TOPOLOGICALLY INSEPARABLE FUNCTIONS II: INFINITARY CASE

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Abstract. Given a set A and a function $\boldsymbol{f} : A \longrightarrow A$, we study the set of all functions $g : A \longrightarrow A$ that are continuous for all topologies for which $\boldsymbol{f}$ continuous.

We prove that in a sense to be made precise in the text, for any essentially infinitary function $\boldsymbol{f}$, any non–constant such g equals $\boldsymbol{f}^n$, for some $n \in \mathbb{N}$. We also prove a similar result for the clone of n–ary functions from $A^n \longrightarrow A$.

Introduction

This is the second of a two part paper. Some results of the previous paper will be stated here without proof. The reader should consult [1] for the proofs as well as for motivations of the problem.

In this paper we study the case of a function $\boldsymbol{f} : A \longrightarrow A$ where A is an infinite set. In the first section we will restrict our attention to unary functions topologically inseparable from $\boldsymbol{f}$. In the last section we will study the clone of finitary functions inseparable from $\boldsymbol{f}$. We define

$$S(A, \boldsymbol{f}) \;=\; \{g : A \to A : g \text{ is continuous for all topologies for which } \boldsymbol{f} \text{ is continuous}\}.$$

$S(A, \boldsymbol{f})$ is a semigroup that contains all constant functions as well as the iterates $\boldsymbol{f}^n$ of $\boldsymbol{f}$. We will let $S(A, \boldsymbol{f})^*$ be the set of all non constant elements of $S(A, \boldsymbol{f})$, and since constant functions are always continuous, we will usually work in the context of this set.We also let $S_0^c(A, \boldsymbol{f}) = S_0(A, \boldsymbol{f}) \cup \{\text{constant functions on A}\}$.

The main result of this paper is that if $\boldsymbol{f}$ is infinitary in a way that we will make precise, then $S_0^c(A, \boldsymbol{f})$ and $S(A, \boldsymbol{f})$ coincide.

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Preliminary Results

We first recall that the relation on A defined by

$$x \sim y \text{ iff there exist } n, m \in \mathbb{N} \text{ such that } \boldsymbol{f}^n(x) = \boldsymbol{f}^m(y)$$

is an equivalence relation. The equivalence classes are called *connected components* or *orbits*. The reader can easily figure out how the orbits look. Some examples appear in the following Diagram.

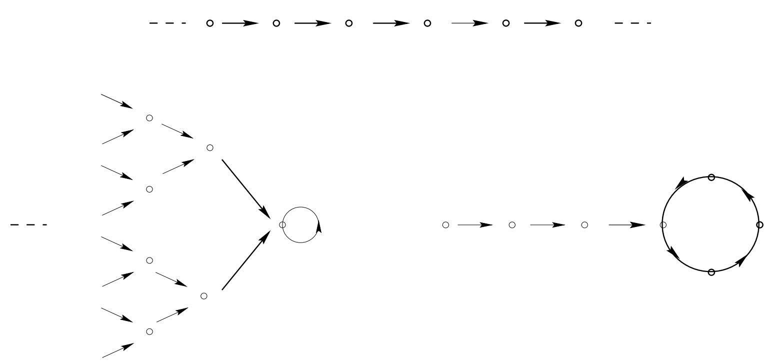

Diagram 1

If there exists $x \in A$ and $n \in \mathbb{N}$ such that $\boldsymbol{f}^n(x) = x$, we will say that the orbit of x has a *cycle*. The number of points in the cycle is called its *length*. An orbit cannot have more than one cycle. On each orbit we can define the following preorder relation

$$x \preceq y \quad \text{iff} \quad \boldsymbol{f}^n(x) = y, \qquad \text{for some } n \in \mathbb{N}\,.$$

If we restrict the relation to points on the branches, it is a partial order. We will refer to it as the *partial order induced by* $\boldsymbol{f}$ *on the orbit.*

By a *chain* in this order we understand a sequence of points $\{x_i : i \in I\}$, where I is an interval of integers, such that for each $i \in I$, $x_{i+1} = \boldsymbol{f}(x_i)$. In particular, a chain is a $\preceq$–totally ordered set such that if $x \prec y$ are two points in it, then $y = \boldsymbol{f}^n(x)$. We will say that n is the *distance from* x *to* y.

An orbit is *unbounded* if it contains a chain that is isomorphic to $\mathbb{Z}, \mathbb{N}$ or $\mathbb{N}^-$, where $\mathbb{N}^-$ is $\mathbb{N}$ with reversed order; otherwise the orbit is *bounded*. Notice that a bounded orbit need not be finite. Also, an unbounded orbit may contain a cycle.

A function $\boldsymbol{f}$ is *totally bounded* if all its orbits are bounded, the set of the lengths of its cycles is bounded and $\boldsymbol{f}$ does not have arbitrarily long chains.

If $\Sigma \subseteq \mathcal{P}(A)$, $\tau(\boldsymbol{f}, \Sigma)$ is the minimal topology that contains Σ and under which $\boldsymbol{f}$ is continuous. It is the topology whose base is the set

$$\{\boldsymbol{f}^{-n_1}(\mathcal{U}_1) \cap \cdots \cap \boldsymbol{f}^{-n_k}(\mathcal{U}_k) : k \in \mathbb{N}, n_i \in \mathbb{N}, \mathcal{U}_i \in \Sigma, \\ \text{for } i = 1, \ldots, k\} \cup \{\emptyset, A\}.$$

Sketch of the proof of the Main Theorem. The proof proceeds by classifying the functions $\boldsymbol{f}$ in various "levels of unboundedness" and then giving a proof that if $\boldsymbol{f}$ belongs to a level that is in some sense unbounded, then the only non–constant functions that are topologically inseparable from $\boldsymbol{f}$ are the trivial ones, that is, its iterates. We are left with functions that are bounded in every interesting way, these are essentially finitary and can be treated with the tools developed in [1]. The cases are as follows.

Let $\boldsymbol{f} : A \longrightarrow A$ be a function.

(1) There is an orbit that contains an infinite ascending maximal chain isomorphic either to $\mathbb{Z}$ or to $\mathbb{N}$. (Section 1.)

(2) No orbit contains an infinite ascending chain but there is an orbit that contains an infinite descending maximal chain isomorphic to $\mathbb{N}^-$.

    In this case all orbits contain a cycle. (Section 2.)

(3) No orbit contains an infinite chain but there are arbitrarily long chains. (Section 3.) There are two sub–cases.
    (a) There is a single orbit that contains arbitrarily long (yet finite) chains.
    (b) No single orbit contains arbitrarily long chains.
    In this case there is a sequence of orbits $\{\mathcal{O}_n : n \in \mathbb{N}\}$ such that for each n, $\mathcal{O}_n$ contains a chain of length greater than n. (Section 4.)

(4) The lengths of all chains is bounded and there are orbits whose cycle has arbitrarily large length.

    In this case we have a sequence of orbits $\{\mathcal{O}_n : n \in \mathbb{N}\}$ such that for each n, the cycle of $\mathcal{O}_n$ has length greater than n.

(5) The function is totally bounded, that is, the lengths of all chains and all cycles of $\boldsymbol{f}$ are bounded by a fixed number. (Reduced to previous paper [1].)

