MLQ

MATHEMATICAL LOGIC QUARTERLY

A Journal for Mathematical Logic, Foundations of Mathematics, and Logical Aspects of Theoretical Computer Science

www.mlq-journal.org

Algebraization of logics defined by literal-paraconsistent or literal-paracomplete matrices

Eduardo Hirsh and **Renato A. Lewin**

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306 – Correo 22, Santiago, Chile

Received 29 January 2007, revised 26 April 2007, accepted 4 May 2007
Published online 7 March 2008

Key words Algebraizable logic, matrix semantics, paraconsistency, paracompleteness.
MSC (2000) 03G99, 03G25, 03B53

Math. Log. Quart. **54**, No. 2, 153 – 166 (2008) / **DOI** 10.1002/malq.200710021

WILEY

REPRINT

Math. Log. Quart. **54**, No. 2, 153 – 166 (2008) / **DOI** 10.1002/malq.200710021

Algebraization of logics defined by literal-paraconsistent or literal-paracomplete matrices

Eduardo Hirsh* and **Renato A. Lewin****

Facultad de Matemáticas, Pontificia Universidad Católica de Chile,
Casilla 306 – Correo 22, Santiago, Chile

Received 29 January 2007, revised 26 April 2007, accepted 4 May 2007
Published online 7 March 2008

Key words Algebraizable logic, matrix semantics, paraconsistency, paracompleteness.
MSC (2000) 03G99, 03G25, 03B53

We study the algebraizability of the logics constructed using literal-paraconsistent and literal-paracomplete matrices described by Lewin and Mikenberg in [11], proving that they are all algebraizable in the sense of Blok and Pigozzi in [3] but not finitely algebraizable. A characterization of the finitely algebraizable logics defined by LPP-matrices is given.

We also make an algebraic study of the equivalent algebraic semantics of the logics associated to the matrices $\mathcal{M}^3_{2,2}$, $\mathcal{M}^3_{2,1}$, $\mathcal{M}^3_{1,1}$, $\mathcal{M}^3_{1,3}$, and $\mathcal{M}^4$ appearing in [11] proving that they are not varieties and finding the free algebra over one generator.

1 Introduction and preliminaries

Literal paraconsistent-paracomplete matrices, or LPP-matrices, were introduced in [11] in an attempt to give a meaningful semantics, via the classic method of matrices, to a large family of paraconsistent and paracomplete logics. The main characteristic of these logics is that paraconsistency and/or paracompleteness occurs only in the most elementary levels, that of propositional letters and their (iterated) negations, called *literals*, but more complex formulas behave classically with respect to negations. The idea behind these matrix semantics arose in [13] and [4]. Needless to say many of these logics have appeared in the literature under different names and formalizations.

In Section 2 we give a summary about LPP-matrices and the logics defined by classes of LPP-matrices.

In Section 3 we prove that any logic which is defined by a set of these matrices is algebraizable in the sense of Czelakowski [6] and Herrmann [8, 9] but that, in general, they are not finitely algebraizable in the sense of Blok and Pigozzi in [2] and [3]. We then give a characterization of the logics that are finitely algebraizable (or algebraizable in the latter sense). For a thorough study of the differences and relations between these two approaches, the reader is advised to consult [6], where broad references to these topics can also be found.

Finally, in Section 4, we study the classes of algebras that arise in the process of algebraization of logics defined by single matrices $\mathcal{M}^3_{2,2}$, $\mathcal{M}^3_{2,1}$, $\mathcal{M}^3_{1,1}$, $\mathcal{M}^3_{1,3}$, and $\mathcal{M}^4$ appearing in [11]. All these are well known, most notably $\mathcal{M}^3_{2,2}$ defines Sette's logic P^1. The algebraic counterpart of this logic was studied in [12] as P^1-algebras and later in [16] under the name of Sette algebras. In this section we will carry out a study similar to that of [12] for the logics defined by the other four matrices.

Except for the proofs of the background on LPP-matrices, the paper is self contained. Nevertheless some concepts, like the negation structure of a matrix, are not explained in detail here. The reader will find them in [11].

* e-mail: ehirsh@mat.puc.cl
** Corresponding author: e-mail: rlewin@mat.puc.cl

1.1 Matrix logics

An $\mathcal{L}$-*matrix* is a pair $\mathcal{A} = \langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and F is a subset of the universe A of $\boldsymbol{A}$, the elements of F are called *the designated elements of* $\mathcal{A}$. A *valuation into a matrix* $\mathcal{A}$ is a function $v : \mathrm{Var} \longrightarrow A$. A valuation v can be extended recursively to $\bar{v} : \mathcal{F}m \longrightarrow A$ in the usual way. We will also write α^v for $\bar{v}(\alpha)$.

Given a matrix $\mathcal{A}$, we define the relation $\vDash_{\mathcal{A}}$ between a set Γ of formulas and a formula φ as follows: $\Gamma \vDash_{\mathcal{A}} \varphi$ if and only if for any valuation v, if $\psi^v \in F$ for all $\psi \in \Gamma$, then $\varphi^v \in F$.

For a class $\mathbb{M}$ of matrices we define the relation $\vDash_{\mathbb{M}}$ as follows: $\Gamma \vDash_{\mathbb{M}} \varphi$ if and only if $\Gamma \vDash_{\mathcal{A}} \varphi$ for all $\mathcal{A} \in \mathbb{M}$.

Given a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ in the usual sense (see for instance [2]), a matrix $\mathcal{A}$ is a *matrix model of* $\mathcal{S}$ if $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Gamma \vDash_{\mathcal{A}} \varphi$. Then F is called an $\mathcal{S}$-*filter*. We observe that if T is an $\mathcal{S}$-theory, then $\langle \mathcal{F}m, T\rangle$ is a matrix model of $\mathcal{S}$. These matrices are called *Lindenbaum matrices for* $\mathcal{S}$, but these are not the kind of matrices that we will study here.

Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ be a deductive system. Then a class of matrices $\mathbb{M}$ is said to be a *matrix semantics of* $\mathcal{S}$ if for all $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}m$, $\Gamma \vdash_{\mathcal{S}} \varphi$ if and only if $\Gamma \vDash_{\mathbb{M}} \varphi$.

On the other hand, it is easy to check that for any class $\mathbb{M}$ of matrices, the relation $\vDash_{\mathbb{M}}$ given above defines a consequence relation. It is also immediate that if $\mathbb{M} \subseteq \mathbb{L}$, then $\vDash_{\mathbb{L}} \subseteq \vDash_{\mathbb{M}}$.

For more on this topic, consult [6].

1.2 The Leibniz congruence and reduced matrices

We recall an important concept in algebraic logic.

Definition 1.1 [2, Definition 1.4] Let $\boldsymbol{A}$ be an algebra and $F \subseteq A$. We define the binary relation on A:

$$\boldsymbol{\Omega}_{\boldsymbol{A}} F = \{\langle a, b\rangle \,:\, \varphi^{\boldsymbol{A}}(a, \bar{c}) \in F \Leftrightarrow \varphi^{\boldsymbol{A}}(b, \bar{c}) \in F, \text{ for all } \varphi(p, q_1, \dots, q_n) \in \mathcal{F}m \text{ and all } \bar{c} \in A^n\},$$

where $\varphi^{\boldsymbol{A}}$ is the interpretation of the formula $\varphi(p, q_1, \dots, q_n)$ in $\boldsymbol{A}$ replacing the propositional letters $p, q_1, \dots, q_n$ by $a, c_1, \dots, c_n$, as usual.

A congruence Θ on $\boldsymbol{A}$ is called *compatible with the subset* F *of* A if for all $a, b \in A$, if $a \in F$ and $\langle a, b\rangle \in \Theta$, then $b \in F$.

Theorem 1.2 [2, Theorem 1.5] *Given an algebra* $\boldsymbol{A}$ *and any* $F \subseteq A$, $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ *is the largest congruence on* $\boldsymbol{A}$ *compatible with* F.

The congruence $\boldsymbol{\Omega}_{\boldsymbol{A}} F$ is *the Leibniz relation on* $\boldsymbol{A}$ *over* F. The corresponding operator on the power set of A, denoted $\boldsymbol{\Omega}_{\boldsymbol{A}}$, is called *the Leibniz operator on* A. If $\boldsymbol{A}$ is the formula algebra of $\mathcal{F}m$, the Leibniz operator is simply denoted $\boldsymbol{\Omega}$.

