliminate $\neg(p_n \wedge p_n)$ from the premises since it follows from $\neg(p_n \vee \neg p_n)$ by Lemma 6.1, 6.

Finally, we observe that $\neg(p_n \vee \neg p_n) = \neg p_n^\bullet$, so we are left with

$$\Sigma, p_n^\bullet \vdash \alpha, \quad \Sigma, \neg(p_n)^\bullet \vdash \alpha.$$

Using the fact that by Theorem 3.2, $\vdash (p_n^\bullet)^\bullet$, the same argument used two paragraphs above shows that $\Sigma \vdash \alpha$.

A similar argument repeated n times will lead to $\vdash \alpha$. ☐

6.1.2 $\mathcal{M}^3_{1,3} = \langle \{0, \frac{1}{2}, 1\}, F_1, \sim_3 \rangle$

The following is an axiomatization for the logic $\mathcal{S}_{1,3}$ defined by this matrix. This system appears in [14] under the name I^1 and was introduced in [19].

The axioms are:

(A$_{1,3}$.1) The axioms of LPPL.

(A$_{1,3}$.2) $\alpha^\bullet \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$.

(A$_{1,3}$.3) $(\neg\alpha)^\bullet$.

Modus Ponens is the only rule of inference.

Lemma 6.4 *The following hold in system $\mathcal{S}_{1,3}$:*

1. $\beta \vdash \neg\neg\beta$.
2. $\neg(\beta \vee \neg\beta), \neg\beta \vdash \gamma$.
3. $\beta, \neg\beta \vdash \gamma$.
4. $\beta, \neg\gamma \vdash \neg(\beta \to \gamma)$.
5. $\neg(\beta \vee \neg\beta) \vdash \neg\neg\beta$.

Proof. The proofs are straightforward using the axioms of LPPL, Theorems 3.1 and 3.2. Axiom (A$_{1,3}$.2) is used in the proof of 1., axiom (A$_{1,3}$.3) is used in the proof of 2. ☐

Lemma 6.5 *Let α be a formula and let $p_1, \dots, p_n$ be the propositional variables that appear in α. For any valuation v define*

$$\alpha' = \begin{cases} \alpha & \textit{if } v(\alpha) = 1, \\ \neg\neg\alpha & \textit{if } v(\alpha) = \frac{1}{2}, \\ \neg\alpha & \textit{if } v(\alpha) = 0, \end{cases} \qquad \overline{\alpha} = \begin{cases} \alpha^{\bullet} & \textit{if } v(\alpha) = 1, \\ \neg(\alpha \vee \neg\alpha) & \textit{if } v(\alpha) = \frac{1}{2}, \\ \alpha^{\bullet} & \textit{if } v(\alpha) = 0. \end{cases}$$

Then

$$\overline{p_1}, \dots, \overline{p_n}, p_1', \dots, p_n' \vdash \alpha'.$$

Proof. The proof is similar to that of Lemma 6.2 using Lemma 6.4. □

Theorem 6.6 (Completeness Theorem for $\mathcal{S}_{1,3}$) *Let α be an $\mathcal{M}^3_{1,3}$-tautology. Then in $\mathcal{S}_{1,3}$, $\vdash \alpha$.*

Proof. The proof is the same as that of Theorem 6.3 except that in this case the third assertion is

$$\Sigma, \neg(p_n \vee \neg p_n), \neg\neg p_n \vdash \alpha,$$

from which we can eliminate $\neg\neg p_n$ from the premises, since it follows from $\neg(p_n \vee \neg p_n)$ by Lemma 6.4, 5. □

6.1.3 $\mathcal{M}^3_{2,1} = \langle \{0, \frac{1}{2}, 1\}, F_2, \sim_1 \rangle$

The following is an axiomatization for the logic $\mathcal{S}_{2,1}$ defined by this matrix. This system appears in [8] and [14] under the name P^1_2. Moreover, an axiomatization and its completeness using Kalmár's method appear in [15].

The axioms are:

(A$_{2,1}$.1) The axioms of LPPL.

(A$_{2,1}$.2) $\beta^\circ \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$.

(A$_{2,1}$.3) $\alpha \leftrightarrow \neg\neg\alpha$.

Modus Ponens is the only rule of inference.

We observe that by Theorem 3.2, for any complex formula A, A° holds in $\mathcal{S}_{2,1}$. We will use this without further mention.

Lemma 6.7 *The following hold in system $\mathcal{S}_{2,1}$:*

1. $\beta^\circ, \neg\beta \vdash \beta \to \gamma$.
2. $\gamma^\circ \vdash \beta \to (\neg\gamma \to \neg(\beta \to \gamma))$.
3. $\neg(\alpha \wedge \alpha) \vdash \neg\alpha$.
4. $\beta \to \alpha, \neg\beta \to \alpha \vdash \alpha$.
5. $\vdash \alpha \vee \neg\alpha$.