Even though the proofs are somewhat complicated, there is a general pattern. Given a function $g \in S(A, \boldsymbol{f})$ and two points, a and b, we build an appropriate topology τ such that these two points belong to the same basic open sets, we will say that a and b are τ–*inseparable*. Then, knowing what the value of $g(a)$ is, we prove that the inseparability of the two points forces $g(b)$ to take some desired value.

The following lemmas were proven in [1] and will allow us to simplify our study.

Lemma 1. ([1] Lemma 1). *Let $g \in S(A, \boldsymbol{f})^*$, then for all $x \in A$, there exists an $n \in \mathbb{N}$ such that $g(x) = \boldsymbol{f}^n(x)$.*

This n depends on x. The purpose of this study is to find necessary and sufficient conditions for the uniqueness of n.

Lemma 2. ([1] Lemma 2). *If $g \in S(A, \boldsymbol{f})^*$, then x and $g(x)$ belong to the same orbit.*

Lemma 3. ([1] Lemma 22). *Let $g \in S(A, \boldsymbol{f})^*$. If there is a point mapped by g into a point in the same branch (i.e. not in the cycle of its orbit), then there exists an $n \in \mathbb{N}$ such that for any point x not in the same orbit, $g(x) = \boldsymbol{f}^n(x)$.*

Lemma 4. *If $g \in S(A, \boldsymbol{f})^*$, then g is monotone, i.e. if $x \prec y$, then $g(x) \preceq g(y)$.*

Proof. Let $\mathcal{U} = \{\boldsymbol{f}^n(g(x)) : n \in \mathbb{N}\}$ and let us consider the topology $\tau(\boldsymbol{f}, \mathcal{U})$. The important condition here is that $g(x)$ is the $\preceq$–least element of $\mathcal{U}$. Moreover, for any k, $\boldsymbol{f}^{-k}(\mathcal{U})$ is a $\preceq$–upward closed set.

We observe that if $n \leq m$, then $\boldsymbol{f}^{-n}(\mathcal{U}) \subseteq \boldsymbol{f}^{-m}(\mathcal{U})$, so

$$\boldsymbol{f}^{-m_1}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-m_k}(\mathcal{U}) = \boldsymbol{f}^{-min\{m_1,\ldots,m_k\}}(\mathcal{U}).$$

so the $\boldsymbol{f}^{-k}(\mathcal{U})$ (together with A), are a base for $\tau(\boldsymbol{f}, \mathcal{U})$.

Now $x \in g^{-1}(\mathcal{U})$, so if this latter is A, $y \in g^{-1}(\mathcal{U})$ and thus $g(x) \preceq g(y)$. If not then $x \in \boldsymbol{f}^{-m}(\mathcal{U})$ for some m, and since $x \prec y$, $y \in \boldsymbol{f}^{-m}(\mathcal{U})$, so $y \in g^{-1}(\mathcal{U})$ and again we get that $g(x) \preceq g(y)$. □

Lemma 5. *Suppose that $\boldsymbol{f}$ has an infinite descending chain $\{x_i : i \in \mathbb{N}^-\}$ and let $g : A \longrightarrow A$ be a non–constant function such that for some $a \in A$,*

$$a \prec g(x_i) \text{ for all } i \in \mathbb{N}^-.$$

Then $g \notin S(A, \boldsymbol{f})^*$.

Proof. Suppose on the contrary that $g \in S(A, \boldsymbol{f})^*$ and let $a \in A$ be the $\preceq$–least point such that for some $i_0 \in I$, $g(x_{i_0}) = a$. By Lemma 4, for all $i \leq i_0$, $a \preceq g(x_i) \preceq g(x_{i_0}) = a$.

Let $\mathcal{U} = A - \{a\}$ and consider the topology $\tau(\boldsymbol{f},\mathcal{U})$. The idea here is that all basic open sets contain a "tail" of the infinite descending chain.

Observe that $g^{-1}(\mathcal{U}) \neq \emptyset$, A, so

$$g^{-1}(\mathcal{U}) = \bigcup_{j\in J} \bigcap_{k\in F_j} \boldsymbol{f}^{-k}(\mathcal{U}),$$

where $J \subseteq \mathbb{N}$ and F_j is finite for all $j \in J$.

It is clear that $x_i \notin g^{-1}(\mathcal{U})$ for all $i \leq i_0$. Now let

$$\boldsymbol{f}^{-m_1}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-m_k}(\mathcal{U}),$$

with $m_1 < m_2 < \cdots < m_k$, be one of the basic components of $g^{-1}(\mathcal{U})$. Then

$$\mathcal{V} = \bigcap_{k=0}^{m_k} \boldsymbol{f}^{-k}(\mathcal{U}) \subseteq \boldsymbol{f}^{-m_1}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-m_k}(\mathcal{U}),$$

so the points in $\mathcal{V}$ are such that their distance to a is less than m_k. We now recall that the x_i's are an infinite descending chain so we may choose a point x_r that is at a greater distance from a, for instance, such that $\boldsymbol{f}^{m_k+1}(x_r) = x_{i_0} \preceq a$. Then $x_r \in \mathcal{V} \subseteq g^{-1}(\mathcal{U})$ and thus $g(x_r) \neq m$, contradicting our assumption. □

Lemma 6. *Let* $\boldsymbol{f} : A \longrightarrow A$ *be unbounded. Then for any* $g \in S(A, \boldsymbol{f})^*$, *the set*

$\{n \in \mathbb{N} : g(x) = \boldsymbol{f}^n(x)$ *for some* $x \in A$ *and it is the least such an* $n\}$

is bounded.

Proof. For a proof by contradiction, assume that there exists a strictly increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of non–negative integers and a sequence $\{x_i\}_{i\in\mathbb{N}}$ of points in A such that

$$g(x_i) = \boldsymbol{f}^{n_i}(x_i),$$

where $n_i < l_i$, if x_i belongs to a cycle of length l_i.

Since $\boldsymbol{f}$ is not totally bounded, there are three cases to be considered.

(1) The points x_i all belong to an infinite chain. If the chain is infinite descending, then by Lemma 5, the sequence $\{g(x_i)\}_{i\in I}$ contains an infinite descending subsequence. If the chain is infinitely ascending then $\{g(x_i)\}_{i\in\mathbb{N}}$ contains an infinitely ascending subsequence. In any case, by refining the sequence $\{x_i\}_{i\in\mathbb{N}}$,

we can assume that for all $i \in I$,

$$g(x_i) \prec x_{i+1} \prec g(x_{i+1}) \ .$$

(2) The points of the sequence $\{x_i\}_{i\in\mathbb{N}}$ (nor any of its subsequences) do not belong to a single infinite chain nor to the cycles of their orbits. This case implies that there exist arbitrarily long finite chains.
(3) The points x_i, and thus the points $g(x_i)$, belong to the cycle of their orbit. In this case there have to exist infinitely many orbits with arbitrarily long cycles. We may assume that each x_i belongs to a different orbit.

We will now build a topology such that the basic open sets do not contain arbitrarily large gaps, so that two points separated by a large enough distance cannot both be out of an open set.

Let $\mathcal{U} = A - \{g(x_i) : i \in \mathbb{N}\}$ and consider the topology $\tau(\boldsymbol{f},\mathcal{U})$. We can arrange by deleting some of the x_i's that $g^{-1}(\mathcal{U}) \neq \emptyset$ so

$$g^{-1}(\mathcal{U}) = \bigcup_{j\in J} \bigcap_{k\in F_j} \boldsymbol{f}^{-k}(\mathcal{U}),$$

where $J \subseteq \mathbb{N}$ and F_j is finite for all $j \in J$.

