TOPOLOGICALLY INSEPARABLE FUNCTIONS I: FINITARY CASE

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Abstract. Given a finite set A and a distinguished function $\boldsymbol{f} : A \longrightarrow A$, we study the set of all functions $g : A \longrightarrow A$ that are continuous for all topologies for which $\boldsymbol{f}$ is continuous. The main result is a characterization of the functions $\boldsymbol{f}$ such that this set is trivial, that is, contains only the constant functions and the iterates of $\boldsymbol{f}$.

Introduction

The Problem. Let A be a set and $\boldsymbol{f} : A \to A$ be a fixed function. Associated with this function in a very natural way, we have the (algebraic) semigroup $S_0(A, \boldsymbol{f})$ generated by $\boldsymbol{f}$

$$S_0(A, \boldsymbol{f}) = \{\boldsymbol{f}^n : n \in \mathbb{N}\},$$

where $\boldsymbol{f}^0$ is the identity function and for $n \in \mathbb{N}$, $\boldsymbol{f}^{n+1} = \boldsymbol{f} \circ \boldsymbol{f}^n$.

For any given topology τ over A, we have another associated semigroup, $S(\tau)$, the semigroup of all τ–continuous functions on A.

Suppose now that our distinguished function $\boldsymbol{f}$ is τ–continuous. Then any function in $S_0(A, \boldsymbol{f})$, as well as all constant functions will be τ–continuous, in some sense, very trivially so. Are there any other functions that are also τ–continuous? A simple cardinality argument shows that this is usually the case. What happens if we change to another topology τ' for which $\boldsymbol{f}$ is continuous? Then, the constant functions and the members of $S_0(A, \boldsymbol{f})$ will still be continuous but the other continuous functions will in general be different. A very natural question arises then:

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Are there any non–trivial functions that are continuous for every topology for which $\boldsymbol{f}$ *is continuous?*

In other words, are there functions, other than the obvious ones mentioned above, whose continuity is implied by the continuity of $\boldsymbol{f}$?

For a very simple example, let $\boldsymbol{f} : \{0,1,2\} \longrightarrow \{0,1,2\}$ be the cycle $(0\ 1\ 2)$. Then there are only three iterations of $\boldsymbol{f}$, namely, $\boldsymbol{f}^0$, $\boldsymbol{f}^1$ and $\boldsymbol{f}^2$. Nevertheless, it is easy to check that no matter what the topology on $\{0,1,2\}$ is, if $\boldsymbol{f}$ is continuous, then *any* function from $\{0,1,2\}$ into $\{0,1,2\}$ is continuous. The reason for this is that if $\boldsymbol{f}$ is continuous then the topology has to be either trivial or discrete. So there are non–trivial functions that are forced to be continuous if $\boldsymbol{f}$ is continuous.

We define

$$\begin{aligned} S(A,\boldsymbol{f}) &= \{g : A \to A : g \text{ is continuous for all topologies for} \\ &\qquad\qquad \text{which } \boldsymbol{f} \text{ is continuous}\} \\ &= \bigcap \{S(\tau) : \boldsymbol{f} \text{ is } \tau\text{–continuous}\}. \end{aligned}$$

$S(A,\boldsymbol{f})$ is a semigroup that contains $S_0(A,\boldsymbol{f})$. We call it *the semigroup of functions topologically inseparable from* $\boldsymbol{f}$. We let $S(A,\boldsymbol{f})^*$ be the set of all non constant elements of $S(A,\boldsymbol{f})$, and since constant functions are always continuous, regardless of the function $\boldsymbol{f}$, we will usually work in the context of this set. We also let $S_0^c(A,\boldsymbol{f}) = S_0(A,\boldsymbol{f}) \cup \{\text{constant functions on A}\}$.

This is the first of two papers in which we determine necessary and sufficient conditions on $\boldsymbol{f}$ so that $S_0^c(A,\boldsymbol{f}) = S(A,\boldsymbol{f})$. In this first one we restrict our attention to the case when A is a finite set. In the second paper (see [1]) we study the infinite case. The techniques used in the finite and the infinite cases are quite different, this is the main reason why we divided this study into two separate articles. In the second paper we also study the clone of all n–ary functions on A that are topologically inseparable from $\boldsymbol{f}$ and we extend our results to it.

Motivations. Even though the problem is quite natural and stands on its own, it has universal–algebraic origins and motivations and is inspired by [2].

Given a topological algebra $\boldsymbol{A} = \langle A; f_i \rangle_{i \in I}$, there are (at least) two clones naturally associated to it. One is the clone $Clo(\boldsymbol{A})$, of all terms and the other is the clone $Clo(A,\tau)$, where τ is the underlying topology, of all n–ary τ–continuous functions on A .

We observe that since $\boldsymbol{A}$ is a topological algebra, any term defined function $g : A^n \to A$, (i.e., $g = \sigma^{\boldsymbol{A}}$ for some n–ary term σ), is continuous (in the appropriate product topology); the identity and the constant functions are also continuous. Again, all these are trivially so. So $Clo(\boldsymbol{A})$ is isomorphically embedded in $Clo(A, \tau)$. Are these two clones "the same", that is, isomorphic?

The example above may be interpreted in this context as the case of a mono–unary algebra $\langle \{0, 1, 2\}; \boldsymbol{f} \rangle$. It amounts to the fact that $Clo_1(\boldsymbol{A})$ is not the same as $Clo_1(A, \tau)$.

The answer to the problem stated in these two papers may give us an idea of how to solve the following question. Given a class of algebras, is its clone of terms representable by the clone of continuous functions of a certain topological space? We know for instance that the clone of terms for Boolean algebras is represented by the clone of continuous functions on $\{0, 1\}$ with the discrete topology.

1. Preliminary Results

We first recall that the relation on A defined by

$$x \sim y \text{ iff there exist } n, m \in \mathbb{N} \text{ such that } \boldsymbol{f}^n(x) = \boldsymbol{f}^m(y)$$

is an equivalence relation. The equivalence classes are called *connected components* or *orbits*. The reader can easily figure out what the orbits may look. Some examples appear in the following diagram.