Proof. The proofs are straightforward. Axiom (A$_{2,1}$.2) is used in the proof of 1. and axiom (A$_{2,1}$.3) is used in the proof of 1. and 3. □

Lemma 6.8 *Let α be a formula and let $p_1, \dots, p_n$ be the propositional variables that appear in α. For any valuation v define*

$$\alpha' = \begin{cases} \alpha & \textit{if } v(\alpha) = 1, \\ \alpha & \textit{if } v(\alpha) = \frac{1}{2}, \\ \neg\alpha & \textit{if } v(\alpha) = 0, \end{cases} \qquad \overline{\alpha} = \begin{cases} \alpha^{\circ} & \textit{if } v(\alpha) = 1, \\ \alpha \wedge \neg\alpha & \textit{if } v(\alpha) = \frac{1}{2}, \\ \alpha^{\circ} & \textit{if } v(\alpha) = 0. \end{cases}$$

Then

$$\overline{p_1}, \dots, \overline{p_n}, p_1', \dots, p_n' \vdash \alpha'.$$

Proof. The proof is similar to that of Lemma 6.2 using Lemma 6.7. □

Theorem 6.9 (Completeness Theorem for $\mathcal{S}_{2,1}$) *Let α be an $\mathcal{M}^3_{2,1}$-tautology. Then in $\mathcal{S}_{2,1}$, $\vdash \alpha$.*

Proof. The proof is like the previous ones but in this case, by Lemma 6.8, we have

$$\Sigma, p_n^\circ, p_n \vdash \alpha, \quad \Sigma, p_n^\circ, \neg p_n \vdash \alpha, \quad \Sigma, p_n \wedge \neg p_n, p_n \vdash \alpha.$$

We observe that from the first two, using Lemma 6.7, 4., we can cut p_n and $\neg p_n$.

From the third one we can eliminate p_n since it follows from $p_n \wedge \neg p_n$ by axiom (A3) and (MP). Finally, we observe that by Theorem 3.2, we can replace $p_n \wedge \neg p_n$ by $\neg(p_n^\circ)$, so we are left with

$$\Sigma, p_n^\circ \vdash \alpha, \quad \Sigma, \neg(p_n^\circ) \vdash \alpha.$$

Using Lemma 6.7, 4. again, $\Sigma \vdash \alpha$.

A similar argument repeated n times will lead to $\vdash \alpha$. □

6.1.4 $\mathcal{M}^3_{2,2} = \langle\{0, \frac{1}{2}, 1\}, F_2, \sim_2\rangle$

We will give an axiomatization for the system $\mathcal{S}_{2,2}$ that is appropriate in our context. This system is Sette's logic P^1, see [18]. It appears in [8] as system P^1_1 and in [11] as P^1. See also [12] and [17].

The axioms are

(A$_{2,2}$.1) The axioms of LPPL.

(A$_{2,2}$.2) $\beta^\circ \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$.

(A$_{2,2}$.3) $(\neg\alpha)^\circ$.

Modus Ponens is the only rule of inference.

Lemma 6.10 *The following hold in system $\mathcal{S}_{2,2}$:*

1. $\beta^\circ \vdash \beta \to \neg\neg\beta$.
2. $\beta^\circ, \neg\beta \vdash \beta \to \gamma$.
3. $\gamma^\circ \vdash \beta \to (\neg\gamma \to \neg(\beta \to \gamma))$.
4. $\neg(\alpha \wedge \alpha) \vdash \neg\alpha$.
5. $\beta \to \alpha, \neg\beta \to \alpha \vdash \alpha$.
6. $\vdash \alpha \vee \neg\alpha$.

Proof. The proofs are straightforward. □

Lemma 6.11 *Let α be a formula and let $p_1, \dots, p_n$ be the propositional variables that appear in α. For any valuation v define*

$$\alpha' = \begin{cases} \alpha & \textit{if } v(\alpha) = 1, \\ \alpha & \textit{if } v(\alpha) = \frac{1}{2}, \\ \neg\alpha & \textit{if } v(\alpha) = 0, \end{cases} \qquad \overline{\alpha} = \begin{cases} \alpha^\circ & \textit{if } v(\alpha) = 1, \\ \alpha \wedge \neg\alpha & \textit{if } v(\alpha) = \frac{1}{2}, \\ \alpha^\circ & \textit{if } v(\alpha) = 0. \end{cases}$$

Then

$$\overline{p_1}, \dots, \overline{p_n}, p'_1, \dots, p'_n \vdash \alpha'.$$

Proof. The proof is similar to that of Lemma 6.2 using Lemma 6.10. □

Theorem 6.12 (Completeness Theorem for $\mathcal{S}_{2,2}$) *Let α be an $\mathcal{M}^3_{2,2}$-tautology. Then in $\mathcal{S}_{2,2}$, $\vdash \alpha$.*

Proof. The proof is similar to that of Theorem 6.3. □

6.2 A four element matrix

We will study one more example which is interesting because it is the smallest matrix that is both paraconsistent and paracomplete. It is also one of the four matrices that verify the important theorem $\vdash \alpha \leftrightarrow \neg\neg\alpha$.