It is clear that $x_i \notin g^{-1}(\mathcal{U})$ for all $i \in \mathbb{N}$. Now let

$$\boldsymbol{f}^{-m_1}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-m_k}(\mathcal{U}),$$

with $m_1 < m_2 < \cdots < m_k$, be one of the basic components of $g^{-1}(\mathcal{U})$ and choose x_i be such that $g(x_i) = \boldsymbol{f}^{n_i}(x_i)$, where $n_i > m_k$. Then

$$x_i \in \bigcap_{j=0}^{m_k} \boldsymbol{f}^{-j}(\mathcal{U}) \subseteq \boldsymbol{f}^{-m_1}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-m_k}(\mathcal{U}) \subseteq g^{-1}(\mathcal{U}),$$

a contradiction. $\square$

Corollary 7. *Assume $g \in S(A, \boldsymbol{f})^*$. Then there exists an $n \in \mathbb{N}$ such that for infinitely many points $x \in A$, $g(x) = \boldsymbol{f}^n(x)$.*

Let us now consider a function $\boldsymbol{f}$ with an orbit that contains a cycle of length l, as depicted in Diagram 2.

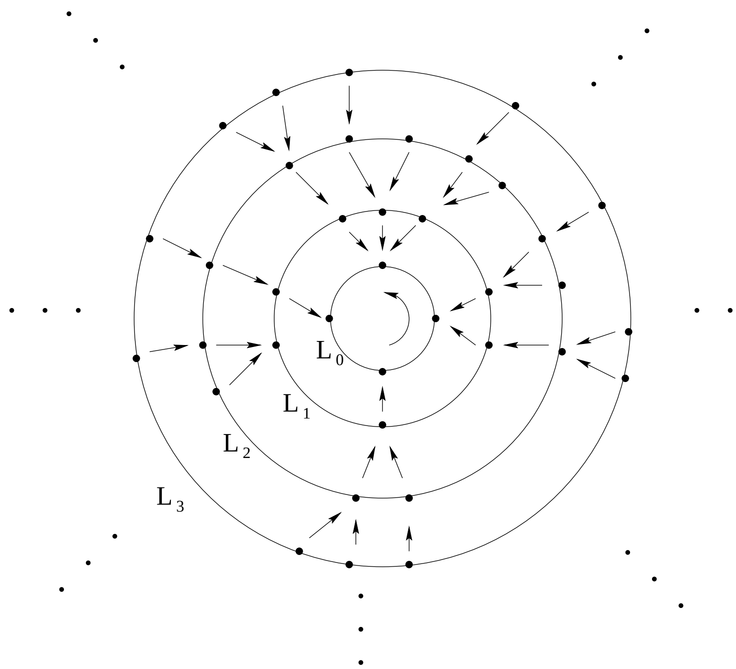

Diagram 2

We can define levels, this time they will look like concentric circles.

$$\begin{aligned} L_0 &= \{x : x = \boldsymbol{f}^l(x)\} = \text{ points in the cycle.} \\ L_1 &= \{x : \boldsymbol{f}(x) \in L_0\} - L_0 \\ &\vdots \\ L_{n+1} &= \{x : \boldsymbol{f}(x) \in L_n\} \\ &\vdots \end{aligned}$$

This is a base for a topology for which $\boldsymbol{f}$ is continuous.

Lemma 8. ([1] Lemma 20). *If $\boldsymbol{f} : A \to A$ has an orbit $\mathcal{O}$ containing a cycle and if $g \in S(A, \boldsymbol{f})^*$, then g maps levels of $\mathcal{O}$ into levels of $\mathcal{O}$.*

1. There exists an infinite ascending chain

Suppose there exists an orbit $\mathcal{O}$ that contains no cycle and has a maximal chain $\mathcal{C}$ that is infinite. Then $\mathcal{C}$ is isomorphic with $\mathbb{Z}$ or $\mathbb{N}$. As we will see, this last option is irrelevant in what follows.

The function $\boldsymbol{f}$ will look as in Diagram 3. The c_i, $i \in \mathbb{Z}$ (or $i \in \mathbb{N}$ if we assume that the maximal chain has no initial point), are the elements of the maximal chain.

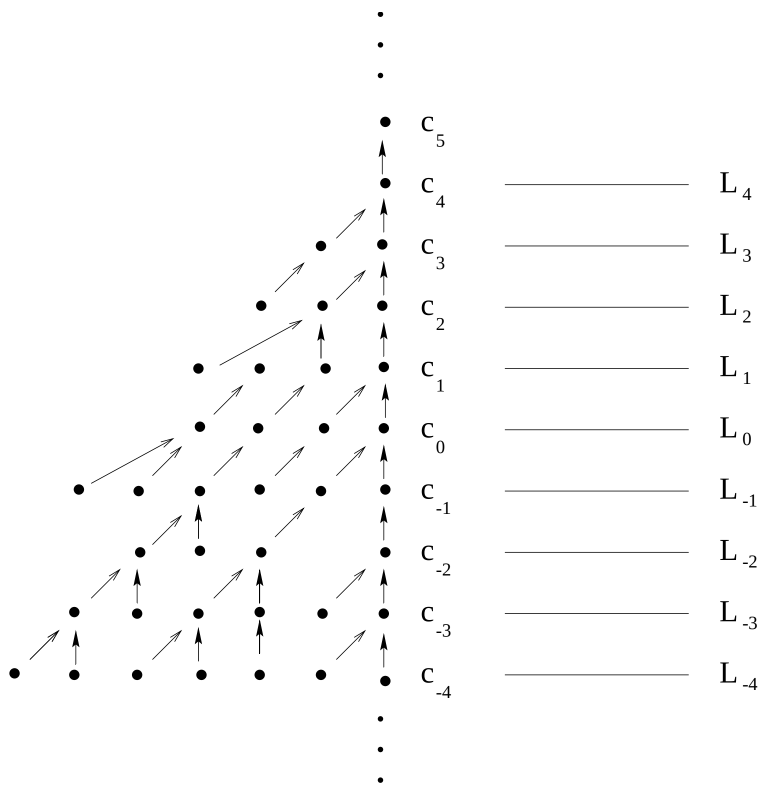

Diagram 3

We can define levels in the following way

$$L_n = \{x : \boldsymbol{f}^k(x) = c_{n+k}\} \ .$$

Notice that there is a denumerable number of levels in each orbit. Of course the levels may have any cardinality.

Lemma 9. *Let* $\boldsymbol{f}$ *be as in Diagram 2. If* $g \in S(\mathcal{O}, \boldsymbol{f})^*$, *then for all* k, $g(L_k) \subseteq L_m$, *for some* m*.*

Proof. The idea of the proof is that the levels are basic open sets in the topology generated by them and since each point a belongs to a single level, whenever a is in an open set, the whole level must be included in it.

It is clear that for $i \neq j$, $L_i \cap L_j = \emptyset$ and that $\boldsymbol{f}^{-m}(L_n) = L_{n-m}$ so $\Sigma = \{L_i : i \in \mathbb{Z}\}$ is a base for the topology $\tau(\boldsymbol{f}, \Sigma)$.