The following lemma is a slight generalization given in [13] of [2, Lemma 1.6] and is usually the easiest way of characterizing $\boldsymbol{\Omega}$.

Lemma 1.3 *Let* $\boldsymbol{A}$ *be an algebra and* $F \subseteq A$. *Let* Θ *be a binary relation on* A *that is definable by a set of formulas over the matrix* $\langle \boldsymbol{A}, F\rangle$ *with parameters and without equality.*

(i) *If* Θ *is reflexive, then* $\boldsymbol{\Omega}_{\boldsymbol{A}} F \subseteq \Theta$.

(ii) *If, in addition,* Θ *is a congruence on* $\boldsymbol{A}$ *that is compatible with* F, *then* $\boldsymbol{\Omega}_{\boldsymbol{A}} F = \Theta$.

We say that the matrix $\mathcal{M}$ is a *reduced matrix* if $\boldsymbol{\Omega}_{\boldsymbol{A}} F = \mathrm{Id}$. It is easy to see that

$$\mathcal{M}|_{\boldsymbol{\Omega}_{\boldsymbol{A}} F} = \langle \boldsymbol{A}|_{\boldsymbol{\Omega}_{\boldsymbol{A}} F}, F|_{\boldsymbol{\Omega}_{\boldsymbol{A}} F}, \sim\rangle$$

is a reduced matrix.

It is a well known fact (see for instance [6]) that $\mathcal{M}$ and $\mathcal{M}|_{\boldsymbol{\Omega}_{\boldsymbol{A}} F}$ define the same deductive system, so if the matrices $\mathcal{M}$ and $\mathcal{M}'$ are such that both contain elements with the same negation types, they have the same reduction, so they give rise to the same deductive system. It is enough then to study reduced matrices.

2 The logics

All concepts in this section were introduced in [11]. The reader is referred to that paper for details and for the proofs of all theorems mentioned here. A more comprehensive bibliography can be consulted there.

2.1 Language

Let $\mathcal{F}m$ be the set of formulas built in the usual recursive way from a denumerable set $\text{Var} = \{p_1, p_2, \dots\}$ of propositional variables and the connectives, $\wedge$, $\vee$, $\rightarrow$, and $\neg$.

The literals of $\mathcal{F}m$ is the set of all formulas of the form $\neg^k p$, where $\neg^0 p = p$ and $\neg^{k+1} p = \neg(\neg^k p)$, $p \in \text{Var}$. Formulas that contain a binary connective will be called *complex*.

We will use letters p, q, r, etc., as metalinguistic variables for propositional variables, roman capital letters P, Q, R, etc., as variables for complex formulas, and Greek letters α, β, γ, etc., as variables for general formulas.

2.2 Literal-paraconsistent-paracomplete matrices

Let A be a set such that $\{0,1\} \subseteq A$, F a subset of A such that $1 \in F$ and $0 \notin F$, and $\sim : P \longrightarrow P$ a function such that $\sim 1 = 0$ and $\sim 0 = 1$. We define *the literal-paraconsistent-paracomplete matrix*, or *LPP-matrix*, $\langle \boldsymbol{A}, F, \sim \rangle$ with the following operations:

$$a \vee b = \begin{cases} 1 & \text{if } a \in F \text{ or } b \in F, \\ 0 & \text{otherwise,} \end{cases}$$

$$a \wedge b = \begin{cases} 1 & \text{if } a \in F \text{ and } b \in F, \\ 0 & \text{otherwise,} \end{cases}$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \notin F \text{ or } b \in F, \\ 0 & \text{otherwise.} \end{cases}$$

We observe that the LPP-matrix $\langle \boldsymbol{A}, F, \sim \rangle$ is just a matrix in the usual sense of the previous paragraph, where the algebra over the universe A has three binary operations defined as above and one unary operation $\sim$, which we want to distinguish because of its relevance in the definition. Observe, too, that the name "literal-paraconsistent-paracomplete" is slightly misleading since not all these matrices are both paraconsistent and paracomplete. An LPP-matrix is *paraconsistent* if for some $a \in A$ both $a \in F$ and $\sim a \in F$; it is *paracomplete* if both $a \notin F$ and $\sim a \notin F$. As a matter of fact, the matrix $\langle \{1,0\}, \{1\}, \sim \rangle$, where $\sim 0 = 1$ and $\sim 1 = 0$, defines classical logic and is neither paraconsistent nor paracomplete. As an example, the matrices $I^n P^k$ introduced in [7] are LPP-matrices.

2.3 Axiomatization of LPPL

The following is a sound and complete deductive system for the logic defined by the class of all LPP-matrices $\langle \boldsymbol{A}, F, \sim \rangle$, with no conditions on A, F, or $\sim$. This system will be called *literal-paraconsistent-paracomplete logic*, or LPPL.

Define the deductive system LPPL with Modus Ponens (MP) as its only rule and the following axioms:

(A1) $\alpha \rightarrow (\beta \rightarrow \alpha)$.

(A2) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$.

(A3) $(\alpha \wedge \beta) \rightarrow \alpha$.

(A4) $(\alpha \wedge \beta) \rightarrow \beta$.

(A5) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \wedge \gamma)))$.

(A6) $\alpha \rightarrow (\alpha \vee \beta)$.

(A7) $\beta \rightarrow (\alpha \vee \beta)$.

(A8) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$.

(A9) Axiom for negation: $(\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P)$, where P and Q are complex formulas.

We observe that this system was proposed by Puga and da Costa in [15] as an axiomatization of an imaginary logic of Vasiliev. The matrix proposed in that paper (denoted $\boldsymbol{M}$) is not an LPP-matrix.

Theorem 2.1 (Deduction theorem) *The following holds in* LPPL*: If* $\Gamma, \varphi \vdash_{\mathsf{LPPL}} \psi$*, then* $\Gamma \vdash_{\mathsf{LPPL}} \varphi \rightarrow \psi$.

Theorem 2.2 *Let $\varphi(p_1, p_2, \ldots, p_n)$ be a classical tautology. Then*

1. $\vdash_{\mathsf{LPPL}} \varphi(A_1, A_2, \ldots, A_n)$*, for complex formulas* $A_1, A_2, \ldots, A_n$*;*

2. *if φ does not contain negations, then* $\vdash_{\mathsf{LPPL}} \varphi(\alpha_1, \alpha_2, \ldots, \alpha_n)$*, for formulas* $\alpha_1, \alpha_2, \ldots, \alpha_n$ (*which may include negations*).

2.4 Completeness

Definition 2.3

1. A set Σ of formulas is *non-trivial* if there exists a formula that is not deducible from Σ.

2. A set of formulas Σ is *satisfiable* if there exists a valuation v into an LPP-matrix $(P, F, \sim)$ such that $\sigma^v \in F$ for all $\sigma \in \Sigma$.

Theorem 2.4 *If Σ is a non-trivial set of formulas, then it is satisfiable.*

Theorem 2.5 *If for any LPP-matrix $\mathcal{M}$, $\Gamma \vDash_{\mathcal{M}} \varphi$, then $\Gamma \vdash_{\mathsf{LPPL}} \varphi$.*

2.5 Negation structures and the Leibniz congruence

We will characterize the Leibniz operator in the context of LPP-matrices.

Let $\mathcal{M} = \langle \boldsymbol{A}, F, \sim \rangle$ be an LPP-matrix. *The negation structure of $\mathcal{M}$* is a function

$$\mathrm{nstr}_{\mathcal{M}} : A \longrightarrow \{0,1\}^{\mathbb{N}}$$

such that $\mathrm{nstr}_{\mathcal{M}}(a)(k) = 1$ if and only if $\sim^k a \in F$.

The negation type of $a \in A$ is the sequence $\mathrm{nstr}_{\mathcal{M}}(a)$. If $\mathcal{M}$ is finite, then each negation type is eventually periodic, one can think of it as a finite sequence of 0's and 1's. For more details about these concepts see [11], particularly Section 7.

Theorem 2.6 $\langle a, b \rangle \in \boldsymbol{\Omega_A} F$ *if and only if* $\mathrm{nstr}_{\mathcal{M}}(a) = \mathrm{nstr}_{\mathcal{M}}(b)$.

Remark 2.7 Notice that in a reduced LPP-matrix there is a single element for each negation type present in $\mathcal{M} = \langle \boldsymbol{A}, F, \sim \rangle$.