Consider the matrix $\mathcal{M}^4 = \langle\{0, \bot, \top, 1\}, F, \sim\rangle$, where $\neg\top = \top$ and $\neg\bot = \bot$, and the filter $F = \{1, \top\}$. We define the system $\mathcal{S}^4$ as follows.

The axioms are:

(A_4,1) The axioms of LPPL.

(A_4,2) $(\alpha^\bullet \wedge \beta^\circ) \rightarrow ((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha))$.

(A_4,3) $\alpha \leftrightarrow \neg\neg\alpha$.

Modus Ponens is the only rule of inference.

Lemma 6.13 *The following hold in system* $\mathcal{S}^4$*:*

1. $\vdash \beta^\circ \leftrightarrow (\neg^n\beta)^\circ$.
2. $\vdash \beta^\bullet \leftrightarrow (\neg^n\beta)^\bullet$.
3. $\neg\beta^\circ \vdash \beta \leftrightarrow \neg\beta$.
4. $\neg\beta^\bullet \vdash \neg(\neg\beta \wedge \neg\beta)$.
5. $\beta \wedge \neg\beta \vdash \neg^n\beta$.
6. $\gamma^\circ, \gamma, \neg\gamma \vdash \alpha$ (*or* $\gamma^\circ, \gamma, \neg(\gamma \wedge \gamma) \vdash \alpha$).
7. $\gamma^\circ, \beta, \neg\gamma \vdash \neg(\beta \rightarrow \gamma)$ (*or* $\gamma^\circ, \beta, \neg(\gamma \wedge \gamma) \vdash \neg(\beta \rightarrow \gamma)$).

Proof. The proofs are straightforward. Axiom (A_4.2) is used in the proof of 6. and axiom (A_4.3) is used in the proof of 1. and 3. ☐

Lemma 6.14 *Let* α *be a formula and let* $p_1, \dots, p_n$ *be the propositional variables that appear in* α*. For any valuation* v *define*

$$\alpha' = \begin{cases} \alpha & \textit{if } v(\alpha) = 1, \\ \alpha & \textit{if } v(\alpha) = \top, \\ \neg(\alpha \wedge \alpha) & \textit{if } v(\alpha) = \bot, \\ \neg\alpha & \textit{if } v(\alpha) = 0, \end{cases} \qquad \overline{\alpha} = \begin{cases} \alpha^\circ \wedge \alpha^\bullet & \textit{if } v(\alpha) = 1, \\ \neg\alpha^\circ \wedge \alpha^\bullet & \textit{if } v(\alpha) = \top, \\ \alpha^\circ \wedge \neg\alpha^\bullet & \textit{if } v(\alpha) = \bot, \\ \alpha^\circ \wedge \alpha^\bullet & \textit{if } v(\alpha) = 0. \end{cases}$$

Then

1. $\overline{p_1}, \dots, \overline{p_n}, p'_1, \dots, p'_n \vdash \overline{\alpha}$,
2. $\overline{p_1}, \dots, \overline{p_n}, p'_1, \dots, p'_n \vdash \alpha'$.

Proof. The proof is similar to that of Lemma 6.2 using Lemma 6.13. ☐

Theorem 6.15 (Completeness Theorem for $\mathcal{S}^4$) *Let* α *be an* $\mathcal{M}^3_{2,2}$*-tautology. Then in* $\mathcal{S}^4$, $\vdash \alpha$.

Proof. The proof is similar to that of Theorem 6.9 using Lemma 6.14. ☐

6.3 Matrices with negation of length at most 2

Let us consider matrices such that for any $a \in P$, $\sim\sim a = a$, that is, matrices that can model the axiom

$$\alpha \leftrightarrow \neg\neg\alpha.$$

For these matrices there are only four possible negation types, namely $(1, 1, 1, \dots)$, $(1, 0, 1, 0, \dots)$, $(0, 1, 0, 1, \dots)$ and $(0, 0, 0, \dots)$. Since $(1, 0, 1, 0, \dots)$ and $(0, 1, 0, 1, \dots)$ are the types of 1 and 0, respectively, they are always present, so there are only four (reduced) matrices, namely, if both, only one or none of the other two types are present.

If both types are present, then the matrix is $\mathcal{M}^4$.

If only the type $(1, 1, 1, 1, \dots)$ appears, then the matrix is $\mathcal{M}^3_{2,1}$.

