

MATRIX SEMANTICS FOR ANNOTATED LOGICS

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ABSTRACT. A matrix semantics for paraconsistent systems \mathcal{SAL}_τ of annotated logics is developed. Systems \mathcal{SAL}_τ are extensions of the systems \mathcal{PTS} , previously proven algebraizable by the authors. The results reported here will be useful in the study of the class of algebras that arise in the process of algebraizing the systems.

1. INTRODUCTION

Annotated logics were introduced in [8] by V. S. Subrahmanian as logical foundations for computer programming. Blair and Subrahmanian [1] proved that these systems were paraconsistent and showed that they could form the basis of a programming language for reasoning about data bases that contain inconsistent information. Subsequently, numerous applications in Artificial Intelligence, like inheritance networks, object oriented data bases, etc. have been developed.

One of the intended meanings of an annotated formula p_λ is “ p has credibility greater or equal than λ ”, where λ belongs to a complete lattice \mathcal{T} .

A complete study from the model theoretical and proof theoretical points of view has been done in [4]. They show that almost all classical basic results in model theory can be adapted to the systems they call \mathcal{PT} . However, since there are several kinds of axioms (some for complex formulas, others for atomic formulas and still others for arbitrary formulas), these systems are not structural in the sense that their consequence relation is not closed under substitutions.

In [6], the authors introduce a structural version \mathcal{SPT} of annotated logics and proved that they are equivalent to the original \mathcal{PT} systems, in the sense that everything provable in a system of one type has a translation that is provable (using some additional premises) in the corresponding system of the other type.

The main result of that paper is that for a finite lattice \mathcal{T} of annotation constants, the system \mathcal{SPT} is algebraizable in the sense of

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Blok and Pigozzi, [2] and [3]. The systems SPT are not complete with respect to the semantics given in that paper. In this paper we propose a matrix semantics for \mathcal{SAL}_τ , *structural annotated logic based on τ* , which are extensions of SPT .

Since any extension of an algebraizable system is itself algebraizable, for finite τ , \mathcal{SAL}_τ is algebraizable. We think that this will be useful to develop a theory of annotated algebras, i.e., the class of algebras that arise in the process of algebraizing \mathcal{SAL}_τ .

In [7] the authors develop closely related systems.

2. THE SYSTEMS \mathcal{SAL}_τ

In this section we will define structural annotated deductive systems \mathcal{SAL}_τ that are extensions of the systems SPT introduced in [6].

Let τ be a (fixed) finite lattice and $\sim : \tau \mapsto \tau$ an arbitrary function. Let \perp, \top denote the least and the greatest elements of the lattice.

The function \sim , intended to act as a negation for annotated formulas, is absolutely arbitrary.

Remark 2.1. Since the lattice τ has finite cardinality n , for any $\lambda \in \tau$,

$$\{\sim^m \lambda : m \in \omega\},$$

where ω is the set of natural numbers, is a finite set. So for any λ , there exists a least integer t_λ such that for some p

$$\sim^{t_\lambda} \lambda = \sim^{t_\lambda+p} \lambda.$$

Of course there will also exist the least such an integer p . Call it p_λ .

Observe that if $r \geq t_\lambda$ and s is a multiple of p_λ , then

$$\sim^r \lambda = \sim^{r+s} \lambda,$$

so defining

$$\begin{aligned} T &= \max\{t_\lambda : \lambda \in \tau\}, \\ P &= m.c.m.\{2\} \cup \{p_\lambda : \lambda \in \tau\}. \end{aligned}$$

we have, for any $\lambda \in \tau$,

$$\sim^T \lambda = \sim^{T+P} \lambda.$$

In the definition of P above, we have included 2 in the minimum common multiple for technical reasons which will become clear when we state axiom \neg_4 .

2.1. The Language. The language of \mathcal{SAL}_τ will consist of the binary logical symbols $\wedge, \vee, \rightarrow$ and the unary symbols \neg, \circ and f_λ , for each $\lambda \in \mathcal{T}$, a denumerable set $\mathcal{P} = \{p_i : i \in \omega\}$ of propositional letters and parentheses. The set \mathcal{F} of formulas is defined recursively as follows:

- (1) Propositional letters are formulas.
- (2) If A is a formula, then A° is a formula.
- (3) If A is a formula and λ is an annotation constant, then $f_\lambda A$ is a formula.
- (4) If A is a formula, then $\neg A$ is a formula.
- (5) If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulas.
- (6) An expression is a formula if and only if it is obtained by finitely many applications of the above rules.

The formulas of the form $\neg^{k_n} f_{\lambda_n} \neg^{k_{n-1}} f_{\lambda_{n-1}} \cdots \neg^{k_1} f_{\lambda_1} \neg^{k_0} p$, where for $0 \leq i \leq n$, $k_i \in \omega$ and $\lambda_i \in \mathcal{T}$, are called *hyperliterals*. Observe that for $n = 0$, p and $\neg^k p$ are hyperliteral. A formula that is not a hyperliteral will be called a *complex formula*.

The intended meaning for the formula A° is “ A is well behaved”, in particular, regarding negations.