3 Algebraization

In this section we will study the algebraizability of the deductive system LPPL and various other deductive systems defined by sets of LPP-matrices. The theoretical framework is that of the extensions of Blok and Pigozzi's theory, presented in [2] and later in [3], to broader classes of systems developed by, among others, J. Czelakowski in [6] and B. Herrmann in [8, 9].

Theorem 3.1 *The deductive system* LPPL *is algebraizable.*

Proof. For formulas φ, ψ let $\delta(\varphi) \approx \varepsilon(\varphi)$, where $\delta(\varphi) = \varphi \wedge \varphi$ and $\varepsilon(\varphi) = \varphi \rightarrow \varphi$, be the defining equation and let

$$\varphi \,\Delta_0\, \psi = \varphi \leftrightarrow \psi, \qquad \varphi \,\Delta_k\, \psi = \neg^k \psi \leftrightarrow \neg^k \varphi, \quad \text{for } k \geq 1.$$

We must show that δ, ε, and Δ satisfy conditions of [2, Theorem 4.7], but without the finiteness of Δ as suggested in [9].

1. $\vdash_{\mathsf{LPPL}} \varphi \,\Delta\, \varphi$ holds by Theorem 2.2.
2. $\varphi \,\Delta\, \psi \vdash_{\mathsf{LPPL}} \psi \,\Delta\, \varphi$ holds by the definition of Δ.
3. $\varphi \,\Delta\, \psi, \psi \,\Delta\, \xi \vdash_{\mathsf{LPPL}} \varphi \,\Delta\, \xi$ holds by Theorem 2.2.
4. We must prove:
   (a) $\varphi \,\Delta\, \psi \vdash_{\mathsf{LPPL}} \neg\varphi \,\Delta\, \neg\psi$. This holds by the definition of Δ.
   (b) $\varphi_1 \,\Delta\, \psi_1, \varphi_2 \,\Delta\, \psi_2 \vdash_{\mathsf{LPPL}} (\varphi_1 * \varphi_2) \,\Delta (\psi_1 * \psi_2)$, where $*$ is any binary connective.
      i. $\varphi_1 \,\Delta_0\, \psi_1, \varphi_2 \,\Delta_0\, \psi_2 \vdash_{\mathsf{LPPL}} (\varphi_1 * \varphi_2) \,\Delta_0 (\psi_1 * \psi_2)$ holds by Theorem 2.2 since these are tautological consequences that do not involve negations.

ii. Also, since $(\varphi_1 * \varphi_2)$ and $(\psi_1 * \psi_2)$ are complex formulas, by Theorem 2.2, negations behave classically in LPPL, so we get $(\varphi_1 * \varphi_2)\, \Delta_0(\psi_1 * \psi_2) \vdash_{\mathsf{LPPL}} (\varphi_1 * \varphi_2)\, \Delta_k(\psi_1 * \psi_2)$, for $k \geq 1$.

5. We must prove $\varphi \dashv\vdash_{\mathsf{LPPL}} \delta(\varphi)\, \Delta\, \varepsilon(\varphi)$, i. e. $\varphi \dashv\vdash_{\mathsf{LPPL}} (\varphi \wedge \varphi)\, \Delta(\varphi \to \varphi)$. We first prove that this holds for Δ_0.

From right to left, $(\varphi \wedge \varphi)\, \Delta_0(\varphi \to \varphi) \vdash_{\mathsf{LPPL}} \varphi$ is obvious since $\varphi \to \varphi$ is a theorem of LPPL, so by MP we get $(\varphi \wedge \varphi)$. Also, $((\varphi \wedge \varphi) \to \varphi)$ is an axiom of LPPL, so by MP we get φ.

From left to right we must prove $\varphi \vdash_{\mathsf{LPPL}} (\varphi \wedge \varphi)\, \Delta_0(\varphi \to \varphi)$.

Since $(\varphi \to \varphi)$ is a theorem, $\varphi \vdash_{\mathsf{LPPL}} (\varphi \wedge \varphi) \to (\varphi \to \varphi)$ holds by axiom (A1) and MP.

On the other hand, $\varphi \vdash_{\mathsf{LPPL}} \varphi \wedge \varphi$ by (A5) and MP, so by (A1) and MP, $\varphi \vdash_{\mathsf{LPPL}} (\varphi \to \varphi) \to (\varphi \wedge \varphi)$.

Finally, for $k \geq 1$, $(\varphi \wedge \varphi)\, \Delta_0(\varphi \to \varphi) \vdash_{\mathsf{LPPL}} (\varphi \wedge \varphi)\, \Delta_k(\varphi \to \varphi)$, as both $(\varphi \wedge \varphi)$ and $(\varphi \to \varphi)$ are complex formulas, and thus $\varphi \vdash_{\mathsf{LPPL}} (\varphi \wedge \varphi)\, \Delta_k(\varphi \to \varphi)$. □

Corollary 3.2 *Let $\mathbb{M}$ be a set of LPP-matrices and let $\mathcal{S}_{\mathbb{M}}$ be the deductive system defined by $\mathbb{M}$. Then $\mathcal{S}_{\mathbb{M}}$ is algebraizable.*

Proof. If $\Sigma \vdash_{\mathsf{LPPL}} \varphi$, then $\Sigma \vdash_{\mathbb{M}} \varphi$, so the same defining equations and equivalence formulas used for LPPL will work for $\mathcal{S}_{\mathbb{M}}$. □

The previous theorem and corollary prove that all deductive systems defined by a class of matrices are algebraizable but we do not know if they are finitely algebraizable. The next theorem gives us necessary and sufficient conditions for this to happen. We need two lemmas first.

Lemma 3.3 *Let T be a theory in LPPL and define $\Theta = \{(\varphi, \psi) \,:\, \varphi\, \Delta\, \psi \subseteq T\}$. Then $\mathbf{\Omega} T = \Theta$.*

Proof. By the proof of Theorem 3.1, Θ is a congruence in the formula algebra.

It is obvious that Θ is compatible with T, for if $\varphi \in T$ and $(\varphi, \psi) \in \Theta$, then $\varphi\, \Delta\, \psi \subseteq T$, so in particular, $\varphi \leftrightarrow \psi \in T$ and by Modus Ponens, $\psi \in T$.

Finally, since Θ is reflexive and defined by a set of formulas, by Lemma 1.3, $\mathbf{\Omega} T = \Theta$. □

Lemma 3.4 *Let $\mathcal{M}$ be a reduced LPP-matrix with an unbounded negation structure. Then*

$$\{\neg^k p \leftrightarrow \neg^k q \,:\, k < n\} \nvdash_{\mathcal{M}} (\neg^n p \leftrightarrow \neg^n q).$$

Proof. Let $\mathcal{M}$ be a reduced LPP-matrix with an unbounded negation structure, that is, for every $m \in \mathbb{N}$ there is an element $a \in A$ such that the non-periodic part of $\mathrm{nstr}_{\mathcal{M}}(a)$, the negation type of a, is longer than m. Intuitively this means that the chain of negations $a, \sim a, \sim\sim a, \ldots, \sim^k a, \ldots$ has length greater than m. Observe that this does not mean that a single element has a negation type of infinite length, but certainly it includes that case.

Recall that a *finite partial negation type* is a sequence $\boldsymbol{t} = (t_0, \ldots, t_{n-1})$ of 0's and 1's (see [11]). A finite partial negation type is *realized in the matrix* $\mathcal{M}$ if and only if there exists $a \in A$ such that $\sim^k a \in F$ if and only if $t_k = 1$. The number of finite partial negation types of length n is 2^n.

In a chain $a, \sim a, \sim\sim a, \ldots, \sim^{m-1} a$, with $n < m$, there are $m - n$ partial negation types of length n that are realized, namely, those of $a, \sim a, \sim\sim a, \ldots, \sim^{m-n-1} a$. So if $m - n > 2^n$, some partial negation type of length n will be realized at least twice in the chain $a, \sim a, \sim\sim a, \ldots, \sim^{n-1} a$. This means that there are two elements b and $c \in A$ such that $\mathrm{nstr}_{\mathcal{M}}(b)(k) = t_k = \mathrm{nstr}_{\mathcal{M}}(c)(k)$, for $0 \leq k < n$. Moreover, we can take the first pair of elements b and c such that $\mathrm{nstr}_{\mathcal{M}}(b)(n) \neq \mathrm{nstr}_{\mathcal{M}}(c)(n)$, for if these do not exist, b and c would have the same negation types contradicting the fact that $\mathcal{M}$ is reduced.