If only the type $(0, 0, 0, 0, \dots)$ appears, then the matrix is $\mathcal{M}^3_{1,1}$.

If none of them appears, then the matrix is $\langle\{1, 0\}, \{1\}, \sim\rangle$, that is, the one that defines classical logic.

7 Maximality

We will denote by $\mathcal{S}_{\mathbb{M}}$ the logic associated to a class $\mathbb{M}$ of LPP-matrices, and if $\mathbb{M} = \{\mathcal{M}\}$, we write $\mathcal{S}_{\mathcal{M}}$.

If $\mathcal{S}_{\mathbb{M}}$ is a logic defined by a class of matrices, we will denote by $\mathcal{S}_{\mathbb{M}}(\varphi)$ the logic obtained by adding the new axiom φ. Observe that this implies that all substitution instances of the new formula must be added, but if φ is a CPC-tautology and α is complex, then $\varphi(\alpha)$ is an $\mathcal{M}$-tautology, so the only new formulas in $\mathcal{S}_{\mathbb{M}}(\varphi)$ are instances $\varphi(\neg^n p)$ of φ and their consequences. As an example, the logic $\mathcal{S}_{\mathbb{M}}(\alpha^\circ \wedge \alpha^\bullet)$ is CPC. The following lemma will be useful in the sequel.

Lemma 7.1 *Assume* $\Gamma \vDash_{\mathrm{CPC}} \psi$ *and let*

$$\Sigma = \{(\neg^k p)^\circ \wedge (\neg^k p)^\bullet \ : \ (\neg^k p) \textit{ is a literal that appears in } \Gamma \cup \{\psi\}\}.$$

Then $\Sigma \cup \Gamma \vdash_{\mathsf{LPPL}} \psi$.

Proof. Simply observe that under the hypotheses, for any LPP-matrix $\mathcal{M}$, $\Sigma \cup \Gamma \vDash_{\mathcal{M}} \psi$. □

In particular this implies that if $(p^\circ \wedge p^\bullet)$ is a consequence of $S_{\mathcal{M}}$, then this system is CPC.

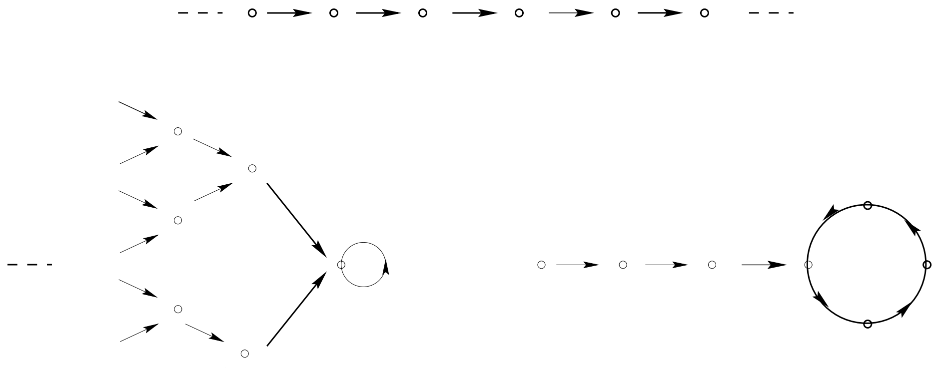
A logic is *maximal* if adding as an axiom any classical tautology that is not an $\mathcal{M}$-tautology, the logic becomes CPC.

Assume $\mathbb{M}$ is a family of LPP-matrices and $\mathcal{M} \in \mathbb{M}$. Then $\mathcal{S}_{\mathbb{M}} \subseteq \mathcal{S}_{\mathcal{M}}$, and thus, if $\mathcal{S}_{\mathbb{M}}$ is maximal, so is $\mathcal{S}_{\mathcal{M}}$. In this section we will characterize the logics defined by a single matrix $\mathcal{M}$ such that $S_{\mathcal{M}}$ is maximal. We will see that maximality depends on certain features of the negation structure of $\mathcal{M}$.

Before going into the topic we should say a few words about the description of the negation structure of a matrix.

The function $\sim\colon A \longrightarrow A$ defines an equivalence relation given by $a \equiv b$ if and only if $\sim^n a = \sim^m b$, for some non-negative integers n, m. The equivalence classes are called *connected components*, *negation orbits* or simply *orbits*.

The reader can easily figure out how the orbits look like. Some examples appear in the following diagram.



If there exists $a \in A$ and a non-negative integer n such that $\sim^n a = a$, we will say that *the orbit of* a *has a cycle*. The number of points in the cycle is called its *length*. The elements of the cycle are said to be *periodic*; its *period* is the length of the cycle. An orbit cannot have more than one cycle. For example, in the matrix $\mathcal{M}^4$ the orbits are $\{0,1\}$, $\{\top\}$ and $\{\bot\}$ they are all cycles. The matrix $\mathcal{M}^3_{2,2}$ has a single orbit with one cycle, as shown:

$$\tfrac{1}{2} \longrightarrow 0 \leftrightarrows 1,$$

where the arrows indicate the action of $\sim$.