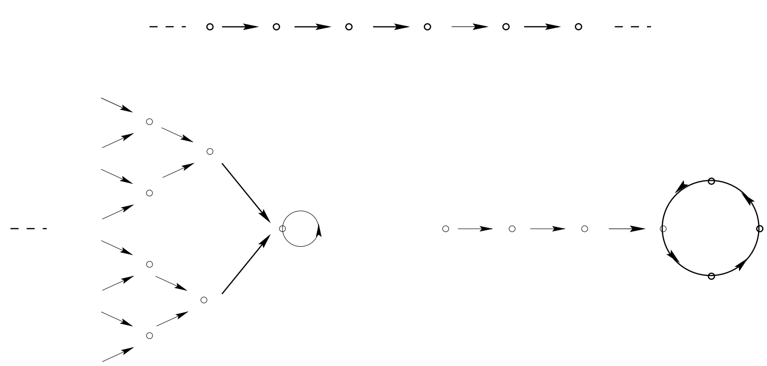

Diagram 1

Since A is finite, for each $x \in A$ there exist n, $m \in \mathbb{N}$, $n \neq m$, such that $\boldsymbol{f}^n(x) = \boldsymbol{f}^m(x)$. The set of all points such that for some $n \in \mathbb{N}$, $n \neq 0$, $\boldsymbol{f}^n(x) = x$, is called the *cycle* of the orbit. The number of points in the cycle is called its *length*. An orbit cannot have more than one cycle. The points that are not in the cycle are the *branches* of the orbit.

On each orbit we can define the following relation

$$x \preceq y \quad \text{iff} \quad \boldsymbol{f}^n(x) = y, \qquad \text{for some } \ n \in \mathbb{N}\ .$$

If we restrict the relation to points outside the cycle, then it is a partial order which we call the *partial order induced by* $\boldsymbol{f}$ *on the orbit.*

If $\Sigma \subseteq \mathcal{P}(A)$, $\tau(\boldsymbol{f}, \Sigma)$ is the minimal topology that contains Σ and for which $\boldsymbol{f}$ is continuous. It is the topology whose base is the set

$$\mathcal{B} = \{\boldsymbol{f}^{-n_1}(\mathcal{U}_1)\cap\cdots\cap\boldsymbol{f}^{-n_k}(\mathcal{U}_k) : k, n_i \in \mathbb{N},\ \mathcal{U}_i \in \Sigma, \ \ \text{for all} \ \ i\}\cup\{\emptyset, A\}.$$

We will write $\tau(\boldsymbol{f},\mathcal{U})$ instead of $\tau(\boldsymbol{f}, \{\mathcal{U}\})$.

Sketch of the proof of the Main Theorem: We proceed by proving two main cases.

(1) All orbits of $\boldsymbol{f}$ consist of a cycle with no branch attached to it. (Section 2.)

We first study the case when there is a single cycle. We prove a decomposition theorem, and then reduce the general case to the single cycle case.

(2) Some or all the orbits contain branches. (Section 3.)

We prove that except for a special case, the problem depends on the structure of the cycles and not on the branches, so it is essentially reduced to the previous case.

Even though the proofs are somewhat complicated, there is a general pattern. Given a function $g \in S(A, \boldsymbol{f})$ and two points, a and b, we build an appropriate topology τ such that these two points belong to the same basic open sets, in which case we say that a and b are τ*–inseparable.* Then, knowing what the value of $g(a)$ is, we prove that the inseparability of the two points forces $g(b)$ to take some desired value.

Lemma 1. *Let* $g \in S(A, \boldsymbol{f})^*$. *Then for all* $x \in A$, *there exists an* $n \in \mathbb{N}$ *such that* $g(x) = \boldsymbol{f}^n(x)$.

Proof. Let $g : A \to A$, $x \in A$ and consider $\mathcal{U} = \{g(x)\}$. Since g is $\tau(\boldsymbol{f},\mathcal{U})$–continuous, $x \in g^{-1}(\mathcal{U})$ and g is not constant, $g^{-1}(\mathcal{U}) \neq \emptyset, A$, thus

$$x \in g^{-1}(\mathcal{U}) = \bigcup_{j\in J}\bigcap_{k\in F_j} \boldsymbol{f}^{-k}(\mathcal{U}),$$

where $J \subseteq \mathbb{N}$ and for all $j \in J$, F_j is finite. That is, for some $j \in J$, $x \in \boldsymbol{f}^{-n_1}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-n_m}(\mathcal{U})$, i.e. $\boldsymbol{f}^n(x) \in \mathcal{U} = \{g(x)\}$ for $n \in \{n_1, \ldots, n_m\}$, thus $g(x) = \boldsymbol{f}^n(x)$ for some integer n. □

Lemma 2. *If* $g \in S(A, \boldsymbol{f})^*$ *, then* x *and* $g(x)$ *belong to the same orbit.*

Proof. By Lemma 1, x and $g(x)$ are in the same orbit. $\square$

Remark 1.

(1) *Observe that if* τ *is a topology over one of the orbits, then* $\tau \cup \{A\}$ *is a topology over* A *. Moreover, if* $\boldsymbol{f}$ *restricted to that orbit is* τ*–continuous, by Lemma 2,* $\boldsymbol{f}$ *is* $\tau \cup \{A\}$*–continuous. That is, if* $g \in S(A, \boldsymbol{f})$ *, then* g *must be* $\tau \cup \{A\}$*–continuous. This allows us to study each orbit separately and then glue them together. Of course, studying one orbit at a time is equivalent to studying a function* $\boldsymbol{f}$ *that consists of a single orbit.*

(2) *If the length of the cycle of* $\mathcal{O}$ *is* l *, then* $\boldsymbol{f}^n(x) = \boldsymbol{f}^{n+kl}(x)$ *, for any* x *in the cycle and any* $k \in \mathbb{N}$*.*

(3) *Moreover, if* g *is such that for all* $x \in \mathcal{O}$*,* $g(x)$ *is in the cycle (in particular if the orbit has no branch) and if* $g(x) = \boldsymbol{f}^n(x)$ *, then* $g(x) = \boldsymbol{f}^{n+kl}(x)$ *.*

2. Cycles without branches

In this section we will study the simplest case, $\boldsymbol{f}$ is a cycle or the union of several cycles. We will represent a cycle as an orbit $\mathcal{O} = \{1, 2, \ldots, l\}$, with

$$\boldsymbol{f}(i) = i + 1,$$

where addition is modulo the length l of the cycle.

Definition 1. *If* $\boldsymbol{f} : A \longrightarrow A$ *is a single cycle* $\mathcal{O} = \{1, 2, \ldots, l\}$ *of length* l *and* $n \neq 1$ *is a divisor of* l *, for* $i = 1, \ldots, n$*, define*

$$\mathcal{U}_i = \{i + k\frac{l}{n} : k = 1, \ldots, n\},$$

where addition is modulo l *. The* $\frac{l}{n}$ *(disjoint) sets* $\mathcal{U}_i$ *are called* n–complexes*, and the topology they generate is called the* n–complex topology on A *and will be denoted* τ_n*.*

We can depict the points of $\mathcal{O}$ evenly distributed on a circle, labeled 1 through l and $\boldsymbol{f}$ acting counterclockwise on them. Each n–complex is a regular n–gon. The following is an immediate consequence of the definition.