Let L_m be the level of $g(a)$, for certain $a \in L_k$, for some $k \in \mathbb{Z}$. Then $a \in g^{-1}(L_m) = \bigcup_{j \in J} L_j$, and since a is in a unique level L_k, $k \in J$, $g(L_k) \subseteq L_m$. $\square$

The last lemma shows that the values of the function g are determined by the values of g on $\mathcal{C}$. Thus for functions without cycles, it is enough to study all functions $\boldsymbol{f} : \mathbb{Z} \longrightarrow \mathbb{Z}$ continuous for all topologies τ such that the successor function s is τ–continuous, i.e. $S(\mathbb{Z}, s)$. It is also clear that the above remains true if $\mathcal{C}$ has a first element; in this case we must study $S(\mathbb{N}, s)$.

Theorem 10. *If* $g \in S(\mathbb{Z}, s)^*$, *then* $g = s^n$, *for some integer* $n \in \mathbb{N}$.

Proof. We consider only the points less than 0. By Corollary 7 there are infinitely many $x \in \mathbb{Z}$, $x \leq 0$, such that for some fixed n, $g(x) = s^n(x)$. We will assume from now on that this is the minimal such an n.

There are two steps in the proof, we first check that for the infinite interval of all points less than a fixed point, the function g takes the value $s^n(x)$, i.e. g "eventually" equals s^n.

The second step is to check that this forces all other points to behave in the same way.

In order to prove the first assertion, observe that since n is minimal, there is a finite number of points for which g takes a value less than n, so they belong to a bounded interval $[a, b]$.

Let c be a point in the infinite interval $(-\infty, a)$ such that $g(c) = s^n(c)$ and $g(c-1) = s^{n+1}(c)$. We observe that unless for all points in $(-\infty, a)$ $g = s^n$, there exists one such point since by monotonicity,

$$c - 1 + n = s^n(c-1) < g(c-1) \leq g(c) = s^n(c) = c + n.$$

Choose a point $d < c$ such that $d + m = c$, with $m > 2$ such that $g(d) = s^n(d)$, and define

$$\mathcal{U} = \{g(c) - j(m-1) : j \in \mathbb{N}\}.$$

So we see that all elements in $\mathcal{U}$ are congruent modulo $m - 1$.

It is immediate that for $k_1, k_2 \in \mathbb{N}$, $\boldsymbol{f}^{-k_1}(\mathcal{U}) \cap \boldsymbol{f}^{-k_2}(\mathcal{U}) = \emptyset$ unless $k_1 \equiv k_2 \pmod{m-1}$, so the basic open sets of $\tau(s,\mathcal{U})$ are the sets $\boldsymbol{f}^{-k}(\mathcal{U})$.

As a consequence, given any positive integer k, if $x \in \boldsymbol{f}^{-k}(\mathcal{U})$ then $x - m + 1 \in \boldsymbol{f}^{-k}(\mathcal{U})$, in other words, the topology $\tau(s,\mathcal{U})$ cannot separate x and $x - m + 1$.

Next we observe that $c - 1 \in g^{-1}(\mathcal{U})$ and since g is continuous, $d = c - m = c - 1 - m + 1 \in g^{-1}(\mathcal{U})$ which is a contradiction since

$$g(d) = d + n = (c - m) + n = g(c) - m \notin \mathcal{U}.$$

This proves that for all points less than some c, we have $g = s^n$.

Let d be the least point such that $g(d) = s^m(d)$, for $m > n$. Obviously $c < d$, and define

$$\mathcal{V} = \{g(d) - jr : j \in \mathbb{N}\} = \{d + m - jr : j \in \mathbb{N}\},$$

where r and $m-n$ are relatively prime. The remarks made in the previous paragraphs about $\mathcal{U}$ can be repeated here for $\mathcal{V}$, so the topology $\tau(s, \mathcal{V})$ does not separate the points d and $d-r$ and since $d \in g^{-1}(\mathcal{V})$, so is $d-r$. But this is a contradiction since

$$g(d) = s^n(d) = d + n \notin \mathcal{V},$$

or else

$$d + n \equiv d + m \pmod{r},$$

which implies that r divides $m-n$, contrary to our choice of r.

This completes the proof that there cannot exist a point $d > c$ such that $g(d) \neq s^n(x)$.

□

Corollary 11. *If $g \in S(\mathbb{N}, s)^*$, then $g = s^n$, for some integer $n \in \mathbb{N}$.*

Proof. The proof of Theorem 10 can be adapted to the case $S(\mathbb{N}, s)$ by simply considering all points greater than 0 and letting n be the maximal number for which there are infinitely many points for which $g = s^n$. Then proceed as in the theorem changing $<$ to $>$. □

Remark 1. *The first part of the proof, namely, that for all points less than some fixed point, the function $g = s^n$ also works in the case when there is a chain isomorphic to $\mathbb{N}^-$. This will be used several times later on.*

Theorem 12. *Let $\boldsymbol{f} : A \longrightarrow A$ be a function that has an infinite ascending chain. Then for all $g \in S(A, \boldsymbol{f})$, there exists an $n \in \mathbb{N}$ such that $g = \boldsymbol{f}^n$.*

Proof. Let the chain $\mathcal{C}$ belong to orbit $\mathcal{O}$. By Theorem 10 (or its immediate Corollary 11 if $\mathcal{C}$ has an initial point), there exists an $n \in \mathbb{N}$ such that $g \restriction \mathcal{C} = \boldsymbol{f}^n$. By Lemma 9 and the remarks following it, the values of g on $\mathcal{C}$ force that $g \restriction \mathcal{O} = \boldsymbol{f}^n$. Finally, by Lemma 3, for any point $x \notin \mathcal{O}$, $g(x) = \boldsymbol{f}^n(x)$, so the theorem is proved. □

2. There exists an infinite descending chain

Theorem 13. *Let $\boldsymbol{f} : A \longrightarrow A$ be a function that has an infinite descending chain. Then for all $g \in S(A, \boldsymbol{f})$, there exists an $n \in \mathbb{N}$ such that $g = \boldsymbol{f}^n$.*

Proof. We may assume that there is no infinite ascending chain, so the infinite descending chain $\mathcal{C}$ belongs to an orbit that contains a cycle. We will let l be the length of the cycle.

By remark 1 we know that there is a point $a \in \mathcal{C}$ such that for $x \preceq a$, $g(x) = \boldsymbol{f}^n(x)$. Let b be any point in the orbit such that $b \not\preceq a$.

We will now build a topology such that if b belongs to a basic open set, then so does some point $c \preceq a$.

Define
$$\mathcal{U} = \{x : x \preceq a\} \cup \{g(b)\}.$$
Then for any m, $\boldsymbol{f}^{-m}(\mathcal{U}) \cap \mathcal{C} \neq \emptyset$ and as a matter of fact, $\boldsymbol{f}^{-m}(\mathcal{U})$ contains an infinite descending subchain of $\mathcal{C}$.

Since $b \in g^{-1}(\mathcal{U})$, it is not empty, furthermore, we can arrange (by changing a) that $g^{-1}(\mathcal{U}) \neq A$ so

$$b \in \boldsymbol{f}^{-m_1}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-m_k}(\mathcal{U}),$$
for some $m_1 < m_2 < \cdots < m_k$.