All this implies that $\{\neg^k p \leftrightarrow \neg^k q \,:\, k < n\}$ is satisfied in $\mathcal{M}$ interpreting p and q as b and c, respectively. On the other hand, it implies that $\neg^n p \leftrightarrow \neg^n q$ is not satisfied by b and c, so

$$\{\neg^k p \leftrightarrow \neg^k q \,:\, k < n\} \nvdash_{\mathcal{M}} (\neg^n p \leftrightarrow \neg^n q).$$ □

Remark 3.5 Observe that for a fixed n the previous lemma remains true if instead of "unbounded" we ask that the negation structure of $\mathcal{M}$ contains a chain of length greater than $2^n + n$.

The following theorem gives necessary conditions for a logic defined by a class of LPP-matrices to be finitely algebraizable.

Theorem 3.6 *Let $\mathbb{M}$ be a set of LPP-matrices and let $\mathcal{S}_{\mathbb{M}}$ be the deductive system defined by $\mathbb{M}$. Then $\mathcal{S}_{\mathbb{M}}$ is finitely algebraizable if and only if the class $\mathbb{M}^*$ of all reduced matrices in $\mathbb{M}$ is a finite set of finite LPP-matrices.*

Proof. We begin by observing that the condition on the set $\mathbb{M}^*$ of reduced LPP-matrices is equivalent to the existence of an N such that for every $\mathcal{M} \in \mathbb{M}$ and every $a \in A$ the negation type of a is eventually periodic and the length T of the longest chain and the length P of the largest period are such that $T + P < N$.

To prove the theorem from right to left, observe that if N is a bound for all negation structures of the matrices in $\mathbb{M}$ and $n \geq N$, then

$$\{\neg^k\varphi \leftrightarrow \neg^k\psi \,:\, k \leq N\} \vdash_{\mathbb{M}} (\neg^m\varphi \leftrightarrow \neg^m\psi),$$

so $\varphi\,\Delta_0\,\psi, \varphi\,\Delta_1\,\psi, \ldots, \varphi\,\Delta_N\,\psi$ generates all equivalence formulas in Theorem 3.1, so $\mathcal{S}_{\mathbb{M}}$ is finitely algebraizable.

Assume now that there is no bound to the negation structures of the matrices in $\mathbb{M}$. Let $n \in \mathbb{N}$. Then there is a reduced matrix $\mathcal{M} \in \mathbb{M}$ whose negation structure is as large as we need. In our case, it should be large enough so that Lemma 3.4 holds.

For each $n \in \mathbb{N}$ let $T_n = \mathrm{Cn}\{\neg^k p \leftrightarrow \neg^k q \,:\, k < n\}$, and let $T = \bigcup_{n\in\mathbb{N}} T_n$. It is immediate that $p\,\Delta\,q \subseteq T$, and thus by Lemma 3.3, $(p, q) \in \mathbf{\Omega}T$.

Assume on the other hand that $(p, q) \in \bigcup_{n\in\mathbb{N}} \mathbf{\Omega}T_n$. Then for some $m \in \mathbb{N}$, $(p, q) \in \mathbf{\Omega}T_m$, or what is the same, $p\,\Delta\,q \subseteq T_m$. This is clearly impossible by Lemma 3.4.

Since $\mathbf{\Omega}$ does not preserve directed sets of theories, by [2, Theorem 4.2], $\mathcal{S}_{\mathbb{M}}$ is not finitely algebraizable. □

Corollary 3.7 LPPL *is not finitely algebraizable*

Corollary 3.8 *There is an LPP-matrix $\mathcal{M}$ such that the associated deductive system $\mathcal{S}_{\mathcal{M}}$ is not finitely algebraizable.*

Proof. Let $\mathcal{M}$ be a reduced matrix with a single infinite non-periodic, non-Boolean negation orbit such that $\mathcal{S}_{\mathcal{M}}$ is maximal. By [11, Theorem 26], any finite negation type $\boldsymbol{s} = (s_1, \ldots, s_n)$ that is realized in $\mathcal{M}$ is realized infinitely many times. Nevertheless, there must exist two elements a and b that realize this type but they differ in the next value of the type, otherwise these elements would have the same negation type and the matrix would not be reduced. All this implies that $\{\neg^k p \leftrightarrow \neg^k q \,:\, k < n\} \nvdash_{\mathcal{M}} (\neg^n p \leftrightarrow \neg^n q)$.

The proof now proceeds as that of Theorem 3.6 using this result instead of Lemma 3.4. □

4 The systems $\mathcal{S}_{1,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{2,2}$, and $\mathcal{S}^4$

In [11], the authors study carefully four systems defined by single matrices which are minimal in some sense. Most of them have appeared in the literature under various names and formalizations.

In this section we will present the deductive systems $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,2}$, and $\mathcal{S}^4$ with their axiomatization and the corresponding (weak) completeness theorems. The algebraizability of these systems was proven directly in [10].

4.1 Three element matrices

There are three possible functions $\sim$, namely, either $\sim_1 \frac{1}{2} = \frac{1}{2}$ or $\sim_2 \frac{1}{2} = 1$ or $\sim_3 \frac{1}{2} = 0$. Likewise, there are two possible filters, namely, $F_1 = \{1\}$ and $F_2 = \{1, \frac{1}{2}\}$. Of the six combinations only four of them are reduced:

$$\mathcal{M}^3_{1,1} = \langle\{0, \tfrac{1}{2}, 1\}, F_1, \sim_1\rangle, \quad \mathcal{M}^3_{1,3} = \langle\{0, \tfrac{1}{2}, 1\}, F_1, \sim_3\rangle,$$
$$\mathcal{M}^3_{2,1} = \langle\{0, \tfrac{1}{2}, 1\}, F_2, \sim_1\rangle, \quad \mathcal{M}^3_{2,2} = \langle\{0, \tfrac{1}{2}, 1\}, F_2, \sim_2\rangle.$$

The first two matrices are the smallest LPP-matrices that are paracomplete (but not paraconsistent); the second two are the smallest LPP-matrices that are paraconsistent (but not paracomplete.)

4.1.1 $\mathcal{M}^3_{1,1} = \langle\{0, \frac{1}{2}, 1\}, F_1, \sim_1\rangle$

The following is an axiomatization for the logic $\mathcal{S}_{1,1}$ defined by this matrix. This system appears in [14] under the name I^1_2.

We let $\alpha^\bullet = \alpha \vee \neg\alpha$.

($A_{1,1}$.1) The axioms of LPPL.

($A_{1,1}$.2) $\alpha^\bullet \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$.

($A_{1,1}$.3) $\beta \leftrightarrow \neg\neg\beta$.

Modus Ponens is the only rule of inference.

Theorem 4.1 [11, Theorem 10] *Let α be an $\mathcal{M}^3_{1,1}$-tautology. Then in $\mathcal{S}_{1,1}$, $\vdash \alpha$.*

4.1.2 $\mathcal{M}^3_{1,3} = \langle\{0, \frac{1}{2}, 1\}, F_1, \sim_3\rangle$

The following is an axiomatization for the logic $\mathcal{S}_{1,3}$ defined by this matrix. This system appears as I^1 in [7] and in [14]. It was introduced in [18].

($A_{1,3}$.1) The axioms of LPPL.

($A_{1,3}$.2) $\alpha^\bullet \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$.

($A_{1,3}$.3) $(\neg\alpha)^\bullet$.

Modus Ponens is the only rule of inference.

Theorem 4.2 [11, Theorem 13] *Let α be an $\mathcal{M}^3_{1,3}$-tautology. Then in $\mathcal{S}_{1,3}$, $\vdash \alpha$.*

4.1.3 $\mathcal{M}^3_{2,1} = \langle\{0, \frac{1}{2}, 1\}, F_2, \sim_1\rangle$

The following is an axiomatization for the logic $\mathcal{S}_{2,1}$ defined by this matrix. This system appears in [5] and [14] under the name P^1_2.

We let $\alpha^\circ = \neg(\alpha \wedge \neg\alpha)$.

($A_{2,1}$.1) The axioms of LPPL.

($A_{2,1}$.2) $\beta^\circ \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$.

($A_{2,1}$.3) $\alpha \leftrightarrow \neg\neg\alpha$.

Modus Ponens is the only rule of inference.