It is obvious that in a finite matrix, all orbits have a cycle, so any element a is eventually periodic, that is, for some n, $\sim^n a$ belongs to the cycle of its orbit.

Every matrix contains the cycle $0 \leftrightarrows 1$. We will refer to it as *the Boolean cycle*. The orbit to which this cycle belongs is called *the Boolean orbit*.

On each orbit we can define the following preorder relation: $x \preceq y$ if and only if $\sim^n x = y$, for some non-negative integer n.

If a is such that for any b, $a \neq \sim b$, that is, a is not the negation of another point, we say *a is an initial point*. The set $\{\sim^n a \,:\, n \text{ is non-negative integer}\}$ is a *branch* of its orbit.

If we restrict the relation to points on the branches but not on a cycle, it is a partial order. We will refer to it as *the partial order induced by $\sim$ on the orbit*.

By a *chain* in this order we understand a sequence of points $\{x_i \,:\, i \in I\}$, where I is an interval of non-negative integers, such that for each $i \in I$, $x_{i+1} = \sim x_i$. In particular, a chain is a $\preceq$-totally ordered set such that if $x \prec y$ are two points in it, then $y = \sim^n x$.

Remark 7.2 Initial points play an interesting role in the following sense. Let $\mathcal{M}$ be a matrix and let $a \in A$ be an initial point. Then it is immediate that

1. the algebra $\boldsymbol{A}^{-a}$, whose universe is $A - \{a\}$ and the operations are the restrictions to $A - \{a\}$ of the operations of $\boldsymbol{A}$, is a subalgebra of $\boldsymbol{A}$;
2. the matrix $\mathcal{M}^{-a} = \langle \boldsymbol{A}^{-a}, F - \{a\}, \sim\rangle$ is a submatrix of $\mathcal{M}$.

That is, removing an initial point produces a submatrix of the original one. The same can be said about a complete orbit. If we remove a complete orbit (other than the Boolean orbit), we obtain a submatrix of the original one.

An orbit is *unbounded* if it contains a chain that is isomorphic to $\mathbb{Z}$, $\mathbb{N}$ or $\mathbb{N}^-$, where $\mathbb{N}^-$ is $\mathbb{N}$ with reversed order; otherwise the orbit is *bounded*. Notice that a bounded orbit need not be finite. Also, an unbounded orbit may contain a cycle.

Finally, the negation structure of the matrix is described by these connected components in which each point a is labeled by 1 if $a \in F$ and by 0 if $a \notin F$.

7.1 Maximal logics, finite case

Let $\mathcal{M} = \langle \boldsymbol{A}, F, \sim\rangle$ be a finite, reduced LPP-matrix. Then the negation type of each $a \in A$ is finite, and thus, eventually periodic.

For any $a \in A - \{0, 1\}$ define

$$\nu_{a,k}(p) = \begin{cases} (\neg^k p \wedge \neg^k p) & \text{if } \operatorname{nstr}_{\mathcal{M}}(a)(k) = 1, \\ \neg(\neg^k p \wedge \neg^k p) & \text{if } \operatorname{nstr}_{\mathcal{M}}(a)(k) = 0, \end{cases}$$

$$\theta_a(p) = \textstyle\bigwedge_{k=0}^{n-1} \nu_{a,k}(p), \quad \Theta_0(p) = \textstyle\bigwedge_{k=0}^{n-1} ((\neg^k p)^\circ \wedge (\neg^k p)^\bullet),$$

where n is the cardinality of A.

Observe that for any valuation v, since $\mathcal{M}$ is reduced, $(\theta_a)^v = 1$ if and only if $v(p) = a$, and $(\Theta_0)^v = 1$ if and only if $v(p) \in \{0, 1\}$. Thus in this context, the formula θ_a defines the paraconsistent-paracomplete element a and the formula Θ_0 defines the Boolean elements.

We first study necessary conditions for the maximality of the logic $S_{\mathcal{M}}$.

Case I: Assume there exists $a \in A$ such that $a \neq \sim b$ for all $b \in A$. As observed before, these are initial points of the negation function $\sim$. In this case, $\mathcal{M}^{-a} = \langle \boldsymbol{A} - \{a\}, F - \{a\}, \sim\rangle$ is also a reduced (sub)matrix of $\mathcal{M}$, so all $\mathcal{M}$-tautologies are also $\mathcal{M}^{-a}$-tautologies.