Lemma 3. *The set of* n*–complexes forms a base for the* n*–complex topology. Each point in* A *belongs to a single* n*–complex.*

Theorem 4. *Let* $\boldsymbol{f} : A \longrightarrow A$ *be a single cycle* $\mathcal{O} = \{1, 2, \ldots, l\}$ *of length* l *. Then the only non–trivial topologies for which* $\boldsymbol{f}$ *is continuous are the* n*–complex topologies, where* $n \neq 1$ *divides* l *.*

Proof. If n is a divisor of l and $\mathcal{U}_i$ is an n–complex, then

$$\boldsymbol{f}^{-1}(\mathcal{U}_i) = \mathcal{U}_{i-1},$$

where $i-1$ is substraction modulo $\frac{l}{n}$. This shows that $\boldsymbol{f}$ is continuous for any n–complex topology.

Let τ be a topology that is not an n–complex topology, for which $\boldsymbol{f}$ is continuous. Let $\mathcal{U} = \{k_1, k_2, \ldots, k_r\}$ be a non–empty open set of minimal cardinality, where $k_1 < k_2 < \cdots < k_r$. If $\mathcal{U}$ is not an n–complex, for some i , we have that k_i , k_{i+1} and k_{i+2} are such that $k_{i+1} - k_i < k_{i+2} - k_{i+1}$ or $k_{i+1} - k_i > k_{i+2} - k_{i+1}$. Since both cases can be treated similarly, we can assume that the first of these occurs. Then

$$\begin{aligned} k_i &\in \mathcal{U} \cap \boldsymbol{f}^{-(k_{i+1}-k_i)}(\mathcal{U}) \quad \text{but} \\ k_{i+1} &\notin \mathcal{U} \cap \boldsymbol{f}^{-(k_{i+1}-k_i)}(\mathcal{U}). \end{aligned}$$

So $\mathcal{U} \cap \boldsymbol{f}^{-(k_{i+1}-k_i)}(\mathcal{U})$ is a non-empty open set whose cardinality is strictly less than that of $\mathcal{U}$, a contradiction.

A similar argument can be used if $\mathcal{U}$ has only two elements.

So the open sets of minimal cardinality of τ must be such that the points they label are evenly distributed on the circle, so they are n–complexes for some divisor n of l . This implies that τ contains the n–complex topology.

Let $\mathcal{U}$ be an open set of minimal cardinality that is not a union of n–complexes. Then

$$\mathcal{U} = \mathcal{V} \cup \bigcup_{k \in F} \mathcal{U}_k,$$

where the $\mathcal{U}_k$ are n–complexes, F may be empty, $\mathcal{V}$ does not contain an n–complex and $\mathcal{V} \cap \bigcup_{k \in F} \mathcal{U}_k = \emptyset$.

For $x \in \mathcal{V}$, let $\mathcal{U}_x$ be the n–complex that contains x and let

$$\mathcal{W} = \mathcal{U} \cap \mathcal{U}_x = ((\bigcup_{k \in F} \mathcal{U}_k) \cap \mathcal{U}_x) \cup (\mathcal{V} \cap \mathcal{U}_x).$$

Then

(1) $\mathcal{W}$ is an open set.
(2) Since $x \in \mathcal{W}$, it is not empty.
(3) $(\bigcup_{k \in F} \mathcal{U}_k) \cap \mathcal{U}_x = \emptyset$.

(4) Since $\mathcal{V}$ contains no n–complexes, neither does $\mathcal{V}\cap\mathcal{U}_x$.
(5) $\mathcal{V}\cap\mathcal{U}_x \subsetneqq \mathcal{V}$.

These show that $\mathcal{W}$ is a non–empty open set that contains no n–complex and its cardinality is strictly smaller than that of $\mathcal{U}$, a contradiction.

So there cannot exist an open set that is not a union of n–complexes.

□

The preceding theorem implies that in order to prove that $g \in S(A, \boldsymbol{f})$ we just have to check the n–complex topologies, for all divisors of l. The next theorem implies that it is enough to check the n–complex topologies when n is a power of a prime.

Theorem 5. *Let n be a divisor of l.*
Assume that $n = p\,q$, with p and q relatively prime. Then

$$\tau_n = \tau_p \cap \tau_q.$$

Proof. It is straightforward that if $k|n$, then $\tau_n \subseteq \tau_k$, since n–complexes are unions of k–complexes. So

$$\tau_n \subseteq \tau_p \cap \tau_q.$$

Since $\tau_p\cap\tau_q$ is a non–trivial topology for which $\boldsymbol{f}$ is continuous, by the previous theorem, it is an m–complex topology for some divisor m of l.

Let $\mathcal{U}_i = \{x : x = i + k\frac{l}{m},\ 0 \le k < m\}$ be a basic open set in τ_m.
Since $\mathcal{U}_i \in \tau_p$,

$$\mathcal{U}_i = \bigcup_j \mathcal{V}_j,$$

where the $\mathcal{V}_j$'s are p–complexes. Now since $i \in \mathcal{U}_i$, $i \in \mathcal{V}_j$ for some j, and since $\mathcal{V}_j \subseteq \mathcal{U}_i$, $i + \frac{l}{p} \in \mathcal{U}_i$ and there exists k such that

$$i + \frac{l}{p} = i + k\ \frac{l}{m},$$

so $p|m$.

Similarly $q|m$, so $n = p\,q|m$, and thus $\tau_m \subseteq \tau_n$. □

Lemma 6. *Let $g \in S(\mathcal{O}, \boldsymbol{f})^*$ and n be a divisor of l. Then n–complexes are mapped by g into n–complexes. Equivalently, if $i \equiv j \pmod{\frac{l}{n}}$, then $g(i) \equiv g(j) \pmod{\frac{l}{n}}$.*

Proof. The idea of this proof is that points that belong to the same n–complex cannot be separated. Consider the n–complex topology on

$\mathcal{O}$. Given i , let $\mathcal{V}$ be the n–complex that contains it and let $\mathcal{U}$ be the n–complex that contains $g(i)$. Then

$$i \in g^{-1}(\mathcal{U}) = \bigcup_{k \in F} \mathcal{U}_k.$$

So $\mathcal{V} \subseteq g^{-1}(\mathcal{U})$ and thus $g(\mathcal{V}) \subseteq \mathcal{U}$.