Theorem 4.3 [11, Theorem 16] *Let α be an $\mathcal{M}^3_{2,1}$-tautology. Then in $\mathcal{S}_{2,1}$, $\vdash \alpha$.*

4.1.4 $\mathcal{M}^3_{2,2} = \langle\{0, \frac{1}{2}, 1\}, F_2, \sim_2\rangle$

We will give an axiomatization for the system $\mathcal{S}_{2,2}$ that is appropriate in our context. This system is Sette's logic P^1, see [17]. It appears in [5] as system P^1_1 and in [7] as P^1. See also [12] and [16].

($A_{2,2}$.1) The axioms of LPPL.

($A_{2,2}$.2) $\beta^\circ \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$.

($A_{2,2}$.3) $(\neg\alpha)^\circ$.

Modus Ponens is the only rule of inference.

Theorem 4.4 [11, Theorem 19] *Let α be an $\mathcal{M}^3_{2,2}$-tautology. Then in $\mathcal{S}_{2,2}$, $\vdash \alpha$.*

4.2 A four element matrix

The matrix $\mathcal{M}^4 = \langle\{0, \perp, \top, 1\}, F, \sim\rangle$, where $\neg\top = \top$, $\neg\perp = \perp$, and F denotes the filter $\{1, \top\}$, is the smallest LPP-matrix that is both paraconsistent and paracomplete.

We define the system $\mathcal{S}^4$ as follows.

(A_4.1) The axioms of LPPL.

(A_4.2) $(\alpha^\bullet \wedge \beta^\circ) \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$.

(A_4.3) $\alpha \leftrightarrow \neg\neg\alpha$.

Modus Ponens is the only rule of inference.

Theorem 4.5 [11, Theorem 22] *Let α be an $\mathcal{M}^4$-tautology. Then in $\mathcal{S}^4$, $\vdash \alpha$.*

5 Algebraic properties

In this section we study two properties of the equivalent quasi-variety semantics of the deductive systems $\mathcal{S}_{1,1}$, $\mathcal{S}_{2,1}$, $\mathcal{S}_{1,3}$, $\mathcal{S}_{2,2}$, and $\mathcal{S}_4$.

From now on we will use $\boldsymbol{M}^3_{1,1}$, $\boldsymbol{M}^3_{2,1}$, $\boldsymbol{M}^3_{1,3}$, $\boldsymbol{M}^3_{2,2}$, and $\boldsymbol{M}^4$ for the algebraic reducts of the matrices $\mathcal{M}^3_{1,1}$, $\mathcal{M}^3_{2,1}$, $\mathcal{M}^3_{1,3}$, $\mathcal{M}^3_{2,2}$, and $\mathcal{M}^4$, respectively.

The equivalent quasi-variety of each system is generated by the algebraic reduct of the corresponding LPP-matrix. An algebra in the quasi-variety generated by $\boldsymbol{M}$ will be called an $\boldsymbol{M}$*-algebra*.

If $\boldsymbol{M}$ is $\boldsymbol{M}^3_{1,1}$, $\boldsymbol{M}^3_{2,1}$, $\boldsymbol{M}^3_{1,3}$, or $\boldsymbol{M}^4$, the quasi-variety generated by $\boldsymbol{M}$ is $ISPP_{\mathrm{u}}\{\boldsymbol{M}\} = ISP\{\boldsymbol{M}\}$, because any ultrapower of a single finite algebra is isomorphic to that algebra. So any $\boldsymbol{M}$-algebra is a subalgebra of a direct product of the corresponding generating algebra $\boldsymbol{M}$. This has some consequences.

5.1 Free algebras

We will construct the corresponding free algebras over one generator for these quasi-varieties.

Remark 5.1 Let $\boldsymbol{A}$ be an $\boldsymbol{M}$-algebra. If $\boldsymbol{A}$ is isomorphic to a subalgebra of a power $\boldsymbol{M}^I$ of $\boldsymbol{M}$, we will call *Boolean element* the elements $a \in A$ such that for all $i \in I$ either $a(i) = 0$ or $a(i) = 1$. We will denote by $\mathbf{Bl}(\boldsymbol{A})$ the set of all Boolean elements of the algebra $\boldsymbol{A}$. It is easy to see that $a \in \mathbf{Bl}(\boldsymbol{A})$ if and only if $a \wedge a = a$ and also that for every $a \in A$, $a \wedge a \in \mathbf{Bl}(\boldsymbol{A})$.

Lemma 5.2 *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $\boldsymbol{M}$-algebras and $f : A \longrightarrow B$. Then f is a homomorphism if and only if f verifies the following conditions:*

1. *f restricted to $\mathbf{Bl}(\boldsymbol{A})$ is a homomorphism between $\mathbf{Bl}(\boldsymbol{A})$ and $\mathbf{Bl}(\boldsymbol{B})$.*
2. *$a \notin \mathbf{Bl}(\boldsymbol{A})$ implies $f(\sim a) = \sim f(a)$.*
3. *$a \notin \mathbf{Bl}(\boldsymbol{A})$ implies $f(a \wedge a) = f(a) \wedge f(a)$.*

Proof.

($\Rightarrow$) If f is a homomorphism, it is easy to see that f verifies the second and third items of this lemma. For the first one we have that if $x \in \mathbf{Bl}(\boldsymbol{A})$, then $x \wedge x = x$, hence we have $f(x) = f(x \wedge x) = f(x) \wedge f(x)$ and therefore $f(x) \in \mathbf{Bl}(\boldsymbol{B})$.

($\Leftarrow$) For elements $a, b \in A$ we have to prove that $f(a * b) = f(a) * f(b)$, where $*$ is a binary connective (because $f(\sim a) = \sim f(a)$ is true by hypothesis). Since $a * b$ is a complex formula, by an easy calculation we see that $a * b = (a \wedge a) * (b \wedge b)$, so

$$\begin{aligned} f(a * b) &= f((a \wedge a) * (b \wedge b)) \\ &= f(a \wedge a) * f(b \wedge b) && \text{(since } x \wedge x \in \mathbf{Bl}(\boldsymbol{A})\text{)} \\ &= (f(a) \wedge f(a)) * (f(b) \wedge f(b)) && \text{(by hypothesis)} \\ &= f(a) * f(b). \end{aligned}$$

□

Let b be an element of an $\boldsymbol{M}$-algebra $\boldsymbol{B}$. Since $\boldsymbol{B}$ is isomorphic to a subalgebra of a power $\boldsymbol{M}^I$ of $\boldsymbol{M}$, for some set I, $\boldsymbol{M}^I \cong \boldsymbol{M}^J \times \boldsymbol{M}^K \times \boldsymbol{M}^L$, where

$$J = \{i \in I : b_i = 0\}, \quad K = \{i \in I : b_i = 1\}, \quad \text{and} \quad L = \{i \in I : b_i = \tfrac{1}{2}\}.$$

In the most general setting, $J \neq \emptyset$, $K \neq \emptyset$, and $L \neq \emptyset$, but the cardinalities of these sets are irrelevant when it comes to operations involving only b, that is, b can be regarded as the tuple $(0, 1, \frac{1}{2})$.

5.2 Free $\boldsymbol{M}^3_{1,1}$-algebra over a single generator

Let $\mathcal{F}$ be the $\boldsymbol{M}^3_{1,1}$-algebra generated by x, it is easy to see that the elements of $\mathcal{F}$ are

$$\begin{array}{llll} x = (0, 1, \frac{1}{2}), & \sim x = (1, 0, \frac{1}{2}), & x \wedge x = (0, 1, 0), & \sim x \wedge \sim x = (1, 0, 0), \\ x \wedge \sim x = (0, 0, 0), & x \vee \sim x = (1, 1, 0), & x \rightarrow \sim x = (1, 0, 1), & \sim x \rightarrow x = (0, 1, 1), \\ x \rightarrow x = (1, 1, 1), & \multicolumn{3}{l}{\sim (x \vee \sim x) = (x \rightarrow \sim x) \wedge (\sim x \rightarrow x) = (0, 0, 1).} \end{array}$$

Its diagram is in Figure 1.