By the remark in the previous paragraph, $\boldsymbol{f}^{-m_1}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-m_k}(\mathcal{U})$ contains infinitely many points in $\mathcal{C}$, so let c be one of these points such that $\boldsymbol{f}^n(c)$ belongs to the chain. Since $c \in g^{-1}(\mathcal{U})$

$$g(c) = \boldsymbol{f}^{m_1}(c) = \cdots = \boldsymbol{f}^{m_k}(c),$$
by our choice of c, $k = 1$ and $m_1 = n$, so $g(b) = \boldsymbol{f}^n(b)$. This finishes our proof. □

3. There are arbitrarily long finite chains

Theorem 14. *Let $\boldsymbol{f} : A \longrightarrow A$ be a function that has arbitrarily long chains. Then for each $g \in S(A, \boldsymbol{f})^*$, there exists an $n \in \mathbb{N}$ such that $g = \boldsymbol{f}^n$.*

Proof. We may assume that $\boldsymbol{f}$ has no infinite (ascending or descending) chain. As we mentioned before, there are two sub–cases.

Case 1. There is a sequence of arbitrarily long chains each of which belongs to a different orbit.

Let $\{\mathcal{C}_k : k \in \mathbb{N}\}$ be a set of chains such that $\mathcal{C}_k$ has length greater than k. Let a_k be the extreme of the chain $\mathcal{C}_k$, i.e., the point that has no predecessor. By Lemma 6, there exists an $n \in \mathbb{N}$ such that for infinitely many k 's, $g(a_k) = \boldsymbol{f}^n(a_k)$. This implies that for $k > n$, $g(a_k)$ belongs to the chain $\mathcal{C}_k$. We apply Lemma 3 to prove that $g = \boldsymbol{f}^n$.

Case 2. The arbitrarily long chains belong to a single orbit $\mathcal{O}$.

Since there is no infinite ascending chain, $\mathcal{O}$ must contain a cycle $\mathcal{C}$. Defining levels as in Lemma 8, we observe that since there are arbitrarily long chains, all levels are non–empty, so we have an infinite descending chain of levels. Then L_0 is a "fixed point" of $\boldsymbol{f}$, that is, $\boldsymbol{f}(L_0) = L_0$.

Moreover, by Lemma 8, the set of all levels are a base for a topology for which $\boldsymbol{f}$ is continuous, so if $g \in S(A, \boldsymbol{f})^*$, then g maps levels of $\mathcal{O}$ into levels of $\mathcal{O}$.

We finally observe that since $g(a)$ is determined by the level to which a belongs, this case is similar to the study of $S(\mathbb{N}^-, s)$, where s is the successor function and $s(0) = 0$. This was studied in the previous section. We can apply remark 1 and conclude that for all points that belong to a level beyond some fixed one L_{k_0}, $g(x) = \boldsymbol{f}^n(x)$.

For any point b not in those levels, consider the topology generated by

$$\mathcal{U} = \bigcup_{k_0 \le k} L_k \cup \{g(b)\}.$$

Then use the same argument used in Theorem 13. $\square$

4. There are arbitrarily long cycles

Lemma 15. *Let $\boldsymbol{f} : A \to A$ be the union of bounded orbits $\mathcal{O}_i$, $i \in \mathbb{N}$, such that the lengths l_i of the cycle in $\mathcal{O}_i$ verify $l_0 < l_1 < \cdots$. Suppose $g \in S(A, \boldsymbol{f})^*$. Then there exists an $n \in \mathbb{N}$ such that, $g = \boldsymbol{f}^n$.*

Proof. By Lemma 6, there exists an $n \in \mathbb{N}$ such that for infinitely many $x \in \mathbb{N}$, $g(x) = \boldsymbol{f}^n(x)$.

So let x be such that $g(x) \neq \boldsymbol{f}^n(x)$. For each $i \in \mathbb{N}$ we choose a single $x_i \in \mathcal{O}_i$ such that $g(x_i) = \boldsymbol{f}^n(x_i)$ (we may assume that it exists, if not, drop that orbit.)

Define $\mathcal{U}_i = \{\boldsymbol{f}^k(x_i) : k \in F_i\}$, where $\langle F_i : i \in \mathbb{N}\rangle$ is an enumeration of all finite subsets of $\mathbb{N}$, in which each finite subset appears infinitely many times. Next define

$$\mathcal{U} = \bigcup \mathcal{U}_i \cup \{g(x)\},$$

eliminating $\mathcal{U}_i$ if $x \in \mathcal{O}_i$. Observe that $\mathcal{U} \neq A$. If not for each i, $\mathcal{U}_i = \mathcal{O}_i$. A simple cardinality argument shows that this is impossible since the orbits get bigger and bigger and the finite sets appear infinitely many times in the sequence. Consider $\tau(f,\mathcal{U})$. Since g is continuous,

$$x \in g^{-1}(\mathcal{U}) = \bigcup_{i\in I} \bigcap_{j\in G_i} \boldsymbol{f}^{-j}(\mathcal{U}),$$

where $I \subseteq \mathbb{N}$, G_i is finite for all $i \in I$, so

$$x \in \boldsymbol{f}^{-n_1}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-n_k}(\mathcal{U}), \quad n_1 < n_2 < \cdots < n_k.$$

But then

$$g(x) = \boldsymbol{f}^{n_1}(x) = \boldsymbol{f}^{n_2}(x) = \cdots = \boldsymbol{f}^{n_k}(x).$$

On the other hand $\{n_1, n_2, \ldots, n_k\} = F_j$ for some j, (in fact, for infinitely many) so

$$x_j \in \boldsymbol{f}^{-n_1}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-n_k}(\mathcal{U}),$$

that is, $x_j \in g^{-1}(\mathcal{U})$ so

$$g(x_j) = \boldsymbol{f}^{n_1}(x) = \boldsymbol{f}^{n_2}(x) = \cdots = \boldsymbol{f}^{n_k}(x)$$

but $g(x_j) = \boldsymbol{f}^n(x)$, this implies that

$$n_i \equiv n \mod l_j, \quad i = 1, \ldots, k.$$

Since this is so for infinitely many different l_j's, the only possibility is that

$$n_1 = n_2 = \cdots = n_k = n.$$

□

5. Main Theorem

We will put together the partial results obtained in previous sections.

Theorem 16. *Let* $\boldsymbol{f} : A \to A$. *Then* $\boldsymbol{f}$ *is topologically isolated if and only if one of the following holds:*

(1) $\boldsymbol{f}$ *is not totally bounded or*
(2) $\boldsymbol{f}$ *is totally bounded and for any prime* p, *if* p^k *divides the length of one cycle of* $\boldsymbol{f}$, *then* p^k *divides the length of another cycle or*
(3) $\boldsymbol{f}$ *is a fixed point function such that the lengths of the two longest branches that belong to two different orbits differ at most by 1.*

Proof. The proof of this theorem is by cases whose proofs are scattered in the previous sections. Let $g \in S(A, \boldsymbol{f})$ be a non-constant function.

Suppose $\boldsymbol{f}$ is not totally bounded. Then there are four cases

(1) The function $\boldsymbol{f}$ has an infinite ascending chain. This case is handled by Theorem 12.

(2) The function $\boldsymbol{f}$ has an infinite descending chain. This case is handled by Theorem 13.
(3) The function $\boldsymbol{f}$ has arbitrarily long finite chains. This case is handled by Theorem 14.
(4) The function $\boldsymbol{f}$ has arbitrarily long cycles. This case is handled by Theorem 15.

Suppose $\boldsymbol{f}$ is totally bounded. We will reduce this case to the one studied in [1] by several simple observations.

All orbits contain a cycle.