2.2. Axioms and Inference Rules. The axioms and inference rules for \mathcal{SAL}_τ are as follows:

- (1) Axioms for binary connectives.
 - (\rightarrow_1) $A \rightarrow (B \rightarrow A)$,
 - (\rightarrow_2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,
 - (\rightarrow_3) $((A \rightarrow B) \rightarrow A) \rightarrow A$,
 - (\wedge_1) $(A \wedge B) \rightarrow A$,
 - (\wedge_2) $(A \wedge B) \rightarrow B$,
 - (\wedge_3) $A \rightarrow (B \rightarrow (A \wedge B))$,
 - (\vee_1) $A \rightarrow (A \vee B)$,
 - (\vee_2) $B \rightarrow (A \vee B)$,
 - (\vee_3) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$.
- (2) Axioms for negation:
 - \neg_1) $(A^\circ \wedge B^\circ) \rightarrow ((A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A))$,
 - \neg_2) $A^\circ \rightarrow (A \rightarrow (\neg A \rightarrow B))$,
 - \neg_3) $A^\circ \rightarrow (A \vee \neg A)$.
 - \neg_4) $\neg^{T+P} A = \neg^T A$,

where T and P are the numbers defined in Remark 2.1.

They depend on the lattice and the function \sim .

- (3) Axioms for \circ :
 - \circ_1) $A^\circ \leftrightarrow (\neg A)^\circ$,

$$\circ_2) \quad A^\circ \leftrightarrow (f_\lambda A)^\circ,$$

$$\circ_3) \quad \begin{array}{ll} (A^\circ)^\circ & (A \vee B)^\circ \\ (A \wedge B)^\circ & (A \rightarrow B)^\circ. \end{array}$$

(4) The axioms for annotated variables are:

$$\begin{array}{l} \tau_1) \quad \neg(A^\circ) \rightarrow f_\perp A, \\ \tau_2) \quad \neg(A^\circ) \rightarrow (\neg^k f_\lambda A \leftrightarrow \neg^{k-1} f_{\sim\lambda} A), \text{ for } k \geq 1, \\ \tau_3) \quad (f_\mu A \rightarrow f_\lambda A), \text{ for } \mu \geq \lambda, \\ \tau_4) \quad f_\mu f_\lambda A \leftrightarrow f_\mu A, \\ \tau_5) \quad \neg(A^\circ) \rightarrow (f_\lambda A \leftrightarrow f_\lambda \neg A), \\ \tau_6) \quad A^\circ \rightarrow (f_\lambda A \leftrightarrow A). \end{array}$$

For the next two axioms we will need the following definitions.

$$S_\kappa(A) = f_\kappa A \wedge \bigwedge_{\lambda \not\leq \kappa} \neg(f_\lambda A \wedge f_\lambda A),$$

$$S_\top(A) = A^\circ \wedge f_\top A$$

and

$$T_{(\lambda, \kappa)}(A) = \bigwedge_{\substack{m \leq N \\ \sim^m \lambda \leq \kappa}} \neg^m A \wedge \bigwedge_{\substack{m \leq N \\ \sim^m \lambda \not\leq \kappa}} \neg(\neg^m A \wedge \neg^m A),$$

where $N = \max \{n, P + T\}$.

$$\tau_7) \quad \neg(A^\circ) \rightarrow \bigvee_{\kappa \in \tau} S_\kappa(A),$$

$$\tau_8) \quad S_\kappa(A) \rightarrow \bigvee_{\lambda \in \tau} T_{(\lambda, \kappa)}(A).$$

(5) Inference Rules

R1) Modus Ponens

$$\frac{A, A \rightarrow B}{B}$$

R2) If $J \neq \emptyset$ and for all $j \in J$, $\lambda_j \in \tau$ and $\lambda = \bigvee_{j \in J} \lambda_j$, then

$$\frac{A \rightarrow f_{\lambda_j} B, \text{ for } j \in J}{A \rightarrow f_\lambda B}.$$

2.3. Facts.

- Remark 2.2.** (1) The systems \mathcal{SAL}_τ are structural since all axioms and the two rules of inference are closed under substitutions.
- (2) For each lattice \mathcal{T} , \mathcal{SAL}_τ is an extension of the corresponding system $\text{SP}\mathcal{T}$ of [6]. More precisely, \mathcal{SAL}_τ is obtained from $\text{SP}\mathcal{T}$ by adding axiom τ_7 and τ_8 .
- (3) In the axiomatization of $\text{SP}\mathcal{T}$ there is only a crude consideration of the specific lattice \mathcal{T} and the function \sim that we are using. Since \mathcal{T} and \sim are arbitrary, these axioms take a rather cumbersome form, nevertheless they are not complicated in spirit. Axiom τ_7 states that every hyperliteral formula has a certain degree of credibility κ . Axiom τ_8 codes some of the finer aspects of negation of a hyperliteral formula with respect to a degree of credibility κ .