Observe that $i \equiv j \pmod{\frac{l}{n}}$ if and only if i and j belong to the same n–complex. So this implies that $g(i)$ and $g(j)$ belong to the same n–complex, so $g(i) \equiv g(j) \pmod{\frac{l}{n}}$. □

Theorem 7. *Let* $\boldsymbol{f} : \mathcal{O} \to \mathcal{O}$ *be a cycle,* $\mathcal{O} = \{1, 2, \ldots, l\}$ *of length* $l = p^n$*, where* p *is prime. Then*

$$|S(\mathcal{O}, \boldsymbol{f})| = p^{p^{\frac{p^n-1}{p-1}}}.$$

Proof. The only condition for $g \in S(\mathcal{O}, \boldsymbol{f})$ is that g must be τ–continuous for the p^i–complex topology, for each $i \leq n$. If

$$\mathcal{O} = \{1, 2, \ldots, p, p+1, \ldots, p^2, \ldots, p^n\},$$

and

$$p^i < j \leq p^{i+1},$$

then

$$j \equiv k \pmod{p^i}$$

for a unique k , such that $1 \leq k \leq p^i$. By lemma 6,

$$g(j) \equiv g(k) \pmod{p^i},$$

in other words, $g(j)$ and $g(k)$ belong to the same p^{n-i}–complex, so once we have determined $g(k)$, there are only p^{n-i} possible values for $g(j)$.

For each of the first p points, there are p^n possible values of g . For each of those, if $i > 0$, for the $p^{i+1} - p^i$ points between p^i and p^{i+1} , there are p^{n-i} possible values of g . So the total number of combinations is

$$(p^n)^p \cdot \prod_{i=1}^{n-1} (p^{(n-i)})^{(p^{i+1}-p^i)} = p^{np + \sum_{i=1}^{n-1}(n-i)(p^{i+1}-p^i)}.$$

Finally,

$$np + \sum_{i=1}^{n-1} (n-i)(p^{i+1} - p^i) = p\frac{p^n - 1}{p - 1},$$

as stated. □

Lemma 8. *Let* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *be two orbits without branches of a function* $\boldsymbol{f} : A \to A$ *such that the length* $l_1 > 1$ *of the cycle of* $\mathcal{O}_1$ *divides the length* l_2 *of the cycle of* $\mathcal{O}_2$ *. Let* $g \in S(A, \boldsymbol{f})^*$ *, then there exists an* $n \in \mathbb{N}$ *such that ,* $g \restriction \mathcal{O}_1 = \boldsymbol{f}^n$ *.*

Proof. We let $l_2 = k\, l_1$.

Assume $y \in \mathcal{O}_2$ is such that $g(y) = \boldsymbol{f}^n(y)$ and let $x \in \mathcal{O}_1$. We will define a topology for which this point y and any given point $x \in \mathcal{O}_1$ both belong to the same (unique) basic open set. So let

$$\mathcal{U} = \{\boldsymbol{f}^n(x)\} \cup \{\boldsymbol{f}^n(y), \boldsymbol{f}^{n+l_1}(y), \boldsymbol{f}^{n+2l_1}(y), \ldots, \boldsymbol{f}^{n+(k-1)l_1}(y)\}.$$

Observe that the part of $\mathcal{U}$ that belongs to $\mathcal{O}_2$ is a k–complex in that orbit.

Then $\tau(\boldsymbol{f},\mathcal{U})$ has exactly l_1 disjoint basic open sets. Also, it is easy to check that for any $\mathcal{V} \in \tau(\boldsymbol{f},\mathcal{U})$, $x \in \mathcal{V}$ if and only if $y \in \mathcal{V}$. Thus, since $y \in g^{-1}(\mathcal{U})$, $x \in g^{-1}(\mathcal{U})$, so $g(x) = \boldsymbol{f}^n(x)$.

Finally, since x was arbitrary, $g \restriction \mathcal{O}_1 = \boldsymbol{f}^n$.

□

In the preceding lemma we showed that if there are two cycles one dividing the other, the values of g on the larger orbit totally determine the behavior of g on the smaller one. In the next lemma we study the effects of the smaller cycle on the larger one. We will state it only in a special case. The proof is similar to that of Theorem 7.

Lemma 9. *Let* $\boldsymbol{f} : A \to A$ *be a function with two cycles,* $\mathcal{O}_1$ *of length* p^m *and* $\mathcal{O}_2$ *of length* p^n*, with* $m < n$*. Then*

$$|S(A, \boldsymbol{f})| = p^{p^{m+1}\frac{p^{n-m}-1}{p-1}}.$$

2.1. **Decomposition of a single cycle.** In this section we devise a method that will allow us to decompose a cycle of arbitrary length l into an equivalent function that has several cycles whose lengths are the powers of a prime that divides l. The study of these decomposed cycles is much simpler than that of the original function.

Definition 2. *Let* $\boldsymbol{f} : \mathcal{O} \to \mathcal{O}$ *be a cycle* $\mathcal{O} = \{1, 2, \ldots, l\}$ *of length* $l = n_1 \cdot n_2 \cdots n_m$*, where the* n_k*'s are pairwise relatively prime. For* $k = 1, 2, \ldots, m$*, let* $\boldsymbol{f}_k : \mathcal{O}_k \to \mathcal{O}_k$ *be cycles* $\mathcal{O}_k = \{1, 2, \ldots, n_k\}$*, of length* n_k *. Define*

$$\begin{aligned} \pi_k : \mathcal{O} &\longrightarrow \mathcal{O}_k \\ i &\longmapsto \text{the residue of } i \text{ modulo } n_k. \end{aligned}$$

Remark 2. *Observe that* $i \equiv j \pmod{n_k}$ *if and only if* $\pi_k(i) = \pi_k(j)$ *.*

Lemma 10. *Assume the notation of Definition 2.*

(1) *If* τ *is the* r*–complex topology on* $\mathcal{O}_k$*, for some divisor* r *of* n_k*, then*

$$\check{\tau} = \{{\pi_k}^{-1}(\mathcal{U}) : \mathcal{U} \in \tau\}$$

is a topology over $\mathcal{O}$*. Also, for each* r*–complex* $\mathcal{U} \in \tau$*,* ${\pi_k}^{-1}(\mathcal{U})$ *is an* $\frac{l\,r}{n_k}$*–complex in* $\mathcal{O}$*. Moreover it is a union of* r*–complexes in* $\mathcal{O}$*.*

(2) *If* τ *is the* r*–complex topology on* $\mathcal{O}$*, for some divisor* r *of* n_k*, then*

$$\tau_k = \{\pi_k(\mathcal{U}) : \mathcal{U} \in \tau\}$$

is the r*–complex topology over* $\mathcal{O}_k$*.*

Proof. (1) We know that $\check{\tau}$ is a topology on $\mathcal{O}$. Let $\mathcal{U} \in \tau$ be an r–complex in $\mathcal{O}_k$. Then if $d = \frac{n_1}{r}$, for some $u < d$,

$$\mathcal{U} = \{u, u+d, u+2d, \ldots, u+(r-1)d\}.$$

So

$$\begin{aligned} \pi_k^{-1}(\mathcal{U}) &= \{u + jd + in_k : \ 0 \le j < r, \ 0 \le i < \frac{l}{n_k}\} \\ &= \{u + id : \ 0 \le i < \frac{l}{d}\} \end{aligned}$$

where addition is modulo l, so $\pi_k^{-1}(\mathcal{U})$ is a $\frac{l}{d}$–complex as stated. Moreover

$$\pi_k^{-1}(\mathcal{U}) = \bigcup_{j=0}^{\frac{l}{n_k}-1} \{u + jd + i\frac{l}{r} : 0 \le i < r\}$$

and for each j, these latter are r–complexes in $\mathcal{O}$, so $\pi_k^{-1}(\mathcal{U})$ is a union of r–complexes in $\mathcal{O}$.