If $\boldsymbol{f}$ has infinitely many orbits, since the lengths of the cycles are bounded, many of them will have the same length. We observe that Lemmas [16], [20], [21], [22], [26] and Theorems [17], [18], [19], [25] and [27], of [1], remain true for infinitely many cycles.

If an orbit has infinitely many points, all but a finite number belong to the branches, so by Lemma 8, the value of a a function $g \in S(A, \boldsymbol{f})$ depends on the level and the number of points in that level is of no interest.

The theorem then follows by Theorems [25] and [27] of [1]. □

6. The Clone

In this section we will study the functions of several variables that are continuous in the corresponding product topology whenever $\boldsymbol{f}$ is continuous. The class of all such functions constitutes a clone which we will denote $Clo(A, \boldsymbol{f})$. As usual, $Clo_k(A, \boldsymbol{f}) = \{f \in Clo(A, \boldsymbol{f}) : f \text{ is a } k\text{–ary function}\}$.

Let us first look at a function of two variables $g : A \times A \to A$ and assume $g \in Clo(A, \boldsymbol{f})$.

We will use the auxiliary function $\Delta g : A \to A$, the *diagonal* of g, defined by $\Delta g(x) = g(x, x)$. Also define the *slices* of g, that is, for fixed x_0, y_0, $g_{x_0}(x) = g(x_0, x)$ and $g^{y_0} = g(x, y_0)$.

Observe that for all x, y,

$$g_x(y) = g(x, y) = g^y(x)$$

and

$$g_x(x) = \Delta g(x) = g^x(x).$$

Since g is in $Clo(A, \boldsymbol{f})$, Δg, g_x and g^x are in $S(A, \boldsymbol{f})$, for all $x \in A$. This implies that they are either constant functions or iterates of $\boldsymbol{f}$.

Lemma 17. *If $g \in Clo_2(A, \boldsymbol{f})$ and Δg is constant function, then g is also a constant function.*

Proof. We will first prove that all sections are constant. Let $\Delta g(x) = c$ for all $x \in A$, and consider the topology $\tau(\boldsymbol{f}, \{c\})$. Observe that the basic open sets are of the form

$$\boldsymbol{f}^{-j}(\{c\}) \times \boldsymbol{f}^{-k}(\{c\}), \quad \boldsymbol{f}^{-j}(\{c\}) \times A, \quad A \times \boldsymbol{f}^{-k}(\{c\}), \quad \text{or } A \times A.$$

for some j, $k \in \mathbb{N}$.

In order to prove that g is constant, suppose on the contrary that there exists a, $b \in A$ such that $g(a,b) \neq c$. Then $g^{-1}(\{c\}) \neq A \times A$ so for all $x \in A$, since $(x,x) \in g^{-1}(\{c\})$, it belongs to a basic open set different from $A \times A$, so

$$x \in \boldsymbol{f}^{-n}(\{c\})$$

for some $n \in \mathbb{N}$.

This implies that for all $x \in A$, $c = \boldsymbol{f}^n(x)$ so x and c belong to the same orbit, thus $\boldsymbol{f}$ is a single orbit topologically isolated function, and consequently, it has arbitrarily long chains.

Suppose that there exists $x_0 \in A$ such that $g_{x_0}(x) = \boldsymbol{f}^n(x)$. Then

$$c = \Delta g(x_0) = g_{x_0}(x_0) = \boldsymbol{f}^n(x_0).$$

Suppose $\boldsymbol{f}$ has either one infinite descending chain or arbitrarily long descending chains. Then choose x_1 such that $\boldsymbol{f}^n(x_1)$ belongs to a level in a branch that is strictly below the level of x_0. So if $L(x)$ stands for the level of x, then extending $\preceq$ to the levels in the obvious way,

$$L(x_1) \prec L(\boldsymbol{f}^n(x_1)) \prec L(x_0) \preceq L(c),$$

Then

$$g^{x_1}(x_0) = g_{x_0}(x_1) = \boldsymbol{f}^n(x_1),$$
$$g^{x_1}(x_1) = \Delta g(x_1, x_1) = c,$$

but this contradicts the fact that g^{x_1} must be monotone in the levels, since

$$L(g^{x_1}(x_0)) = L(\boldsymbol{f}^n(x_1)) \prec L(c) = L(g^{x_1}(x_1)).$$

If $\boldsymbol{f}$ has an infinite ascending chain choose x_1 such that $x_0 \prec \boldsymbol{f}^n(x_0) = c \prec x_1$. Then

$$g^{x_1}(x_1) = c \prec \boldsymbol{f}^n(x_1) = g_{x_0}(x_1) = g^{x_1}(x_0),$$

contradicting again the monotonicity of g^{x_1}.

So g_x must be a constant function for all $x \in A$.

Finally, $g(x,y) = g_x(y) = g_x(x) = \Delta g(x) = c$, is a constant function. $\square$

Lemma 18. *Assume that* $\boldsymbol{f}$ *is not a fixed point–function. If* $g \in Clo_2(A, \boldsymbol{f})$ *and* $\Delta g = \boldsymbol{f}^n$*, for some* $n \in \mathbb{N}$*, then* $g(x,y) = \boldsymbol{f}^n(x)$ *or* $g(x,y) = \boldsymbol{f}^n(y)$.

Proof. Let $\Delta g(x) = \boldsymbol{f}^n(x)$ for all $x \in A$.

We will first prove that there cannot exist sections of the form g_x and sections of the form g^x that are constant functions.

Suppose there exists a, $b \in A$ such that for all $x \in A$, $g_a(x) = c$ and $g^b(x) = c'$. Then $c = g(a,b) = c'$.

Moreover, $\boldsymbol{f}^n(a) = g(a,a) = c = g(b,b) = \boldsymbol{f}^n(b)$, that is a and b belong to the same level of the same orbit $\mathcal{O}$. As a matter of fact, if g_x is a constant function or g^x is a constant function, the same argument using g_a or g^b proves that

$$g_x(z) = c = \boldsymbol{f}^n(x) \ \text{ or } \ g^x(z) = c = \boldsymbol{f}^n(x).$$

On the other hand, for any two points x, $y \in A$ such that $\boldsymbol{f}^n(x) \neq c$ and $\boldsymbol{f}^n(y) \neq c$, since

$$g_x(x) = \boldsymbol{f}^n(x) \neq c = g^b(x) = g_x(b),$$

g_x is not a constant function. A similar argument shows that g^y is not a constant function. Thus there exist $m, k \in \mathbb{N}$ such that

$$g_x(z) = \boldsymbol{f}^m(z) \quad \text{and} \quad g^y(z) = \boldsymbol{f}^k(z).$$

so

$$\boldsymbol{f}^m(y) = g_x(y) = g(x,y) = g^y(x) = \boldsymbol{f}^k(x).$$

This implies that any two such points belong to the same orbit.