Theorem 2.1. *Let $\varphi(x_1, x_2, \dots, x_n)$ be a classical tautology. Then*

(1)

$$A_1^\circ, A_2^\circ, \dots, A_n^\circ \vdash \varphi(A_1, A_2, \dots, A_n).$$

(2) *If $\varphi(x_1, x_2, \dots, x_n)$ does not contain negations, then*

$$\vdash \varphi(A_1, A_2, \dots, A_n).$$

Proof. Assuming $A_1^\circ, A_2^\circ, \dots, A_n^\circ$, by Modus Ponens, we obtain all axioms for classical negation, which together with those for \rightarrow , \vee and \wedge , are all we need to prove any classical tautology. If negation is not involved, the axioms for binary connectives suffice. \square

Lemma 2.2. (1) *If A is hyperliteral with propositional letter p , then*

a)

$$\vdash A^\circ \leftrightarrow p^\circ.$$

b)

$$\neg(p^\circ) \vdash f_\lambda A \leftrightarrow f_\lambda p.$$

(2) *If A is complex, then*

$$\vdash A^\circ.$$

Proof. The proof is by induction on the complexity of A .

- (1) a) – If $A = p$, then $\vdash p^\circ \leftrightarrow A^\circ$.
 – If $A = \neg B$.

Then B has to be hyperliteral with propositional letter p , so by induction hypothesis, $\vdash B^\circ \leftrightarrow p^\circ$.

But then, by axiom \circ_1 , we get

$$\vdash (\neg B)^\circ \leftrightarrow p^\circ.$$

– If $A = f_\lambda B$. Similar to the previous case using axiom \circ_2 .

b) – If $A = p$, then $\vdash f_\lambda p \leftrightarrow f_\lambda A$.

– If $A = \neg B$.

By induction hypothesis,

$$\neg(p^\circ) \vdash f_\lambda B \leftrightarrow f_\lambda p,$$

and since by axiom τ_5 ,

$$\neg(B^\circ) \vdash f_\lambda B \leftrightarrow f_\lambda \neg B,$$

using a) and a classical tautology,

$$\neg(p^\circ) \vdash f_\lambda \neg B \leftrightarrow f_\lambda p.$$

– If $A = f_\lambda B$. Similar to the previous case using axiom τ_3 .

(2) – If $A = B^\circ$, $A = B \wedge C$, $A = B \vee C$ or $A = B \rightarrow C$, then by axiom \circ_3 ,

$$\vdash A^\circ.$$

– If $A = \neg B$, for some complex formula B . Then by induction hypothesis,

$$\vdash B^\circ.$$

By axiom \circ_1 and Modus Ponens,

$$\vdash (\neg B)^\circ.$$

– If $A = f_\lambda B$. This is similar to the previous case using axiom \circ_2 .

□

3. MATRIX SEMANTICS FOR \mathcal{SAL}_τ

In this section we define a family **EM** of very simple matrices with respect to which the system \mathcal{SAL}_τ is sound. Based on these, we build a family **M** of more complex matrices that is a matrix semantics for system \mathcal{SAL}_τ . We first give a few basic definitions. See [2], [3] or [5] for details.

Definition 3.1. A *valuation* into the matrix $\mathcal{M} = \langle A, D \rangle$ is the unique homomorphism that extends a function

$$u : \mathcal{P} \longrightarrow A$$

to

$$u^* : \mathcal{F} \longrightarrow A.$$

We often call u a valuation since it totally determines the homomorphism.

The formula φ is a *consequence of Γ in \mathcal{M}* , in symbols $\Gamma \vDash_{\mathcal{M}} \varphi$, if for any valuation $u : \mathcal{P} \longrightarrow A$,

$$u^*(\psi) \in D \text{ for all } \psi \in \Gamma \text{ implies } u^*(\varphi) \in D.$$

We say that a matrix $\mathcal{M} = \langle A, D \rangle$ is a *matrix model* of \mathcal{SAL}_{τ} if

$$\Gamma \vdash_{\mathcal{SAL}_{\tau}} \varphi \text{ implies } \Gamma \vDash_{\mathcal{M}} \varphi.$$

A class \mathbf{M} of matrices is a *matrix semantics* of \mathcal{SAL}_{τ} if

$$\Gamma \vdash_{\mathcal{SAL}_{\tau}} \varphi \text{ if and only if } \Gamma \vDash_{\mathcal{M}} \varphi,$$

for all $\mathcal{M} \in \mathbf{M}$. The latter is usually written $\Gamma \vDash_{\mathbf{M}} \varphi$.

3.1. Soundness. We will now define the class **EM** of matrices, which we will call *elementary \mathcal{SAL}_{τ} -matrices* and prove that the matrices of **EM** are matrix models of \mathcal{SAL}_{τ} .

Let $L = \mathcal{T} \cup \{0, 1\}$ and extend the order of \mathcal{T} as follows. For all $\lambda \in \mathcal{T}$,

$$0 > \lambda > 1.$$

Let \mathcal{I} be a lattice ideal of \mathcal{T} and $D = \mathcal{I} \cup \{1\}$. Observe that D is also an ideal of the lattice L and that $0 \notin D$. We define the following operations on L .