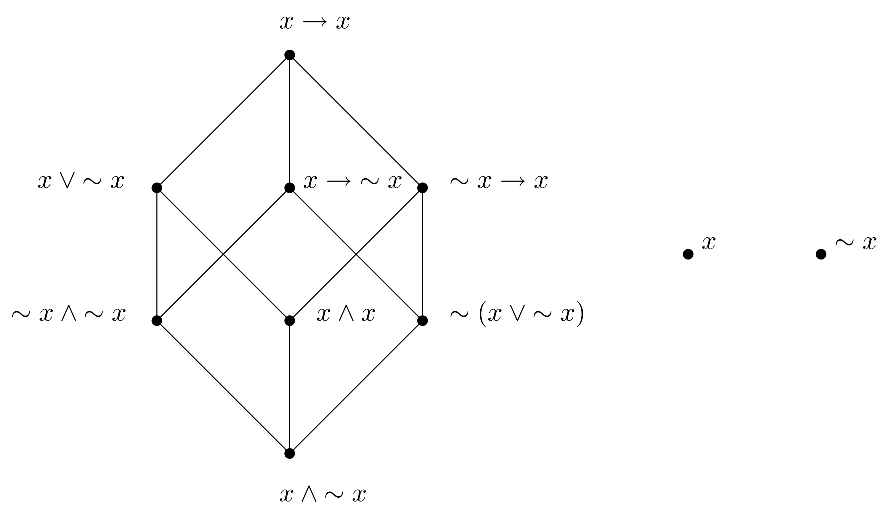


Fig. 1 Free $\boldsymbol{M}^3_{1,1}$-algebra generated by x

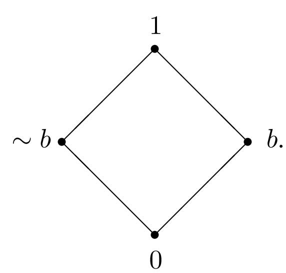
Theorem 5.3 *$\mathcal{F}$ is the free $\boldsymbol{M}^3_{1,1}$-algebra over a single generator x.*

P r o o f. Let b be an element of an $\boldsymbol{M}^3_{1,1}$-algebra $\boldsymbol{B}$, by the remarks at the end of the last subsection, we can say that $b \in {\boldsymbol{M}^3_{1,1}}^J \times {\boldsymbol{M}^3_{1,1}}^K \times {\boldsymbol{M}^3_{1,1}}^L$ and since not all the sets J, K, L are empty, there are seven cases for this element. In all these cases we present the $\boldsymbol{M}^3_{1,1}$-algebra generated by b and the unique homomorphism between $\mathcal{F}$ and the corresponding $\boldsymbol{M}^3_{1,1}$-algebra generated by b such that $f(x) = b$.

1. $L = \emptyset$, $J \neq \emptyset$, and $K \neq \emptyset$.
In this case we have

$$b = (0,1), \quad \sim b = (1,0), \quad b \wedge \sim b = (0,0), \quad b \to b = (1,1).$$

That is,



And the homomorphism f will be:

$$f(x) = f(x \wedge x) = f(\sim x \to x) = b, \quad f(\sim x) = f(\sim x \wedge \sim x) = f(x \to \sim x) = \sim b,$$
$$f(x \wedge \sim x) = f(\sim (x \vee \sim x)) = 0, \qquad f(x \to x) = f(x \vee \sim x) = 1.$$

2. $L = \emptyset$, $J = \emptyset$, and $K \neq \emptyset$.
In this case $b = 1$ and the algebra generated by b in B is the two element Boolean algebra. The homomorphism f will be:

$$f(x) = f(x \wedge x) = f(x \to x) = f(\sim x \to x) = f(x \vee \sim x) = 1 = b,$$
$$f(\sim x) = f(\sim x \wedge \sim x) = f(x \to \sim x) = f(x \wedge \sim x) = f(\sim (x \vee \sim x)) = 0.$$

3. $L = \emptyset$, $J \neq \emptyset$, and $K = \emptyset$. This case is similar to the previous one with $b = 0$.

4. $L \neq \emptyset$ and $J = K = \emptyset$.
In this case we have

$$b = \tfrac{1}{2}, \quad b \to b = 1, \quad b \wedge \sim b = 0.$$

That is,

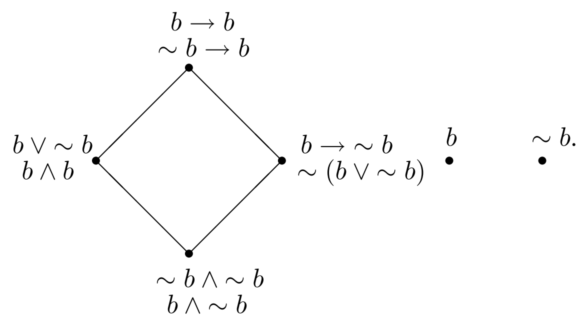
And the homomorphism f will be:

$$f(x) = f(\sim x) = b, \quad f(x \to x) = f(\sim x \to x) = f(x \to \sim x) = f(\sim (x \vee \sim x)) = 1,$$
$$f(x \wedge \sim x) = f(x \wedge x) = f(x \vee \sim x) = f(\sim x \wedge \sim x) = 0.$$

5. $L \neq \emptyset$, $J = \emptyset$, and $K \neq \emptyset$.
In this case we have

$$b = (1, \tfrac{1}{2}), \qquad \sim b = (0, \tfrac{1}{2}), \qquad b \wedge b = (1, 0),$$
$$b \to \sim b = (0, 1), \quad b \wedge \sim b = (0, 0), \quad b \to b = (1, 1).$$

That is,



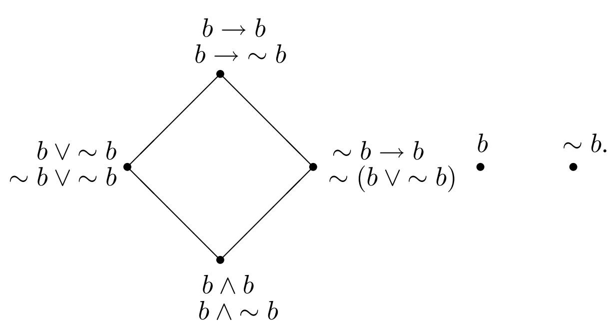
And the homomorphism f will be:

$$f(x) = b, \quad f(\sim x) = \sim b,$$
$$f(x \wedge x) = f(x \vee \sim x) = b \wedge b, \qquad f(x \to \sim x) = f(\sim (x \vee \sim x)) = b \to \sim b,$$
$$f(x \wedge \sim x) = f(\sim x \wedge \sim x) = b \wedge \sim b, \qquad f(x \to x) = f(\sim x \to x) = b \to b.$$

6. $L \neq \emptyset$, $J \neq \emptyset$, and $K = \emptyset$.
In this case we have

$$b = (0, \tfrac{1}{2}), \qquad \sim b = (1, \tfrac{1}{2}), \qquad \sim b \wedge \sim b = (1, 0),$$
$$\sim b \to b = (0, 1), \quad b \wedge \sim b = (0, 0), \quad b \to b = (1, 1).$$

That is,



And the homomorphism f will be:

$$f(x) = b, \quad f(\sim x) = \sim b,$$
$$f(\sim x \wedge \sim x) = f(x \vee \sim x) = \sim b \wedge \sim b, \qquad f(\sim x \to x) = f(\sim (x \vee \sim x)) = b \to \sim b,$$
$$f(x \wedge \sim x) = f(x \wedge x) = b \wedge \sim b, \qquad f(x \to x) = f(x \to \sim x) = b \to b.$$

7. $J \neq \emptyset$, $K \neq \emptyset$, and $L \neq \emptyset$. This case is obvious.

Using Remark 5.1 and Lemma 5.2, it is easy to check that, in all cases, f is an homomorphism; obviously it is unique. □

5.3 Free $M^3_{2,1}$-algebra over a single generator

Let $\mathcal{F}$ be the $M^3_{2,1}$-algebra generated by x, it is easy to see that the elements of $\mathcal{F}$ are

$$x = (0, 1, \tfrac{1}{2}), \qquad \sim x = (1, 0, \tfrac{1}{2}), \qquad x \wedge x = (0, 1, 1),$$
$$y = \sim x \wedge \sim x = (1, 0, 1), \qquad x \wedge \sim x = (0, 0, 1), \qquad x \vee \sim x = (1, 1, 1),$$
$$\sim (x \wedge x) = (1, 0, 0), \qquad \sim (\sim x \wedge \sim x) = (0, 1, 0), \qquad \sim (x \wedge \sim x) = (1, 1, 0),$$
$$\sim (x \vee \sim x) = (0, 0, 0).$$

Its diagram is in Figure 2.