(2) Let $r \mid n_k$. Observe that this implies that $r \nmid n_j$, for $j \neq k$ since the n_k's are relatively prime. Let $\mathcal{U}$ be an r–complex in $\mathcal{O}$. Then for $i, j \in \mathcal{U}$, $i \equiv j \pmod{\frac{l}{r}}$ so $\pi_k(i) \equiv \pi_k(j) \pmod{\frac{n_k}{r}}$.

Also, if $\pi_k(i) = \pi_k(j)$, then $i \equiv j \pmod{n_k}$, so

$$i - j = s\, n_k = s'\, \frac{l}{r},$$

for $s < n_k$, $s' < r$, so $s\ r = s'\ n_1\ \cdots\ n_{k-1},\ n_{k+1},\ \cdots\ n_m$, that is, $r|s'$, which is impossible, so $\pi_k(\mathcal{U})$ has r elements that are congruent modulo $\frac{n_k}{r}$ and thus it is an r–complex in $\mathcal{O}_k$. □

Lemma 11. *Let* $\boldsymbol{f}$ *and* $\boldsymbol{f}_k$ *be as in Definition 2 and let* $g \in S(A, \boldsymbol{f})^*$. *Then for* $k = 1, 2, \ldots, m$ *there exists a unique function* g_k *such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{g} & \mathcal{O} \\ \pi_k \downarrow & & \downarrow \pi_k \\ \mathcal{O}_k & \xrightarrow{g_k} & \mathcal{O}_k \end{array}$$

Moreover, $g_k \in S(\mathcal{O}_k, \boldsymbol{f}_k)$.

Proof. First observe that if $j \in \mathcal{O}_k$, then $j \in \mathcal{O}$, so define

$$g_k(j) = \pi_k(g(j)).$$

We must first check that the g_k are well defined. Let $j \in \mathcal{O}_k$. Then

$${\pi_k}^{-1}(j) = \{j + in_k : i = 1, \ldots, \frac{l}{n_k}\},$$

that is, ${\pi_k}^{-1}(j)$ is an n_k–complex. Note that $j \in {\pi_k}^{-1}(j)$.

Now by Lemma 6, $g({\pi_k}^{-1}(j))$ is also an n_k–complex, so by remark 2,

$$\pi_k(g({\pi_k}^{-1}(j))) = \{s\},$$

for some $s \in \mathcal{O}_k$, so the g_k are well defined.

Let τ be a topology on $\mathcal{O}_k$ such that $\boldsymbol{f}_k$ is τ–continuous, thus τ is generated by r–complexes, for some divisor r of n_k.

Let $\mathcal{U} \in \tau$. Then by Lemma 10 (1) ${\pi_k}^{-1}(\mathcal{U})$ is a union of r–complexes in $\mathcal{O}$. Now since $g \in S(\mathcal{O}, \boldsymbol{f})^*$, $g^{-1}({\pi_k}^{-1}(\mathcal{U}))$ is a union of r–complexes in $\mathcal{O}$ too, so

$$\begin{aligned} g_k^{-1}(\mathcal{U}) &= \{x : g_k(x) \in \mathcal{U}\} \\ &= \{x : \text{ for any } y \in \pi_k^{-1}(x),\ \pi_k(g(y)) \in \mathcal{U}\} \\ &= \{\pi_k(y) : y \in g^{-1}(\pi_k^{-1}(\mathcal{U}))\} \\ &= \pi_k(g^{-1}(\pi_k^{-1}(\mathcal{U}))), \end{aligned}$$

and applying Lemma 10 (2) $g_k^{-1}(\mathcal{U})$ is a union of r–complexes in $\mathcal{O}_k$, so g_k is τ–continuous. □

Lemma 12. *With the notation we have been using, for* $k = 1, 2, \ldots, m$, *let* $r_k \in \mathcal{O}_k$. *Then there exists a unique* $s \in \mathcal{O}$ *such that for all* k

$$\pi_k(s) = r_k.$$

Proof. We use the Chinese Remainder Theorem to find $s < l$ such that for all k

$$s \equiv r_k \pmod{n_k}.$$

□

Definition 3. *Let* $h_k : \mathcal{O}_k \to \mathcal{O}_k$*, for* $k = 1, 2, \ldots, m$*. Define*

$$h_1 * h_2 * \cdots * h_m : \mathcal{O} \longrightarrow \mathcal{O}$$

where $h_1 * h_2 * \cdots * h_m(x)$ *is the unique* y *such that for* $k = 1, 2, \ldots, m$

$$\pi_k(y) = h_k(\pi_k(x)).$$

Lemma 13. *With the same notation as in the previous definitions.*

(1) *If* $g : \mathcal{O} \longrightarrow \mathcal{O}$*, then* $g_1 * g_2 * \cdots * g_m = g$*.*
(2) *If for* $k = 1, 2, \ldots, m$*,* $h_k : \mathcal{O}_k \to \mathcal{O}_k$*, then* $(h_1 * h_2 * \cdots * h_m)_k = h_k$*.*

Proof. Straightforward. □

Lemma 14. *Let* $\boldsymbol{f}$ *and* $\boldsymbol{f}_k$ *be as in Definition 2 and for* $k = 1, 2, \ldots, m$*, let* $g_k : \mathcal{O}_k \to \mathcal{O}_k$ *be in* $S(\mathcal{O}_k, \boldsymbol{f}_k)^*$*. Then* $g_1 * g_2 * \cdots * g_m \in S(\mathcal{O}, \boldsymbol{f})^*$*.*

Proof. Let τ be a p^r–complex topology on $\mathcal{O}$, where p is prime. Since the n_k's are pairwise relatively prime and p^r divides l, then p^r divides a unique n_k. We may assume that this $k = 1$.