Let $\varphi(p) = \neg\theta_a(p)$. Then $\varphi(p)$ is a CPC-tautology but not an $\mathcal{M}$-tautology. Also, $\varphi(p)$ is an $\mathcal{M}^{-a}$-tautology. Then $\mathcal{M}^{-a}$ is a model of $S_{\mathcal{M}}(\varphi)$, so unless $A - \{a\} = \{0, 1\}$, $S(\varphi)$ is not CPC, so $S_{\mathcal{M}}$ is not maximal.

Case II: Assume that $\sim$ has no initial points. Then $\mathcal{M}$ must contain at least one negation cycle

$$\mathcal{O} = \{a, \sim a, \sim^2 a, \ldots, \sim^{n-1} a\},$$

and $\sim^n a = a$ other than the Boolean cycle. Define

$$\Theta_{\mathcal{O}}(p) = \textstyle\bigvee_{a \in \mathcal{O}} \theta_a(p).$$

Since $\mathcal{M}$ is reduced, for any valuation v, $(\Theta_{\mathcal{O}}(p))^v = 1$ if and only if $v(p) \in \mathcal{O}$. In other words, the formula $\Theta_{\mathcal{O}}(p)$ defines the elements of $\mathcal{O}$ in the matrix.

Let $\varphi(p) = \neg\Theta_{\mathcal{O}}(p)$. Then $\varphi(p)$ is a CPC-tautology but not an $\mathcal{M}$-tautology. Then

$$\mathcal{M}^{-\mathcal{O}} = \langle \boldsymbol{A} - \mathcal{O}, F - \mathcal{O}, \sim \rangle$$

is a model of $S_{\mathcal{M}}$, so unless $A - \mathcal{O} = \{0, 1\}$, $S(\varphi)$ is not CPC, so S is not maximal.

Let us now study if the conditions given above are sufficient for the maximality of $S_{\mathcal{M}}$. We begin with the second case. Let $\mathcal{M}$ be a matrix with a single negation cycle $\mathcal{O}$ as in Case II above and let $\varphi(p)$ be any classical tautology but not an $\mathcal{M}$-tautology. Let v be a valuation into $\mathcal{M}$ such that $\varphi^v \notin F$. Obviously this implies that $v(p) \in \mathcal{O}$, so

$$\Theta_{\mathcal{O}}(p) \to \bigvee_{k=0}^{m} \neg\varphi(\neg^k p)$$

is an $\mathcal{M}$-tautology. Since all the subformulas are complex, by Theorem 3.2,

$$\bigwedge_{k=0}^{m} \varphi(\neg^k p) \to \neg\Theta_{\mathcal{O}}(p)$$

is an $\mathcal{M}$-tautology, too.

Now the formula $\bigwedge_{k=0}^{m} \varphi(\neg^k p)$ can be proven in $S_{\mathcal{M}}(\varphi)$, since it is a conjunction of substitution instances of an axiom, so by (MP) the formula $\neg\Theta_{\mathcal{O}}(p)$ can also be proven.

On the other hand, $\neg\Theta_{\mathcal{O}}(p) \to \Theta_0(p)$ is an $\mathcal{M}$-tautology, too, so $\Theta_0(p)$ is a theorem of $S_{\mathcal{M}}(\varphi)$, and thus, any CPC-tautology is a theorem, thus proving the maximality of $S_{\mathcal{M}}$.

Finally, we check if we can reduce the case of a classical tautology but not an $\mathcal{M}$-tautology $\psi(p_1, \ldots, p_n)$ to the previous one, that is, a tautology $\varphi(p)$ with single variable.

Since $\psi(p_1, \ldots, p_n)$ is not an $\mathcal{M}$-tautology, there is a valuation v such that $\psi^v = 0$. We may assume that

$$v(p_1) = a \notin \{0, 1\}.$$

Define $\varphi(p) = \psi(r_1, \ldots, r_n)$, where

$$r_i = \begin{cases} \neg^k p & \text{if } v(p_i) = \sim^k a, \\ (p \to p) & \text{if } v(p_i) = 1, \\ \neg(p \to p) & \text{if } v(p_i) = 0. \end{cases}$$

Then $\varphi(p)$ is a CPC-tautology since it is a substitution instance of the CPC-tautology $\psi(p_1, \ldots, p_n)$. Moreover, $\psi^v = \varphi^v$ and thus, $\varphi(p)$ is not an $\mathcal{M}$-tautology.

The proof for the first case is similar. Simply replace $\Theta_{\mathcal{O}}(p)$ by $\theta_a(p)$. It can also be proven using standard techniques as those in [18].

We have proven the following theorem.