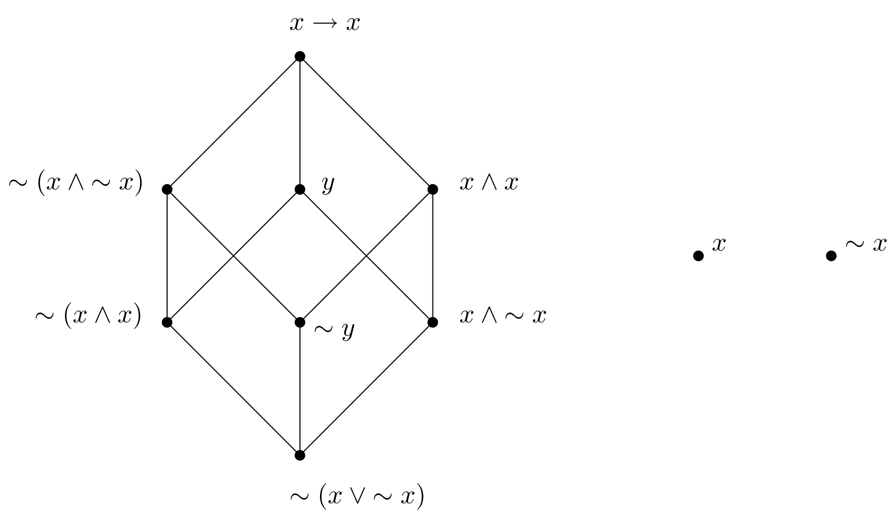


Fig. 2 Free $M^3_{2,1}$-algebra generated by x

Theorem 5.4 *$\mathcal{F}$ is the free $M^3_{2,1}$-algebra over a single generator x.*

Proof. It is analogous to the proof of Theorem 5.3. □

5.4 Free $M^3_{1,3}$-algebra over a single generator

Let $\mathcal{F}$ be the $M^3_{1,3}$-algebra generated by x, it is easy to see that the elements of $\mathcal{F}$ are

$$x = (0, 1, \tfrac{1}{2}), \qquad \sim x = (1, 0, 0), \qquad \sim\sim x = (0, 1, 1),$$
$$x \wedge x = (0, 1, 0), \qquad x \wedge \sim x = (0, 0, 0), \qquad x \vee \sim x = (1, 1, 0),$$
$$x \to \sim x = (1, 0, 1), \qquad x \to x = (1, 1, 1), \qquad \sim (x \vee \sim x) = (x \to \sim x) \wedge (\sim\sim x) = (0, 0, 1).$$

Its diagram is in Figure 3.

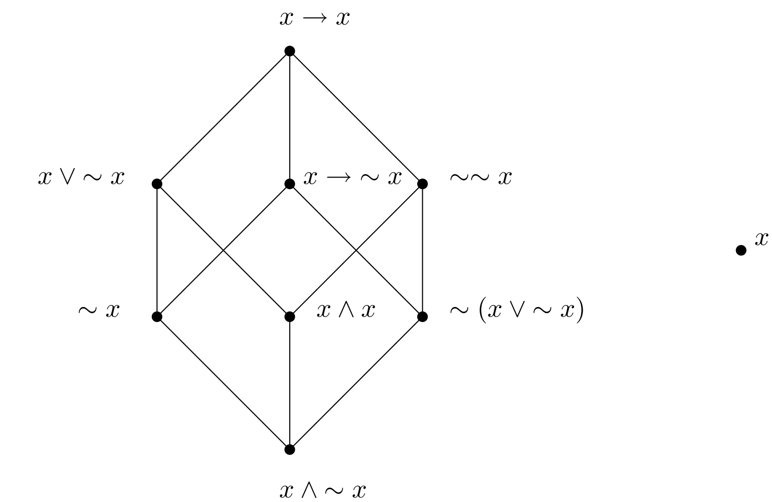


Fig. 3 Free $M^3_{1,3}$-algebra generated by x

Theorem 5.5 *$\mathcal{F}$ is the free $M^3_{1,3}$-algebra over a single generator x.*

P r o o f. It is analogous to the proof of Theorem 5.3. □

5.5 Free $M^3_{2,2}$-algebra over a single generator

Let $\mathcal{F}$ be the $M^3_{2,2}$-algebra generated by x, it is easy to see that the elements of $\mathcal{F}$ are

$$\begin{array}{lll} x = (0,1,\frac{1}{2}), & \sim x = (1,0,1), & \sim\sim x = (0,1,0), \\ x \wedge x = (0,1,1), & x \wedge \sim x = (0,0,1), & \sim (x \wedge x) = (1,0,0), \\ \sim (x \vee \sim x) = (0,0,0), & \sim (x \wedge \sim x) = (1,1,0), & x \to x = (1,1,1). \end{array}$$

Its diagram is in Figure 4.

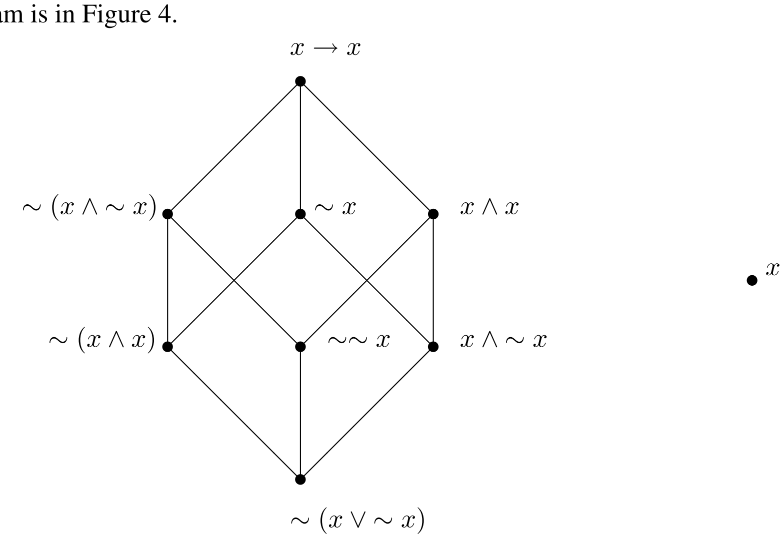


Fig. 4 Free $M^3_{2,2}$-algebra generated by x

Theorem 5.6 *$\mathcal{F}$ is the free $M^3_{2,2}$-algebra over a single generator x.*

P r o o f. It is analogous to the proof of Theorem 5.3. □

5.6 Free M^4-algebra over a single generator

Let $\mathcal{F}$ be the $\boldsymbol{M}^4$-algebra generated by x, it is easy to see that the elements of $\mathcal{F}$ are

$$\begin{array}{lll} x = (0,1,\top,\bot), & \sim x = (1,0,\top,\bot), & y = x \wedge x = (0,1,1,0), \\ v = \sim x \wedge \sim x = (1,0,1,0), & x \wedge \sim x = (0,0,1,0), & x \vee \sim x = (1,1,1,0), \\ w = x \rightarrow \sim x = (1,0,1,1), & \sim x \rightarrow x = (0,1,1,1), & x \rightarrow x = (1,1,1,1), \\ u = (x \rightarrow \sim x) \wedge (\sim x \rightarrow x) = (0,0,1,1) & & \end{array}$$

and its negations. Its diagram is in Figure 5.