Let $\mathcal{U} \in \tau$. Then

$$\begin{aligned}
(g_1 * g_2 * \cdots * g_m)^{-1}(\mathcal{U}) &= \bigcup_{u \in \mathcal{U}} \{x : g_1 * g_2 * \cdots * g_m(x) = u\} \\
&= \bigcup_{u \in \mathcal{U}} \{x : \text{ for all } k,\ \pi_k(u) = g_k(\pi_k(x))\} \\
&= \bigcup_{u \in \mathcal{U}} \bigcap_{k=1}^{n} \pi_k^{-1} g_k^{-1} \{\pi_k(u)\} \\
&= \bigcap_{k=1}^{n} \bigcup_{u \in \mathcal{U}} \pi_k^{-1} g_k^{-1} \{\pi_k(u)\} \\
&= \bigcap_{k=1}^{n} \pi_k^{-1} g_k^{-1} \Big(\bigcup_{u \in \mathcal{U}} \{\pi_k(u)\}\Big) \\
&= \bigcap_{k=1}^{n} \pi_k^{-1} g_k^{-1} \pi_k(\mathcal{U})
\end{aligned}$$

By Lemma 10 (2), $\pi_1(\mathcal{U}))$ is a p^r–complex in $\mathcal{O}_1$ so $g_1^{-1}(\pi_1(\mathcal{U}))$ is a union of p^r–complexes in $\mathcal{O}_1$, since g_1 is continuous, and thus by 10 (1), $\pi_1^{-1}(g_1^{-1}(\pi_1(\mathcal{U})))$ is also a union of p^r–complexes in $\mathcal{O}$.

Now for $k > 1$, if $\mathcal{V}$ is any p^r–complex in τ and $A \subseteq \mathcal{O}_k$,

$$\mathcal{V} \cap \pi_k^{-1}(A) = \bigcup_{a \in A} \mathcal{V} \cap \pi_k^{-1}(\{a\}) \qquad (*)$$

is not empty, there exists $x \in \mathcal{V} \cap \pi_k^{-1}(\{a\})$ for some $a \in A$, that is to say

$$x \equiv a \pmod{n_k}.$$

So since for any $y \in \mathcal{V}$, $y \equiv x \pmod{\frac{l}{p^r}}$ and $\frac{l}{p^r}$ is a multiple of n_k, $y \in \pi_k^{-1}(\{a\})$, that is, $\mathcal{V} \subseteq \pi_k^{-1}(\{a\})$.

This proves that if $\mathcal{V}$ is any p^r–complex in τ, $(*)$ is either empty or equals $\mathcal{V}$. Moreover, if $\mathcal{V}$ is a union of p^r–complexes, then $(*)$ is empty or a union of p^r–complexes in $\mathcal{O}$. Similarly,

$$\mathcal{V} \cap \pi_2^{-1}(A_2) \cap \pi_3^{-1}(A_3) \cap \cdots \cap \pi_m^{-1}(A_m)$$

is either empty or a union of p^r–complexes, for $A_k \subseteq \mathcal{O}_k$, $k \geq 2$.

Now letting

$$\begin{aligned} \mathcal{V} &= \pi_1^{-1}(g_1^{-1}(\pi_1(\mathcal{U}))) \\ A_k &= g_k^{-1}(\pi_k(\mathcal{U})) \end{aligned}$$

for $k > 1$, we get that

$$(g_1 * g_2 * \cdots * g_m)^{-1}(\mathcal{U})$$

is a union of p^r–complexes in $\mathcal{O}$ and thus $g_1 * g_2 * \cdots * g_m$ is τ–continuous.

□

The two lemmas above provide a proof of the following theorem.

Theorem 15. *Let* $\boldsymbol{f} : \mathcal{O} \longrightarrow \mathcal{O}$ *be a single cycle of length* $l = n_1 \cdots n_m$*, where the* n_k*'s are pairwise relatively prime. For* $k = 1, \ldots, m$*, let* $\boldsymbol{f}_k : \mathcal{O}_k \longrightarrow \mathcal{O}_k$ *be cycles of length* n_k*, respectively. Then*

$$g \in S(\mathcal{O}, \boldsymbol{f}) \quad \textit{if and only if} \quad g_k \in S(\mathcal{O}_k, \boldsymbol{f}_k)$$

for $k = 1, \ldots, m$*.*

2.2. **Gluing cycles.** As a special case of what we have proved so far, the study of a single cycle $\boldsymbol{f}$ of length $l = p_1^{k_1} \cdots p_m^{k_m}$, where the p_i's are different prime numbers, is equivalent to the study of m cycles $\boldsymbol{f}_i$ of length $p_i^{k_i}$. In the next lemmas and theorems we study a function $\boldsymbol{f}$ that is the union of several cycles of different lengths.

Lemma 16. *Let* $\boldsymbol{f} : A \longrightarrow A$ *consist of* m *orbits* $\mathcal{O}_k$ *such that for* $k = 1, 2, \ldots, m$ $\boldsymbol{f} \restriction \mathcal{O}_k : \mathcal{O}_k \to \mathcal{O}_k$ *are cycles* $\mathcal{O}_k = \{1, 2, \ldots, n_k\}$ *of length* n_k*, where the* n_k*'s are pairwise relatively prime.*

For any non–empty open set $\mathcal{U}$ *in a topology* τ *on* A

$$\mathcal{U} = \bigcup_{i \in F} \mathcal{U}_i,$$

where $F \subseteq \{1, \ldots, m\}$ *is a non–empty set of indices for which* $\mathcal{U}_i \subseteq \mathcal{O}_i$ *and* $\mathcal{U}_i \neq \emptyset$*. Define*

$$\overline{\mathcal{U}_i} = \bigcup_{j=1}^{m} \mathcal{V}_j\,,$$

where

$$\mathcal{V}_j = \begin{cases} \mathcal{U}_i & , \text{ if } j = i, \\ \mathcal{O}_j & , \text{ if } j \neq i,\ j \in F \\ \emptyset & , \text{ otherwise.} \end{cases}$$

Then

(1) $\mathcal{U} = \bigcap_{i \in F} \overline{\mathcal{U}_i}$.
(2) $\overline{\mathcal{U}_i} \in \tau$.

Proof. (1) is straightforward.