(1)

$$a \vee b = \begin{cases} 1 & , \text{ if } a \in D \text{ or } b \in D, \\ 0 & , \text{ otherwise.} \end{cases}$$

(2)

$$a \wedge b = \begin{cases} 1 & , \text{ if } a \in D \text{ and } b \in D, \\ 0 & , \text{ otherwise.} \end{cases}$$

(3)

$$a \rightarrow b = \begin{cases} 1 & , \text{ if } a \notin D \text{ or } b \in D, \\ 0 & , \text{ otherwise.} \end{cases}$$

(4)

$$a^{\circ} = \begin{cases} 0 & , \text{ if } a \in \mathcal{T} , \\ 1 & , \text{ if } a \in \{0, 1\}. \end{cases}$$

(5)

$$f_{\lambda}a = \begin{cases} \lambda & , \text{ if } a \in \mathcal{T} , \\ a & , \text{ if } a \in \{0, 1\}. \end{cases}$$

(6)

$$\neg a = \begin{cases} \sim a & , \text{ if } a \in \mathcal{T} , \\ 1 - a & , \text{ if } a \in \{0, 1\}. \end{cases}$$

The next lemma will be useful in the sequel.

Lemma 3.1. *Let $\mathcal{M} = \langle L, D \rangle$ be an elementary \mathcal{SAL}_{τ} -matrix model and*

$u : \mathcal{P} \longrightarrow L$ a valuation into L . Then for any formula B the following are equivalent.

- (1) $u^*(B^{\circ}) \notin D$
- (2) B is hyperliteral with propositional letter p , for some $p \in \mathcal{P}$, and $u(p) \in \mathcal{T}$.
- (3) $u^*(B) \in \mathcal{T}$.

Proof.

$$u^*(B^{\circ}) \notin D \text{ iff } (u^*(B))^{\circ} \notin D \text{ iff } u^*(B) \in \mathcal{T} .$$

It is straightforward to check that the latter occurs if and only if B is hyperliteral with propositional letter p , for some $p \in \mathcal{P}$ and $u(p) \in \mathcal{T}$ and that this occurs if and only if $u^*(B) \in \mathcal{T}$. \square

Theorem 3.2. *With the above notations, $\mathcal{M} = \langle L, D \rangle$ is a matrix model of \mathcal{SAL}_{τ} .*

Proof. Since the definition for the binary operations is classical, the axioms for $\rightarrow, \vee, \wedge$ all evaluate to 1 and thus belong to D . Also, the rule R1) is obviously sound.

In what follows, let $a, b, c \in L$. We first check the axioms for negation.

$$\neg_1) (A^{\circ} \wedge B^{\circ}) \rightarrow ((A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A))$$

If $a^{\circ}, b^{\circ} \in D$, then $a, b \in \{0, 1\}$.

Now assume that both $(a \rightarrow b) \in D$ and $(a \rightarrow \neg b) \in D$. Then if $a \in D$, both $b \in D$ and $\neg b \in D$, in any case $0 \in D$, a contradiction. Thus $a \notin D$, so $a = 0$ and $\neg a = 1 \in D$.

$$(a^{\circ} \wedge b^{\circ}) \rightarrow ((a \rightarrow b) \rightarrow ((a \rightarrow \neg b) \rightarrow \neg a)) = 1 \in D$$

$$\neg_2) A^{\circ} \rightarrow (A \rightarrow (\neg A \rightarrow B))$$

If $a^{\circ} \in D$, then $a \in \{0, 1\}$. But then either $a = 0$ or $\neg a = 0$, so no matter what b is,

$$a^\circ \rightarrow (a \rightarrow (\neg a \rightarrow b)) = 1 \in D.$$

- \neg_3) $A^\circ \rightarrow (A \vee \neg A)$
 If $a^\circ \in D$, then $a \in \{0, 1\}$. So either $a = 1$ or $\neg a = 1$.

$$\neg(a^\circ) \rightarrow (a \vee \neg a) = 1 \in D.$$

- \neg_4) $\neg^{T+P}A \leftrightarrow \neg^T A$
 If $a \in \{0, 1\}$, since P is a multiple of 2, $\neg^{T+P}a = \neg^T a$.
 If $a \in \mathcal{T}$, by Remark 2.1,

$$\neg^{T+P}a = \sim^{T+P}a = \sim^T a = \neg^T a,$$

so

$$\neg^{T+P}a \leftrightarrow \neg^T a = 1 \in D.$$

We check the axioms for $^\circ$. For the first two, observe that in our definitions, $a \in \mathcal{T}$ if and only if $f_\lambda a \in \mathcal{T}$ if and only if $\neg a \in \mathcal{T}$.

For the third one, observe that if c is either a° , $a \vee b$, $a \wedge b$ or $a \rightarrow b$, then $c \in \{0, 1\}$, so $c^\circ = 1 \in D$.

We check the axioms for annotated variables.

- τ_1) $\neg(A^\circ) \rightarrow f_\perp A$
 If $\neg(a^\circ) \in D$, then $a \in \mathcal{T}$, so $f_\perp a = \perp \in D$ and

$$\neg(a^\circ) \rightarrow f_\perp a = 1 \in D.$$

- τ_2) $\neg(A^\circ) \rightarrow (\neg^k f_\lambda A \leftrightarrow \neg^{k-1} f_{\sim\lambda} A)$, for $k \geq 1$
 If $\neg(a^\circ) \in D$, then $a \in \mathcal{T}$, so $\neg^k f_\lambda a = \sim^k \lambda$ and $\neg^{k-1} f_{\sim\lambda} a = \neg^{k-1} \sim \lambda = \sim^k \lambda$.