Theorem 7.3 *Let $\mathcal{M}$ be a finite reduced LPP-matrix. Then its associated logic $S_{\mathcal{M}}$ is maximal if and only if $\mathcal{M}$ is a three element matrix or $\mathcal{M}$ has a single (non-Boolean) negation orbit that is a cycle.*

7.2 Maximal logics, infinite case

By a *finite partial negation type* we will understand a finite sequence $\boldsymbol{s} = \langle s_0, s_1, \ldots, s_{n-1} \rangle$ of 0's and 1's. A finite partial negation type is *realized in the matrix* $\mathcal{M}$ if and only if there exists $a \in A$ such that $\sim^k a \in F$ if and only if $s_k = 1$. Letting

$$\nu_k(p) = \begin{cases} (\neg^k p \wedge \neg^k p) & \text{if } s_k = 1, \\ \neg(\neg^k p \wedge \neg^k p) & \text{if } s_k = 0, \end{cases}$$

the formula $\theta_{\boldsymbol{s}}(p) = \bigwedge_{k=0}^{n} \nu_k(p)$ holds for some $a \in A$ if and only if a realizes the finite type $\boldsymbol{s}$.

Observe that the formula $\theta_a(p) = \bigwedge_{k=0}^{n-1} \nu_{a,k}(p)$ defined in the previous subsection corresponds to $\theta_{\boldsymbol{s}}(p)$, where $\boldsymbol{s}$ is the finite partial negation type defined by the first n values of the negation type of a. We will denote this formula with the more descriptive $\theta_a^n(p)$. Even though all matrices are reduced, different elements may realize the same finite partial negation type up to some n.

It is also interesting to note that if the first n values of the negation type of a are not an alternation of 0's and 1's, then $\theta_a^n(p)$ cannot be satisfied by neither 0 nor 1, so $\neg\theta_a^n(p)$ is a CPC-tautology but not an $\mathcal{M}$-tautology.

Let $\mathcal{M} = \langle \boldsymbol{A}, F, \sim\rangle$ be an infinite, reduced LPP-matrix. If $\mathcal{M}$ contains a negation cycle $\mathcal{O}$ of any length, other than the Boolean cycle, then using $\{0,1\} \cup \mathcal{O}$ as a universe and $F \cap (\{0,1\} \cup \mathcal{O})$ as a filter, we can build the submatrix $\mathcal{M}'$. Now arguing as in the previous section, we show that $S_{\mathcal{M}}$ is not maximal, so we will consider only matrices such that the negation structure does not contain a cycle other than the Boolean cycle.

Lemma 7.4 *Let $\mathcal{M}$ be a reduced LPP-matrix such that $S_{\mathcal{M}}$ is maximal. Then for any CPC-tautology $\varphi(p)$ that is not an $\mathcal{M}$-tautology and for any CPC-tautology Θ, there is n depending on Θ such that*

$$\neg\Theta \to \bigvee_{k=0}^{n-1} \neg\varphi(\neg^k p)$$

is an $\mathcal{M}$-tautology.

Proof. Assume $S_{\mathcal{M}}$ is maximal and let $\varphi(p)$ be a CPC-tautology that is not an $\mathcal{M}$-tautology. Then there is a proof from $\mathcal{M}^{\vDash} \cup \{\varphi(\neg^k p) \,:\, k = 0, \dots, n-1\}$ of any CPC-tautology Θ, where $\mathcal{M}^{\vDash}$ is the set of all $\mathcal{M}$-tautologies. This implies that for some n depending on Θ,

$$\mathcal{M}^{\vDash} \vdash \bigwedge_{k=0}^{n-1} \varphi(\neg^k p) \to \Theta.$$

By Theorem 3.2,

$$\mathcal{M}^{\vDash} \vdash \neg\Theta \to \bigvee_{k=0}^{n-1} \neg\varphi(\neg^k p),$$

so this formula is an $\mathcal{M}$-tautology. □

We observe that this implies that any b that does not satisfy the CPC-tautology Θ must satisfy the consequent of this formula, that is to say, for at least one $d \in \{b, \sim b, \dots, \sim^{n-1} b\}$, $\varphi^{\boldsymbol{A}}(d) = 0$.

Theorem 7.5 *Let $\mathcal{M}$ be a reduced LPP-matrix whose negation structure does not contain a non-Boolean cycle. Then $S_{\mathcal{M}}$ is maximal if and only if*

♣ *for any finite partial negation type $\boldsymbol{s} = (s_0, \dots, s_{m-1})$ that is realized in $\mathcal{M}$, there exists n such that for any given $b \in A$, $\boldsymbol{s}$ is realized by at least one $c \in \{b, \sim b, \dots, \sim^{n-1} b\}$.*

Proof. This property states that any finite partial type that is realized by some element in the matrix is realized in every sufficiently long interval in the order induced by the negation. In particular, it guarantees the type is repeated infinitely many times.