To prove (2), let us fix i and let $n = \max\{n_1, \cdots, n_m\}$. Then

$$\begin{aligned} \bigcup_{r=1}^{n} \boldsymbol{f}^{-rn_i}(\mathcal{U}) &= \bigcup_{r=1}^{n} \left(\bigcup_{j \in F} \boldsymbol{f}^{-rn_i}(\mathcal{U}_j) \right) \\ &= \bigcup_{j \in F} \left(\bigcup_{r=1}^{n} \boldsymbol{f}^{-rn_i}(\mathcal{U}_j) \right). \end{aligned}$$

We observe that since n_i is the length of the cycle $\boldsymbol{f} \restriction \mathcal{O}_i$, by remark 1, for $j = i$,

$$\bigcup_{r=1}^{n} \boldsymbol{f}^{-rn_i}(\mathcal{U}_i) = \mathcal{U}_i.$$

Let $j \in F$, $j \neq i$. Choose $x \in \mathcal{U}_j \neq \emptyset$. Then the n_j points $\boldsymbol{f}^{n_i}(x), \boldsymbol{f}^{2n_i}(x), \ldots, \boldsymbol{f}^{n_j \cdot n_i}(x)$ belong to $\mathcal{O}_j$ and are all different, or else for some $0 < r,\ s \leq n_j$,

$$rn_i \equiv sn_i \pmod{n_j},$$

so

$$r \equiv s \pmod{n_j},$$

which is impossible, so

$$\bigcup_{r=1}^{n} \boldsymbol{f}^{-rn_i}(\mathcal{U}_j) = \mathcal{O}_j.$$

This proves that

$$\overline{\mathcal{U}_i} = \bigcup_{r=1}^{n} \boldsymbol{f}^{-rn_i}(\mathcal{U}) \in \tau.$$

$\square$

Theorem 17. *Let* $\boldsymbol{f} : A \longrightarrow A$ *consist of* m *orbits* $\mathcal{O}_k$ *such that for* $k = 1, 2, \ldots, m$, $\boldsymbol{f} \upharpoonright \mathcal{O}_k : \mathcal{O}_k \to \mathcal{O}_k$ *are cycles* $\mathcal{O}_k = \{1, 2, \ldots, n_k\}$ *of length* p_i^k, *where the* p_i*'s are not necessarily distinct primes. Then*

$$g \in S(A, \boldsymbol{f})^* \quad \textit{if and only if} \quad g \upharpoonright \mathcal{O}_k \in S(\mathcal{O}_k, \boldsymbol{f} \upharpoonright \mathcal{O}_k).$$

Proof. Assume $g \upharpoonright \mathcal{O}_k \in S(\mathcal{O}_k, \boldsymbol{f} \upharpoonright \mathcal{O}_k)$ and let τ be a topology on A such that $\boldsymbol{f}$ is τ–continuous. Let $\mathcal{U} \in \tau$, $\mathcal{U} \neq \emptyset$.

By Lemma 16, (1)

$$\mathcal{U} = \bigcap_{i \in F} \overline{\mathcal{U}_i},$$

where F is a non–empty set of indices for which $\mathcal{U}_i = \mathcal{U} \cap \mathcal{O}_i$ is a non–empty open set in the relativized topology. So by Lemma 16, (1) and (2), to check the continuity of g it is enough to find $g^{-1}(\overline{\mathcal{U}_i})$, for each $i \in F$.

So let us fix i. Since $\overline{\mathcal{U}_i} \in \tau$, and $\boldsymbol{f}$ is τ–continuous, τ contains $\tau(\boldsymbol{f}, \overline{\mathcal{U}_i})$. Now any $\mathcal{W} \in \tau(\boldsymbol{f}, \overline{\mathcal{U}_i})$ is either empty, equals A or is of the form

$$\mathcal{W} = \bigcup_k \bigcap_j \boldsymbol{f}^{-n_{kj}}(\overline{\mathcal{U}_i}) = \bigcup_{j=1}^{m} \mathcal{W}_j,$$

where

$$\mathcal{W}_j = \begin{cases} \bigcup_r \bigcap_s \boldsymbol{f}^{-n_{rs}}(\mathcal{U}_i) & \text{, if } j = i, \\ \mathcal{O}_j & \text{, if } j \neq i,\ j \in F \\ \emptyset & \text{, otherwise,} \end{cases}$$

where $\mathcal{W}_i$ could be empty, that is,

$$\mathcal{W} = \overline{\bigcup_r \bigcap_s \boldsymbol{f}^{-n_{rs}}(\mathcal{U}_i)}$$
$$\mathcal{W} = \overline{\bigcup_r \bigcap_s (\boldsymbol{f} \upharpoonright \mathcal{O}_i)^{-n_{rs}}(\mathcal{U}_i)}$$

It is clear that $\boldsymbol{f} \restriction \mathcal{O}_i$ is continuous for $\tau(\boldsymbol{f},\overline{\mathcal{U}_i})$ relativized to $\mathcal{O}_i$. In fact,

$$\tau(\boldsymbol{f},\overline{\mathcal{U}_i}) \cap \mathcal{O}_i = \tau(\boldsymbol{f} \restriction \mathcal{O}_i,\mathcal{U}_i).$$

So we note that since $g \restriction \mathcal{O}_i \in S(\mathcal{O}_i, \boldsymbol{f} \restriction \mathcal{O}_i)$,

$$(g \restriction \mathcal{O}_i)^{-1}(\mathcal{U}_i) \in \tau(\boldsymbol{f} \restriction \mathcal{O}_i,\mathcal{U}_i),$$

so

$$(g \restriction \mathcal{O}_i)^{-1}(\mathcal{U}_i) = \bigcup_k \bigcap_j (\boldsymbol{f} \restriction \mathcal{O}_i)^{-n_{kj}}(\mathcal{U}_i) = \bigcup_k \bigcap_j \boldsymbol{f}^{-n_{kj}}(\mathcal{U}_i)$$

so

$$\overline{(g \restriction \mathcal{O}_i)^{-1}(\mathcal{U}_i)} \in \tau(\boldsymbol{f},\overline{\mathcal{U}_i}).$$

Finally

$$g^{-1}(\overline{\mathcal{U}_i}) = \bigcup_{i \in F} g_i^{-1}(\mathcal{V}_i) \quad = \quad \overline{(g \restriction \mathcal{O}_i)^{-1}(\mathcal{U}_i)} \in \tau(\boldsymbol{f},\overline{\mathcal{U}_i}) \subseteq \tau,$$

and thus g is τ–continuous.

For the other implication, if τ is a topology on $\mathcal{O}_k$ such that $\boldsymbol{f} \restriction \mathcal{O}_k$ is τ–continuous, then $\tau \cup \{A\}$ is a topology over A and $\boldsymbol{f}$ is $\tau \cup \{A\}$–continuous, so for any $\mathcal{U} \in \tau$,

$$g_i^{-1}(\mathcal{U}) = g^{-1}(\mathcal{U}) \in \tau \cup \{A\},$$

but obviously $g_i^{-1}(\mathcal{U}) \neq A$, so g_i is τ–continuous. □

Theorem 18. *Let $\boldsymbol{f} : A \longrightarrow A$ consist of m orbits $\mathcal{O}_k$ such that for $k = 1, 2, \ldots, m$, $\boldsymbol{f} \restriction \mathcal{O}_k : \mathcal{O}_k \to \mathcal{O}_k$ are cycles of length n_k, where the n_k's are powers of not necessarily distinct primes. For each prime number p_i let*

$$\mathcal{O}^{p_i} = \bigcup\{\mathcal{O}_k : \text{ the length of } \mathcal{O}_k \text{ is a power of } p_i\}.$$

Then

$$g \in S(A, \boldsymbol{f})^* \quad \text{if and only if} \quad g \restriction \mathcal{O}^{p_i} \in S(\mathcal{O}^{p_i}, \boldsymbol{f} \restriction \mathcal{O}^{p_i}).$$

Proof. The proof is similar to that of Theorem 17, we only need to modify slightly Lemma 16.