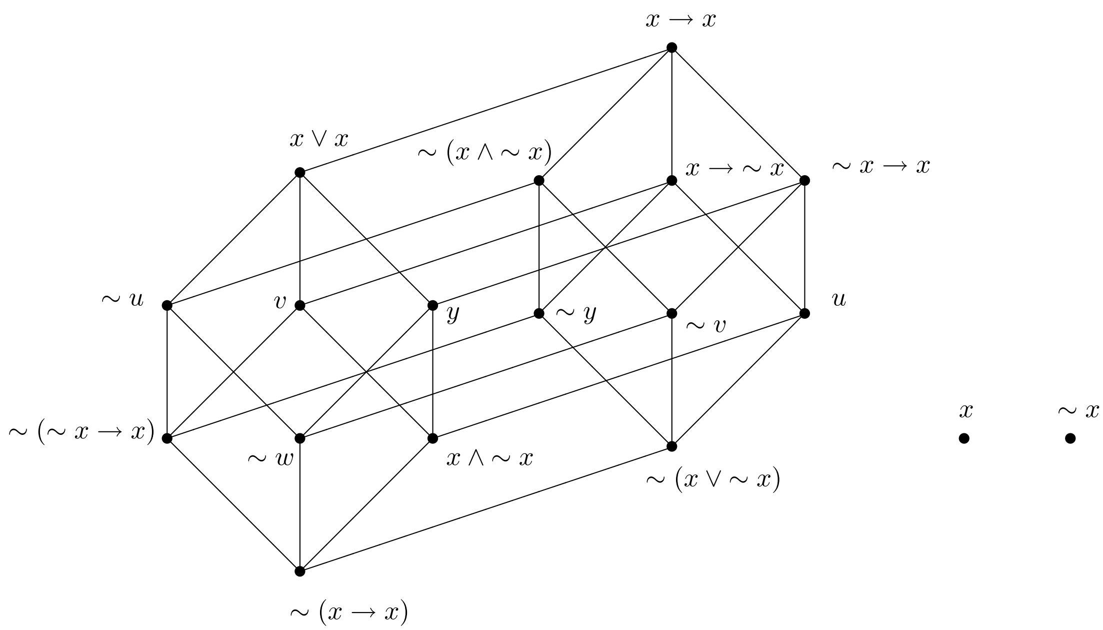


Fig. 5 Free $\boldsymbol{M}^4$-algebra generated by x

Theorem 5.7 *$\mathcal{F}$ is the free $\boldsymbol{M}^4$-algebra over a single generator x.*

P r o o f. It is analogous to the proof of Theorem 5.3. □

5.7 About the equivalent quasi-varieties

We will define the algebras $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ as $\boldsymbol{A}_i = \langle \{b, c\}, \wedge, \vee, \rightarrow, \sim_i \rangle$ with

$*$	b	c	$\sim_1$	$\sim_2$
b	b	b	b	b
c	b	b	b	c

, where $*$ is any of the binary connectives $\wedge$, $\vee$, and $\rightarrow$.

Theorem 5.8 *The quasi-varieties generated by $\boldsymbol{M}^3_{1,1}$, $\boldsymbol{M}^3_{2,1}$, $\boldsymbol{M}^3_{1,3}$, $\boldsymbol{M}^3_{2,2}$, and $\boldsymbol{M}^4$ are not varieties.*

P r o o f. Define $f : \{0, \frac{1}{2}, 1\} \longrightarrow \{b, c\}$ by $f(0) = f(1) = b$ and $f(\frac{1}{2}) = c$.

1. If $\boldsymbol{M}$ is $\boldsymbol{M}^3_{1,3}$ or $\boldsymbol{M}^3_{2,2}$, then it is easy to see that $f : \boldsymbol{M} \longrightarrow \boldsymbol{A}_1$ is a homomorphism.
2. Similarly, if $\boldsymbol{M}$ is $\boldsymbol{M}^3_{1,1}$, $\boldsymbol{M}^3_{2,1}$, or $\boldsymbol{M}^4$, $f : \boldsymbol{M} \longrightarrow \boldsymbol{A}_2$ is a homomorphism.

In order to check that $\boldsymbol{A}_1, \boldsymbol{A}_2$ do not belong to any of the these quasi-varieties, recall that by [2, Theorem 2.17], the quasi-variety that is equivalent to the algebraizable deductive system must satisfy the quasi-identity

$$\delta(x \mathbin{\Delta} y) = \varepsilon(x \mathbin{\Delta} y) \Rightarrow x \approx y.$$

In our cases, $\delta(x \,\Delta\, y) = \varepsilon(x \,\Delta\, y)$ corresponds to the two equations

$$(x \leftrightarrow y) \wedge (x \leftrightarrow y) = (x \leftrightarrow y) \rightarrow (x \leftrightarrow y),$$
$$(\sim x \leftrightarrow \sim y) \wedge (\sim x \leftrightarrow \sim y) = (\sim x \leftrightarrow \sim y) \rightarrow (\sim x \leftrightarrow \sim y),$$

and all components are complex formulas, so after some elementary calculations, this reduces to

$$(x \leftrightarrow y) = (x \rightarrow x) \,\&\, (\sim x \leftrightarrow \sim y) = (x \rightarrow x),$$

so the quasi-identity we need to check is simply

$$(x \leftrightarrow y) = (x \rightarrow x) \,\&\, (\sim x \leftrightarrow \sim y) = (x \rightarrow x) \Rightarrow x \approx y.$$

Now it is just a matter of checking that evaluating x and y by b and c, respectively, this quasi-identity does not hold neither in $\boldsymbol{A}_1$ nor in $\boldsymbol{A}_2$, thus proving that the quasi-varieties generated by $\boldsymbol{M}^3_{1,1}$, $\boldsymbol{M}^3_{2,1}$, $\boldsymbol{M}^3_{1,3}$, $\boldsymbol{M}^3_{2,2}$, and $\boldsymbol{M}^4$ are not varieties. □

Acknowledgements Some of the results reported here are part of the first author's master's thesis at Pontificia Universidad Católica de Chile. Funding for the second author has been provided by FONDECYT 1060726.

References

[1] A. I. Arruda, On the imaginary logic of Vasiliev. In: Non-Classical Logic, Model Theory and Computability (A. I. Arruda, N. C. A. da Costa, and R. Chuaqui, eds.), pp. 3 – 24 (North-Holland, 1977).
[2] W. J. Blok and D. Pigozzi, Algebraizable Logics. Memoirs of the Amer. Math. Soc. 396, Vol. 77 (1989).
[3] W. J. Blok and D. Pigozzi, Abstract algebraic logic and the deduction theorem. To appear in the Bulletin of Symbolic Logic.
[4] S. E. Bowers, R. A. Lewin, and D. Pigozzi, An annotated logic defined by a matrix. In: Unsolved Problems in Mathematics for the 21st Century (J. M. Abe and S. Tanaka, eds.), pp. 261 – 288 (IOS Press, 2001).
[5] W. A. Carnielli and J. Marcos, A taxonomy of C-systems. In: Paraconsistency, the Logical Way to the Inconsistent (W. A. Carnielli, M. Coniglio, and I. M. L. D'Ottaviano, eds.). Lecture Notes in Pure and Applied Mathematics 228, pp. 1 – 94 (Marcel Dekker, 2002).
[6] J. Czelakowski, Protoalgebraic Logics (Kluwer Academic Publishers, 2001).
[7] V. L. Fernández and M. E. Coniglio, Combining valuations with society semantics. J. Applied Non-Classical Logic **13**, 21 – 46 (2003).
[8] B. Herrmann, Algebraizability and Beth's theorem for equivalential logics. Bulletin of the Section of Logic **22**, University of Łodź, 85 – 88 (1993).
[9] B. Herrmann, Equivalential and algebraizable logics. Studia Logica **57**, 419 – 436 (1996).
[10] E. Hirsh, Algebraization of some paraconsistent and paracomplete logics. Master's thesis, Pontificia Universidad Católica de Chile, 2006.
[11] R. A. Lewin and I. F. Mikenberg, Literal-paraconsistent and literal-paracomplete matrices. Math. Logic Quart. **52**, 478 – 493 (2006).
[12] R. A. Lewin, I. F. Mikenberg, and M. G. Schwarze, Algebraization of paraconsistent logic P^1. J. Non-Classical Logics **7**, 79 – 88 (1990).
[13] R. A. Lewin, I. F. Mikenberg, and M. G. Schwarze, On the algebraizability of annotated logics. Studia Logica **59**, 359 – 386 (1997).
[14] J. Marcos, Possible translations semantics. In: Proceedings of CombLog'04, Workshop on Combination of Logics: Theory and Applications, 1049-001 Lisboa, Portugal, Departamento de Matemática, Instituto Superior Técnico, 2004 (W. A. Carnielli, F. M. Dionísio, and P. Mateus, eds.), pp. 119 – 128.
[15] L. Z. Puga and N. C. A. da Costa, On the imaginary logic of N. A. Vasiliev. Z. Math. Logik Grundlagen Math. **34**, 205 – 211 (1988).
[16] A. P. Pynko, Algebraic study of Sette's maximal paraconsistent logic. Studia Logica **54**, 89 – 128 (1995).
[17] A. M. Sette, On the propositional calculus P^1. Mathematica Japonicae **18**, 173 – 180 (1973).
[18] A. M. Sette and W. A. Carnielli, Maximal weakly-intuitionistic logics. Studia Logica **55**, 181 – 203 (1995).