There are two cases. If $\boldsymbol{f}$ has a single orbit, then it must have arbitrarily long chains. Choose $x_0 \in A$ such that $\boldsymbol{f}^{2n+1}(x_0) \neq c$, and $\boldsymbol{f}^{2n+1}(x_0)$ is not in the cycle of the orbit (if there is a cycle.) Since this implies that $\boldsymbol{f}^n(x_0) \neq c$, $g_{x_0}(z) = \boldsymbol{f}^m(z)$. Also, letting $y_0 = \boldsymbol{f}^n(x_0)$, $g^{y_0}(z) = \boldsymbol{f}^k(z)$. So

$$\boldsymbol{f}^{m+n+1}(x_0) = \boldsymbol{f}^m(\boldsymbol{f}^{n+1}(x_0)) = \boldsymbol{f}^m(y_0) = g_{x_0}(y_0) = g^{y_0}(x_0) = \boldsymbol{f}^k(x_0),$$

and since x_0 does not belong to the cycle of the orbit, this means that $k = m + n + 1$, but then

$$\boldsymbol{f}^n(y_0) = \Delta g(y_0) = g^{y_0}(y_0) = \boldsymbol{f}^k(y_0) = \boldsymbol{f}^{m+n+1}(y_0),$$

contradicting the fact that y_0 does not belong to the cycle of the orbit either.

The second case is if $\boldsymbol{f}$ has several orbits. We note that if x does not belong to the orbit $\mathcal{O}$ of c, then $\boldsymbol{f}^n(x) \neq c$, so $g_x(z) = \boldsymbol{f}^m(z)$

and $g^x(z) = \boldsymbol{f}^k(z)$. Moreover all points x such that $\boldsymbol{f}^n(x) \neq c$ belong to the same orbit $\mathcal{O}'$, so there are only two orbits

$$\begin{aligned} \mathcal{O} &= \{x \in A : \boldsymbol{f}^n(x) = c\}, \\ \mathcal{O}' &= \{x \in A : \boldsymbol{f}^n(x) \neq c\}. \end{aligned}$$

One can easily check that c is a fixed point, so $\mathcal{O}$ consists of a cycle of length 1 with chains of length at most n attached to it.

Since $\boldsymbol{f}$ is topologically isolated with only two orbits, one of which has a fixed point and by hypothesis, it is not a fixed–point function, the orbit $\mathcal{O}'$ must contain arbitrarily long chains. (See [1] Theorems 25 and 27.)

Choose $x_0 \in \mathcal{O}'$ such that $\boldsymbol{f}^{n+1}(x_0)$ is not in the cycle of the orbit (if there is a cycle.) Recall that $g_{x_0}(z) = \boldsymbol{f}^m(z)$ for some $m \in \mathbb{N}$ and that $\boldsymbol{f}^m(x_0) = g_{x_0}(x_0) = \Delta g(x_0) = \boldsymbol{f}^n(x_0)$, and since $\boldsymbol{f}^{n+1}(x_0)$ is not in the cycle of the orbit, neither is x_0, so $m = n$.

Similarly, $g^{\boldsymbol{f}(x_0)}(z) = \boldsymbol{f}^n(z)$, so

$$\boldsymbol{f}^n(x_0) = g^{\boldsymbol{f}(x_0)}(x_0) = g_{x_0}(\boldsymbol{f}(x_0)) = \boldsymbol{f}^n(\boldsymbol{f}(x_0)) = \boldsymbol{f}^{n+1}(x_0),$$

which is not possible.

We have reached to a contradiction to all of our cases, so we may conclude that there cannot exist points a and b such that the sections g_a and g^b are both constant functions.

Suppose now that for all x and y, $g_x = \boldsymbol{f}^{m(x)}$ and $g^y = \boldsymbol{f}^{k(y)}$. Then

$$\boldsymbol{f}^{m(x)}(y) = g_x(y) = g^y(x) = \boldsymbol{f}^{k(y)}(x),$$

so x and y belong to the same orbit, that is, $\boldsymbol{f}$ is a single orbit topologically isolated function, and as such it must contain arbitrarily long chains.

Assume there exists x_0 such that $m(x_0) = n + m$, with $m \neq 0$. Using the arguments of the previous paragraph, we conclude that $\boldsymbol{f}$ must contain a cycle $\mathcal{C}$ and that $\boldsymbol{f}^n(x_0) \in \mathcal{C}$. Also, there must exist arbitrarily long descending chains. We may choose y such that $\boldsymbol{f}^{n+m}(y)$ belongs to a level that is strictly below that of x_0. But then

$$\boldsymbol{f}^{n+m}(y) = g_{x_0}(y) = g^y(x_0) = \boldsymbol{f}^{k(y)}(x_0),$$

which contradicts our choice of y.

This contradiction proves that for all x and y, $m(x) = k(y) = n$, but then, fixing x_0, for all x

$$\boldsymbol{f}^n(x_0) = g^x(x_0) = g_{x_0}(x) = \boldsymbol{f}^n(x),$$

which is clearly impossible if there are arbitrarily long chains.

As a result, not all sections can be iterates of $\boldsymbol{f}$, some of them must be constant functions.

Let us assume as a last case that there are points x_0 and x_1 such that $g_{x_0} = \boldsymbol{f}^m$ and g_{x_1} is a constant function. By the first part of this proof, this last assumption implies that for all y, $g^y = \boldsymbol{f}^{k(y)}$. Moreover,

$$\boldsymbol{f}^{k(y)}(x_0) = g^y(x_0) = g_{x_0}(y) = \boldsymbol{f}^m(y)\,,$$

and $\boldsymbol{f}$ is a single orbit function.

If $\boldsymbol{f}$ has arbitrarily long descending chains, choose y such that the level of $\boldsymbol{f}^m(y)$ is strictly less than the level of x_0, so $\boldsymbol{f}^m(y)$ is below the level of $\boldsymbol{f}^{k(y)}(x_0)$.

If the (single) orbit of $\boldsymbol{f}$ has an infinite ascending chain, then it has no cycle and by the remarks above, for all y, $k(y) = n$ and we get

$$\boldsymbol{f}^n(x_0) = g^y(x_0) = g_{x_0}(y) = \boldsymbol{f}^n(y),$$

which contradicts the existence of arbitrarily long chains.

This completes the proof that in this case either all sections of the form g_x are iterates of $\boldsymbol{f}$ and all sections of the form g^x are constant functions or viceversa.

In the first case,

$$g(x,y) = g^y(x) = g^y(y) = \Delta g(y) = \boldsymbol{f}^n(y),$$

in the second case,

$$g(x,y) = g_x(y) = g_x(x) = \Delta g(x) = \boldsymbol{f}^n(x),$$

which is what we wanted to prove. □

Theorem 19. *Assume $\boldsymbol{f}$ is a non–fixed–point topologically isolated function. Let $g : A\times A \to \mathbb{N}$ be such that $g \in Clo_2(A,\boldsymbol{f})$. Then either g is a constant function or $g(x,y) = \boldsymbol{f}^n(x)$ or $g(x,y) = \boldsymbol{f}^n(y)$.*

Proof. Follows immediately from lemmas 17 and 18. □

Theorem 20. *Assume $\boldsymbol{f}$ is a non–fixed–point topologically isolated function. Let $g : A^k \to A$ be such that $g \in Clo(A,\boldsymbol{f})$. Then either g is a constant function or $g = \boldsymbol{f}^n \circ \pi_i$, for some i, $1 \le i \le k$, where π_i is the i^{th} projection function.*

Proof. Let $g \in Clo_k(A,\boldsymbol{f})$. The proof is by induction on k.

The result follows for $k = 1$ from the Main Theorem 16 and for $k = 2$ from Theorem 19 So, let $k > 2$ and assume that the theorem is true for functions in $Clo_j(A,\boldsymbol{f})$, for $j < k$. The strategy here will be to show that g depends on at most two variables and apply Theorem 19.