We begin by proving the following fact. If $S_{\mathcal{M}}$ is maximal, then the negation structure of $\mathcal{M}$ cannot have arbitrarily long alternations of 1's and 0's. Let $a \in A$ be such that the first m values of its negation type are not an alternation of 0's and 1's, and begins with $(1, 0, \dots)$. This element has to exist since the matrix is reduced and infinite. We apply Lemma 7.4 to the CPC-tautologies $\Theta = p^{\circ}$ and $\varphi(p) = \neg\theta_a^m(p)$, where, as defined above, $\theta_a^m(p)$ is the formula that characterizes the first m values of the negation type of a. Let n be the integer mentioned in the lemma.

Assume there are elements whose negation type begins with an alternation $(1, 0, 1, 0, \dots, 1, 0)$ of arbitrary length. Let c be an element whose negation type begins with such an alternation of length greater than $n + m$. We may also assume that if $c = \sim b$, then $b \in F$, for if that is not the case, choose an element with a longer chain of alternations of 0's and 1's. This means that b does not satisfy Θ and thus, by the remark following the previous lemma, some $d \in \{b, c, \sim c, \dots, \sim^{n-1} c\}$ does not satisfy $\varphi(p)$, or what is the same, the first m values of the negation type of d are the same as those of a, but by our choice of the type of c, this is impossible.

The previous paragraph implies that the number of alternations of 0's and 1's in the negation structure of $\mathcal{M}$ is bounded by some integer K, so the CPC-tautology

$$\Theta_0 = \bigwedge\nolimits_{k=0}^{K}((\neg^k p)^\circ \wedge (\neg^k p)^\bullet)$$

is not satisfied by any $a \notin \{0,1\}$. Intuitively this formula defines the Boolean elements in $\mathcal{M}$.

Now let $\boldsymbol{s}$ be a type realized in $\mathcal{M}$. Then $\neg\theta_{\boldsymbol{s}}(p)$ is a CPC-tautology but not an $\mathcal{M}$-tautology. An application of Lemma 7.4 to $\varphi = \neg\theta_{\boldsymbol{s}}(p)$ and $\Theta = \Theta_0$ yields the result.

In order to prove the implication in the other direction, assume the condition ♣ holds. We begin by checking three of its consequences.

The first one is that the alternations of 0's and 1's in the negation structure must be bounded, for if not, there would be arbitrarily long sequences $\{a, \sim a, \dots, \sim^n a\}$ in which some given negation types are not present. As a consequence, the formula Θ_0 from the first paragraph is satisfied only by 0 and 1, so it characterizes the Boolean elements of $\mathcal{M}$.

Next observe that ♣ is equivalent to

$$\neg\Theta_0 \to \bigvee\nolimits_{k=0}^{n-1} \theta_a^m(\neg^k p) \in \mathcal{M}^{\vDash},$$

for every $a \in A - \{0,1\}$.

Since all formulas involved are complex, this is equivalent to

$$(*) \qquad \bigwedge\nolimits_{k=0}^{n-1} \neg\theta_a^n(\neg^k p) \to \Theta_0 \in \mathcal{M}^{\vDash},$$

for every $a \in A - \{0,1\}$.

Finally, ♣ also implies the following assertion: If a finite negation type $\boldsymbol{s} = (s_0, \dots, s_{n-1})$ is realized in $\mathcal{M}$, then for any $a \notin \{0,1\}$ there exists n such that $\sim^n a$ realizes $\boldsymbol{s}$.

Now let $\psi(p_1, \dots, p_r)$ be a CPC-tautology that is not an $\mathcal{M}$-tautology. Then there are elements $a_1, \dots, a_r \in A$ such that $\psi^{\boldsymbol{A}}(a_1, \dots, a_r) = 0$. Since the evaluation of a formula depends only on the negation structure of the elements assigned to its variables, by the assertion of the last paragraph, we may assume that all these elements belong to a single negation chain, that is, for each $i = 1, \dots, r$, $a_i = \sim^{k_i} a$, for some $a \notin \{0,1\}$.

Let $\varphi(p) = \psi(\neg^{k_1} p, \dots, \neg^{k_r} p)$. Then φ is a CPC-tautology, since it is a substitution instance of a CPC-tautology, but it is not in $\mathcal{M}^{\vDash}$, since it fails for a. On the other hand, since the evaluation of φ depends only on the negation type of a up to the number of negations that appear in φ, for sufficiently big n, $\theta_a^n(p) \to \neg\varphi \in \mathcal{M}^{\vDash}$. This implies that $S_{\mathcal{M}}(\psi) \vdash \neg\theta_a^n(p)$, and thus $S_{\mathcal{M}}(\psi) \vdash \neg\theta_a^n(\neg^m p)$, for any m. So $S_{\mathcal{M}}(\psi) \vdash \bigwedge_{k=0}^{n-1} \neg\theta_a^n(\neg^k p)$, and by $(*)$, $S_{\mathcal{M}}(\psi) \vdash \Theta_0$. ☐

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