Thus

$$\neg(a^\circ) \rightarrow (\neg^k f_\lambda a \leftrightarrow \neg^{k-1} f_{\sim\lambda} a) = 1 \in D.$$

- τ_3) $f_\mu A \rightarrow f_\lambda A$, for $\mu \geq \lambda$
 If $a \in \mathcal{T}$ and $f_\mu a \in D$, then $f_\mu a = \mu \in \mathcal{I}$, and since $\mu \geq \lambda$ and \mathcal{I} is an ideal, $\lambda \in \mathcal{I}$, so $f_\lambda a \in D$.

If $a \in \{0, 1\}$, then $a = f_\mu a = f_\lambda a$, so in any case $f_\mu a \rightarrow f_\lambda a = 1 \in D$.

- τ_4) $f_\mu f_\lambda A \leftrightarrow f_\mu A$
 If $f_\mu f_\lambda a \in D$ and $a \in \mathcal{T}$, then $f_\mu f_\lambda a = \mu = f_\mu a$.
 If $a \in \{0, 1\}$, then $f_\mu f_\lambda a = a = f_\mu a$, so in any case,

$$f_\mu f_\lambda A \leftrightarrow f_\mu A = 1 \in D.$$

$\tau_5)$ $\neg(A^\circ) \rightarrow (f_\lambda A \leftrightarrow f_\lambda \neg A)$
 If $\neg(a^\circ) \in D$, then $a, \neg a \in \mathcal{T}$, so $f_\lambda a = \lambda = f_\lambda \neg a$ and thus

$$\neg(a^\circ) \rightarrow (f_\lambda a \leftrightarrow f_\lambda \neg a) = 1 \in D.$$

$\tau_6)$ $A^\circ \rightarrow (f_\lambda A \leftrightarrow A)$
 If $a^\circ \in D$, then $a \in \{0, 1\}$, so $f_\lambda a = a$, and thus

$$a^\circ \rightarrow (f_\lambda a \leftrightarrow a) = 1 \in D.$$

$\tau_7)$ $\neg(A^\circ) \rightarrow \bigvee_{\kappa \in \mathcal{T}} S_\kappa(A)$
 As in the previous cases, if $\neg(a^\circ) \in D$, then $a \in \mathcal{T}$.
 Define $\kappa_0 = \sup(D)$. Observe that since $\perp \in \mathcal{I} \neq \emptyset$, $\kappa_0 \neq 1$.
 If $\kappa_0 \neq \top$,

$$S_{\kappa_0}^{\mathcal{M}}(a) = \kappa_0 \wedge \bigwedge_{\lambda \not\leq \kappa_0} \neg(\lambda \wedge \lambda).$$

Now if $\lambda \not\leq \kappa_0$, then $\neg(\lambda \wedge \lambda) = 1 \in D$, and since $\kappa_0 \in D$,
 $S_{\kappa_0}^{\mathcal{M}}(a) = 1$.

If $\kappa_0 = \top$, $S_{\top}^{\mathcal{M}}(a) = a^\circ \wedge f_\top a = 1 \wedge \top = 1$, in any case,

$$\neg(a^\circ) \rightarrow \bigvee_{\kappa \in \mathcal{T}} S_\kappa(a) = 1 \in D.$$

$\tau_8)$

$$S_\kappa(A) \rightarrow \bigvee_{\lambda \in \mathcal{T}} T_{(\lambda, \kappa)}(A)$$

Assume $S_\kappa^{\mathcal{M}}(a) \in D$, then it is obvious that $a \in \mathcal{T}$.

Also, $f_\kappa a = \kappa \leq \kappa_0$, or else $S_\kappa^{\mathcal{M}}(a) = 0 \notin D$ and for the same reason, if $\lambda \not\leq \kappa$, then $\lambda \not\leq \kappa_0$, so we get that $\kappa = \kappa_0$.

Next we observe that for all λ ,

$$T_{(\lambda, \kappa_0)}^{\mathcal{M}}(\lambda) = \bigwedge_{\substack{m < N \\ \sim^m \lambda \leq \kappa_0}} \neg^m \lambda \wedge \bigwedge_{\substack{m < N \\ \sim^m \lambda \not\leq \kappa_0}} \neg(\neg^m \lambda \wedge \neg^m \lambda) = 1 \in D,$$

so we get that $T_{(a, \kappa_0)}^{\mathcal{M}}(a) \in D$ and thus

$$\bigvee_{\lambda \in \mathcal{T}} T_{(\lambda, \kappa_0)}^{\mathcal{M}}(a) \in D,$$

so

$$S_\kappa^{\mathcal{M}}(a) \rightarrow \bigvee_{\lambda \in \mathcal{T}} T_{(\lambda, \kappa)}^{\mathcal{M}}(a) = 1 \in D.$$

Finally, we check rule R2. If $J \neq \emptyset$ and for all $j \in J$, $\lambda_j \in \mathcal{T}$ and $\lambda = \bigvee_{j \in J} \lambda_j$, then

$$\frac{A \rightarrow f_{\lambda_j} B, \text{ for } j \in J,}{A \rightarrow f_{\lambda} B}.$$

If $a \notin D$, then the rule is sound. So let us assume that $a \in D$. Let $a \rightarrow f_{\lambda_j} b \in D$ for all $j \in J$ and suppose $a \rightarrow f_{\lambda} b \notin D$.