We observe that if we replace some of the variables by elements of A, the resulting section will also belong to $Clo(A, \boldsymbol{f})$ but will contain strictly less variables so we can apply the induction hypothesis to them. If we replace variable x_i by a constant, we will say that the resulting function is an i–section.

Assume first that there exist an i–section and a j–section (for notational convenience we will assume that $i = 1$ and $j = 2$), such that for some a and b,

$$\begin{aligned} g(a, x_2, x_3, \ldots, x_k) &= c\,, \text{ and} \\ g(x_1, b, x_3, \ldots, x_k) &= c'\,, \end{aligned}$$

are constant functions. Then $c = g(a, b_2, x_3, \ldots, x_k) = c'$.

If for any $i \neq 1, 2$ and $d \in A$, $g(x_1, x_2, \ldots, d, \ldots, x_k) = \boldsymbol{f}^n(x_j)$, (observe $j \neq i$), we have

$$\begin{aligned} c = g(a, x_2, \ldots, d, \ldots, x_k) &= \boldsymbol{f}^n(x_i)\,, \text{ if } i \neq 1 \\ c = g(x_1, b, \ldots, d, \ldots, x_k) &= \boldsymbol{f}^n(x_1)\,, \text{ if } i = 1\,. \end{aligned}$$

This implies that for all $x \in A$, $\boldsymbol{f}^n(x) = c$, so $\boldsymbol{f}$ is a single orbit topologically isolated function and the condition is impossible

So for all $i \neq 1, 2$ and all $d \in A$, $g(x_1, x_2, \ldots, d, \ldots, x_k)$ is a constant function, and thus $g(x_1, x_2, \ldots, d, \ldots, x_k) = g(a, b, \ldots, d, \ldots, x_k) = c$. So fixing $a_3, \ldots, a_k$,

$$g(x_1, \ldots, x_k) = g(x_1, x_2, a_3, \ldots, a_k) = k(x_1, x_2)\,,$$

is well defined and depends only on two variables, so applying Theorem 19, we get the desired result.

Assume now that there exist a single i–section (we will assume that $i = 1$), such that for some a

$$g(a, x_2, \ldots, x_k) = c$$

is a constant function. Then for any b, there will exist positive integers $n = n(b)$ and $i = i(b)$ such that $g(x_1, b, x_3, \ldots, x_k) = \boldsymbol{f}^n(x_i)$. If for some b, $i(b) \neq 1$, then for all $d \in A$,

$$c = g(a, b, x_3, \ldots, d, \ldots, x_k) = \boldsymbol{f}^n(d),$$

so again we have a single orbit function with a condition that contradicts the existence of arbitrarily long chains. This implies that for any

b, $g(x_1, b, x_3, \ldots, x_k) = \boldsymbol{f}^{n(b)}(x_1)$, and thus g depends only on the first two variables and we can apply Theorem 19.

Finally, assume that for any j–section and all a the corresponding section is not a constant function. Then there exist positive integers $n = n(a, j)$ and $i = i(a, j)$ such that

$$g(x_1, \ldots, a, \ldots, x_k) = \boldsymbol{f}^n(x_i)\,,$$

where $i \neq j$.

Assume that for some $j \neq l$, and $a, b \in A$, $i(a, j) \neq i(b, l)$. (We will assume that $i = 1$ and $l = 2$). Then

$$\begin{aligned} g(a, x_2, \ldots, x_k) &= \boldsymbol{f}^n(x_r)\,, \\ g(x_1, b, \ldots, x_k) &= \boldsymbol{f}^m(x_s)\,, \end{aligned}$$

$r \neq 1$, $s \neq 2$ and $r \neq s$. We have several cases.

If either $r \neq 2$ or $s \neq 1$, say $r \neq 1, 2$,

$$\boldsymbol{f}^n(x_r) = g(a, b, a, a, \ldots, x_r, \ldots, a) = \boldsymbol{f}^m(a)\,.$$

and again we have a single orbit function with a condition that contradicts the existence of arbitrarily long chains.

Since this is not possible, we must conclude that $r = 2$ and $s = 1$, that is, for all $a, b \in A$,

$$\begin{aligned} g(a, x_2, \ldots, x_k) &= \boldsymbol{f}^n(x_2)\,, \\ g(x_1, b, \ldots, x_k) &= \boldsymbol{f}^m(x_1)\,, \end{aligned}$$

and again we have that g depends only on the first two variables, so we apply Theorem 19. $\square$

We conclude by noting that the hypothesis that $\boldsymbol{f}$ is a non–fixed–point function is necessary. Consider the following example.

Let $\boldsymbol{f} : A \longrightarrow A$, where $A = \{a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n\}$, be the two–orbit fixed–point function defined by the following Diagram.

$$a_n \mapsto a_{n-1} \mapsto \ldots \mapsto a_1 \mapsto a_0 \mapsto a_0\,,$$

$$b_n \mapsto b_{n-1} \mapsto \ldots \mapsto b_1 \mapsto b_0 \mapsto b_0\,.$$

Then by [1] Theorem 25, $\boldsymbol{f}$ is a topologically isolated function, so $S(A, \boldsymbol{f}) = Clo_1(A, \boldsymbol{f})$ is trivial.

Let us call $\mathcal{O}_a$ and $\mathcal{O}_b$ the two orbits.

$$g(x,y)=\begin{cases} a_0 & \text{, if } x\in\mathcal{O}_a \text{ or } y\in\mathcal{O}_a\,, \\ b_0 & \text{, if } x\in\mathcal{O}_b \text{ and } y\in\mathcal{O}_b\,, \end{cases}$$

Then $g\in Clo_2(A,\boldsymbol{f})$. Let τ be a topology for which $\boldsymbol{f}$ is τ–continuous and $\mathcal{U}\in\tau$. Then

$$g^{-1}(\mathcal{U})=\begin{cases} \emptyset & \text{, if } a_0\notin\mathcal{U} \text{ and } b_0\notin\mathcal{U}\,, \\ \mathcal{O}_a\times A\cup A\times\mathcal{O}_a & \text{, if } a_0\in\mathcal{U} \text{ and } b_0\notin\mathcal{U}\,, \\ \mathcal{O}_b\times\mathcal{O}_b & \text{, if } a_0\notin\mathcal{U} \text{ and } b_0\in\mathcal{U}\,, \\ A\times A & \text{, if } a_0\in\mathcal{U} \text{ and } b_0\in\mathcal{U}\,. \end{cases}$$

Now simply observe that if $a_0\in\mathcal{U}$ and $b_0\notin\mathcal{U}$,

$$\mathcal{O}_a\times A\cup A\times\mathcal{O}_a=\boldsymbol{f}^{-n}(\mathcal{U})\times A\cup A\times\boldsymbol{f}^{-n}(\mathcal{U})\,,$$

and that if $a_0\notin\mathcal{U}$ and $b_0\in\mathcal{U}$,

$$\mathcal{O}_b\times\mathcal{O}_b=\boldsymbol{f}^{-n}(\mathcal{U})\times\boldsymbol{f}^{-n}(\mathcal{U})\,,$$

so in any case, g is τ–continuous.

Finally, if $g(x,y)=\boldsymbol{f}^m(x)$, then $m>n$ and

$$a_0=g(b_0,a_0)=\boldsymbol{f}^m(b_0)=b_0\,,$$

which is not the case. Similarly $g(x,y)\neq\boldsymbol{f}^m(y)$, thus $Clo_2(A,\boldsymbol{f})$ contains non–trivial functions.

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