If $b \in \mathcal{T}$, then the last assumption implies that $\lambda = f_{\lambda} b \notin D$. But also, for all $j \in J$, $\lambda_j = f_{\lambda_j} b \in D$. This is a contradiction since \mathcal{I} is an ideal.

If $b \in \{0, 1\}$, then since $J \neq \emptyset$, there is a $j \in J$ such that $b = f_{\lambda_j} b = f_{\lambda} b \in D$ and thus $a \rightarrow f_{\lambda} b = 1$. So in any case, the rule is sound. \square

3.2. Matrix Semantics.

Definition 3.2. Define

$$\hat{\mathcal{T}} = \omega \times \mathcal{T}$$

and

$$\hat{L} = \hat{\mathcal{T}} \cup \{0, 1\}.$$

Definition 3.3. If A is a set ordered by \leq , we say that I is an *order-ideal of A* if and only if I is downward closed, i.e.,

$$\text{if } x \leq y \text{ and } y \in I, \text{ then } x \in I.$$

Lemma 3.3.

(1) For $a, b \in \hat{\mathcal{T}}$ define the following relation.

$$a \leq b \text{ iff } \begin{cases} a = \langle n, \lambda \rangle, b = \langle n, \mu \rangle \text{ and } \lambda \leq \mu \text{ or} \\ a = 1 \text{ or} \\ b = 0. \end{cases}$$

Then \leq is a partial order over \hat{L} .

(2) If for each $j \in J$, \mathcal{I}_j is an ideal of \mathcal{T} , let

$$D = \bigcup_{j \in \omega} \{j\} \times \mathcal{I}_j \cup \{1\}.$$

Then D is an order-ideal of \hat{L} with respect to the order defined in (1).

Definition 3.4. The class \mathbf{M} of $\mathcal{SAL}_{\mathcal{T}}$ -matrices is defined as follows. For any family $\langle \mathcal{I}_j : j \in \omega \rangle$ of ideals of \mathcal{T} , we define a matrix $\mathcal{M} = \langle \hat{L}, D \rangle \in \mathbf{M}$.

(1)

$$a \vee b = \begin{cases} 1 & , \text{ if } a \in D \text{ or } b \in D, \\ 0 & , \text{ otherwise.} \end{cases}$$

(2)

$$a \wedge b = \begin{cases} 1 & , \text{ if } a \in D \text{ and } b \in D, \\ 0 & , \text{ otherwise.} \end{cases}$$

(3)

$$a \rightarrow b = \begin{cases} 1 & , \text{ if } a \notin D \text{ or } b \in D, \\ 0 & , \text{ otherwise.} \end{cases}$$

(4)

$$a^\circ = \begin{cases} 0 & , \text{ if } a \in \hat{\mathcal{T}}, \\ 1 & , \text{ if } a \in \{0, 1\}. \end{cases}$$

(5)

$$f_\lambda a = \begin{cases} \langle n, \lambda \rangle & , \text{ if } a = \langle n, \mu \rangle \in \hat{\mathcal{T}}, \\ a & , \text{ if } a \in \{0, 1\}. \end{cases}$$

(6)

$$\neg a = \begin{cases} 0 & , \text{ if } a = 1, \\ 1 & , \text{ if } a = 0, \\ \langle n, \sim \mu \rangle & , \text{ if } a = \langle n, \mu \rangle \in \hat{\mathcal{T}}. \end{cases}$$

Theorem 3.4. $\mathcal{M} = \langle \hat{L}, D \rangle$ is a matrix model for \mathcal{SAL}_τ .

Proof. The proof is similar to the proof of theorem 3.2 and is based on the fact that for any $a = \langle n, \mu \rangle \in \hat{\mathcal{T}}$,

$$\begin{aligned} a \in D & \quad \text{if and only if} \quad \mu \in \mathcal{I}_n, \\ f_\lambda a \in D & \quad \text{if and only if} \quad \lambda = f_\lambda \mu \in \mathcal{I}_n, \text{ for all } \lambda \in \mathcal{T}, \\ \neg a \in D & \quad \text{if and only if} \quad \sim \mu \in \mathcal{I}_n. \end{aligned}$$

□

Lemma 3.5. Let Δ be a maximal non-trivial consistent set of formulas and $p_i \in \mathcal{P}$. Then there exists an elementary matrix model $\mathcal{M} = \langle L, D \rangle$ and a valuation

$v : \mathcal{P} \rightarrow L$ such that for any hyperliteral formula A whose propositional letter is p_i ,

$$v^*(A) \in D \quad \text{if and only if} \quad A \in \Delta.$$

Proof. We let

$$\kappa_0 = \sup\{\lambda : p^\circ \notin \Delta \text{ and } S_\lambda(p) \in \Delta\},$$

$$D = \{\lambda : \lambda \leq \kappa_0\} \cup \{1\}$$

and

$$\lambda_0 = \sup\{\lambda : p^\circ \notin \Delta \text{ and } T_{(\lambda, \kappa_0)}(p) \in \Delta\}.$$

Define

$$v(p_i) = \begin{cases} 1 & , \text{ if } p_i \in \Delta \text{ and } p_i^\circ \in \Delta, \\ 0 & , \text{ if } \neg p_i \in \Delta \text{ and } p_i^\circ \in \Delta, \\ \lambda_0 & , \text{ if } p_i^\circ \notin \Delta, \end{cases}$$

and for any propositional letter $q \neq p_i$, $v(q)$ is defined arbitrarily.

$$(1) A = \neg^m p_i$$

Since Δ is maximal non-trivial and p_i° is complex, by theorem 2.1, we have two cases, either p_i° or its negation belongs to Δ .

Assume first that $p_i^\circ \in \Delta$. An easy induction on m will prove that

$$v^*(\neg^m p_i) \in D \text{ if and only if } \neg^m p_i \in \Delta.$$

The second case is if $p_i^\circ \notin \Delta$, or equivalently, $\neg(p_i^\circ) \in \Delta$. In this case,

$$v^*(\neg^m p_i) \in D \text{ iff } \neg^m v^*(p_i) \in D \text{ iff } \sim^m \lambda_0 \leq \kappa_0.$$

By axiom \neg_4 , it is clear that we need to worry only about $m \leq N$.

If $\sim^m \lambda_0 \leq \kappa_0$, then

$$\vdash T_{(\lambda_0, \kappa_0)}(p_i) \rightarrow \neg^m p_i,$$

and since $T_{(\lambda_0, \kappa_0)}(p_i) \in \Delta$, $\neg^m p_i \in \Delta$.

If $\sim^m \lambda_0 \not\leq \kappa_0$, then

$$\vdash T_{(\lambda_0, \kappa_0)}(p_i) \rightarrow \neg(\neg^m p_i \wedge \neg^m p_i),$$

so $\neg(\neg^m p_i \wedge \neg^m p_i) \in \Delta$, and thus $\neg^m p_i \notin \Delta$, since otherwise $\neg^m p_i \wedge \neg^m p_i \in \Delta$ and Δ would be trivial.

So in any case,

$$v^*(\neg^m p_i) \in D \text{ if and only if } \neg^m p_i \in \Delta.$$

$$(2) A = f_\lambda B, \text{ where } B \text{ is hyperliteral and its propositional letter is } p_i.$$

Again we study two cases, the first is if $p_i^\circ \in \Delta$. In this case, we observe that for any hyperliteral formula B with propositional letter p_i , $v^*(B) \in \{0, 1\}$, so $f_\lambda v^*(B) = v^*(B)$. Then

$$v^*(f_\lambda B) \in D \text{ iff } f_\lambda v^*(B) = v^*(B) \in D \text{ iff } B \in \Delta,$$

the last equivalence by induction hypothesis.

But then, by axiom τ_6 , the last is equivalent to $A = f_\lambda B \in \Delta$.

The second case is if $\neg(p_i^\circ) \in \Delta$. In this case, $v(p_i) = \lambda_0 \in \mathcal{T}$, so $v^*(B) \in \mathcal{T}$, and thus, $f_\lambda v^*(B) = \lambda$. So $v^*(f_\lambda B) \in D \text{ iff } f_\lambda v^*(B) \in D \text{ iff } \lambda \leq \kappa_0$.

If $\lambda \leq \kappa_0$, then

$$\vdash f_{\kappa_0} p_i \rightarrow f_{\lambda} p_i,$$

and since also

$$\vdash S_{\kappa_0} p_i \rightarrow f_{\kappa_0} p_i,$$

using modus ponens and a suitable tautology, $f_{\lambda} p_i \in \Delta$.

If $\lambda \not\leq \kappa_0$, then

$$\vdash S_{\kappa_0} p_i \rightarrow \neg(f_{\lambda} p_i \wedge f_{\lambda} p_i),$$

so $\neg(f_{\lambda} p_i \wedge f_{\lambda} p_i) \in \Delta$, and thus $f_{\lambda} p_i \notin \Delta$, since otherwise $f_{\lambda} p_i \wedge f_{\lambda} p_i \in \Delta$, contradicting the non-triviality of Δ .

(3) $A = \neg^m f_{\lambda} B$.

The case when $p_i^{\circ} \in \Delta$ can be proved by induction on m as in item (1).

The case when $\neg(p_i^{\circ}) \in \Delta$ can be reduced to the previous one observing that

$$v^*(\neg^m f_{\lambda} B) = v^*(f_{\sim^m \lambda} p_i) = v^*(f_{\sim^m \lambda} B),$$

and that by lemma 2.2, 1), $B^{\circ} \in \Delta$ if and only if $p_i^{\circ} \in \Delta$ so, using axiom τ_2 ,

$$\neg^m f_{\lambda} B \in \Delta \text{ iff } f_{\sim^m \lambda} B \in \Delta.$$

□

Theorem 3.6. *Let Δ be a maximal non-trivial consistent set of formulas. Then there is a matrix semantics $\mathcal{M} = \langle \hat{L}, D \rangle \in \mathbf{M}$ and an valuation $u : \mathcal{P} \rightarrow \hat{L}$ such that for any formula A ,*

$$u^*(A) \in D \text{ if and only if } A \in \Delta.$$

Proof. For each $j \in \omega$, let

$$\kappa_j = \sup\{\lambda : p_j^{\circ} \notin \Delta \text{ and } S_{\lambda}(p_j) \in \Delta\},$$

$$\mathcal{I}_j = \{\lambda : \lambda \leq \kappa_j\}$$

and

$$\lambda_j = \sup\{\lambda : p_j^{\circ} \notin \Delta \text{ and } T_{(\lambda, \kappa_j)}(p_j) \in \Delta\}.$$

Let

$$D = \bigcup_{j \in \omega} \{j\} \times \mathcal{I}_j \cup \{1\}.$$

Finally define

$$u : \mathcal{P} \rightarrow \hat{L}$$

as follows.

$$u(p_i) = \begin{cases} 1 & , \text{ if } p_i \in \Delta \text{ and } p_i^\circ \in \Delta, \\ 0 & , \text{ if } \neg p_i \in \Delta \text{ and } p_i^\circ \in \Delta, \\ \langle i, \lambda_i \rangle & , \text{ if } p_i^\circ \notin \Delta. \end{cases}$$

We first observe that

$$v : \mathcal{P} \longrightarrow L$$

defined by

$$v(p_i) = \begin{cases} 1 & , \text{ if } u(p_i) = 1, \\ 0 & , \text{ if } u(p_i) = 0, \\ \lambda_i & , \text{ otherwise,} \end{cases}$$

and for $i \neq j$,

$$v(p_j) = \lambda_j,$$

is a valuation from \mathcal{P} into L like the one we defined in lemma 3.5. Moreover, if A is hyperliteral with propositional letter p_i , then

$$u(p_i) \in D \text{ if and only if } v(p_i) \in \mathcal{I}_i,$$

and thus, for all hyperliteral A with propositional letter p_i ,

$$u^*(A) \in D \text{ if and only if } v^*(A) \in \mathcal{I}_i.$$

We can now prove by induction that

$$u^*(A) \in D \text{ if and only if } A \in \Delta.$$

(1) Let A be hyperliteral, with propositional letter p_i . Then

$$u^*(A) \in D \text{ if and only if } v^*(A) \in D_i,$$

and thus by lemma 3.5 applied to v , this occurs if and only if $A \in \Delta$.

(2) $A = B^\circ$ for some formula B .

$$\begin{aligned} u^*(B^\circ) \in D & \text{ iff } ((u^*(B))^\circ) \in D \\ & \text{ iff } (u^*(B))^\circ = 1 \\ & \text{ iff } u^*(B) \in \{0, 1\} \\ & \text{ iff } B \text{ is complex} \end{aligned}$$

and by lemma 2.2 (2), the latter implies that $\vdash B^\circ$ so $A = B^\circ \in \Delta$.

If $B^\circ \in \Delta$ and B is hyperliteral with propositional letter p_i , then by lemma 2.2, (1), a), $p_i^\circ \in \Delta$, so $u(p_i) \in \{0, 1\}$, so $u^*(B) \in \{0, 1\}$ and thus

$$u^*(B^\circ) = 1 \in D.$$

- (3) If $A = B \wedge C$, $A = B \vee C$ or $A = B \rightarrow C$, the proof is straightforward.
 (4) $A = \neg B$, for some complex formula B .

$$u^*(A) \in D \text{ iff } \neg u^*(B) \in D \text{ iff } u^*(B) \notin D \text{ iff } B \notin \Delta,$$

The last equivalence is by induction hypothesis. Now since by lemma 2.2, (2), $\neg(B^\circ) \in \Delta$, $B \notin \Delta$ if and only if $\neg B \in \Delta$.

- (5) $A = f_\lambda B$, for some complex formula B .

$$u^*(A) \in D \text{ iff } 1 = u^*(A) = f_\lambda u^*(B) = u^*(B) \text{ iff } B \in \Delta,$$

this last equivalence by inductive hypothesis. But by axiom τ_6 , this occurs if and only if $f_\lambda B \in \Delta$.

□

Lemma 3.7. *Any non-trivial set of formulas is contained in a maximal non-trivial set of formulas.*

Proof. This is proved in the usual way. See for instance [4]. □

The following theorem is proved in the usual way using this lemma and theorem 3.6.

Theorem 3.8. *For any non-trivial set Σ of formulas of \mathcal{SAL}_τ , there is a matrix $\mathcal{M} \in \mathbf{M}$ such that*

$$\mathcal{M} \models \Sigma.$$

or equivalently

$$\Gamma \vdash A \text{ if and only if } \Gamma \models_{\mathbf{M}} A.$$

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