If

$$\mathcal{U} = \bigcup_{i \in F} \mathcal{U}_i,$$

where F is a non–empty set of indices for which $\mathcal{U}_i \subseteq \mathcal{O}_i$ and $\mathcal{U}_i \neq \emptyset$. Let

$$\mathcal{W}_{p_i} = \bigcup\{\mathcal{U}_k : k \in F \text{ and the length of } \mathcal{O}_k \text{ is a power of } p_i\}$$

and define

$$\overline{\mathcal{W}_i} = \bigcup_{j \in F} \mathcal{V}_j,$$

where

$$\mathcal{V}_j = \begin{cases} \mathcal{W}_{p_i} & \text{, if } j = i, \\ \mathcal{O}_j & \text{, if } j \neq i,\ j \in F \\ \emptyset & \text{, otherwise.} \end{cases}$$

We can now copy the proof of Theorem 17 with $\overline{\mathcal{U}_i}$ replaced by $\overline{\mathcal{W}_i}$. □

Theorem 19. *Let* $\boldsymbol{f} : A \longrightarrow A$ *be a function with no branches. Then* $S(A, \boldsymbol{f}) = S_0^c(A, \boldsymbol{f})$ *if and only if for any prime* p *, if* p^k *divides the length of one cycle, it divides the length of another cycle.*

Remark 3. *One should note that the condition above implies that the maximal power of a prime must divide at least two cycles or none at all. This is what will be used in the proof.*

Proof. Let us first recall that by Lemma 14 and Theorem 15, we may assume that the function $\boldsymbol{f}$ is composed of several cycles, each of length a power of a prime. Cycles of length one, that is fixed points of $\boldsymbol{f}$ are no problem as we can easily see.

Suppose that the condition on the cycles holds and let $\mathcal{O}^p$ be the union of all cycles of length a power of the prime p. So the maximal power of p that divides one of the cycles of $\mathcal{O}^p$, divides two of them, so by Lemma 8, for any $g \in S(A, \boldsymbol{f})^*$, there exists $n \in \mathbb{N}$ such that for any $x \in \mathcal{O}^p$,

$$g(x) = \boldsymbol{f}^n(x).$$

Let $p_1, p_2, \ldots, p_r$ be all the primes that divide a cycle of $\boldsymbol{f}$ and $n_1, n_2, \ldots, n_r$ be the associated numbers found in the previous paragraph. By the Chinese Remainder Theorem, we can find a single n such that for all $i = 1, \ldots, r$

$$n \equiv n_i \pmod{p^{ki}},$$

where k_i is the maximal exponent of the prime p that divides a cycle of $\boldsymbol{f}$. By Remark 1 (2) this n is such that

$$g = \boldsymbol{f}^n.$$

Suppose now that the condition is false. Then there exists a cycle $\mathcal{O}_i$ whose length is a power p^k of a prime. By Lemma 9, there are non–trivial functions in $S(\mathcal{O}_i, \boldsymbol{f} \restriction \mathcal{O}_i)$, so by Theorem 15, there exist functions $g \in S(A, \boldsymbol{f})$ that do not belong to $g \in S_0(A, \boldsymbol{f})$, and the theorem is proved.

□

3. Cycles and Branches

Let us now study the case of a function with a cycle of length l and several branches, as depicted in Diagram 2.

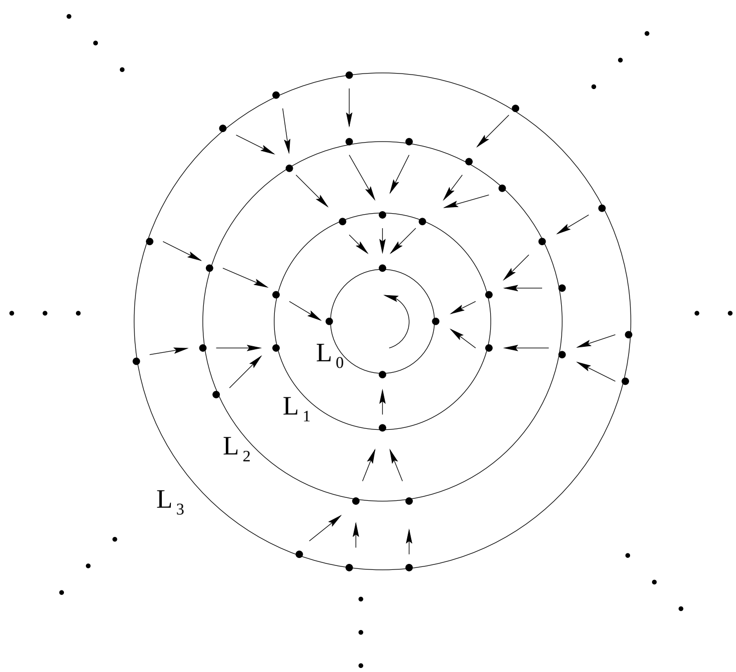

Diagram 2

We can define levels that will look like concentric circles.

$$\begin{aligned} L_0 &= \{x : x = \boldsymbol{f}^l(x)\} = \text{ points on the cycle.} \\ L_1 &= \{x : \boldsymbol{f}(x) \in L_0\} - L_0 \\ &\vdots \\ L_{n+1} &= \{x : \boldsymbol{f}(x) \in L_n\} \\ &\vdots \end{aligned}$$

Lemma 20. *If $\boldsymbol{f} : A \to A$ has an orbit $\mathcal{O}$ containing a cycle and if $g \in S(A, \boldsymbol{f})^*$, then g maps levels of $\mathcal{O}$ into levels of $\mathcal{O}$.*

Proof. It is clear that for $i \neq j$, $L_i \cap L_j = \emptyset$ and that $\boldsymbol{f}^{-m}(L_n) = L_{n+m}$, where L_{n+m} could be the empty set. So $\Sigma = \{L_i : i \in \mathbb{N}\}$ is a base for a topology τ under which $\boldsymbol{f}$ is continuous.

Let L_m be the level of $g(x)$, for certain $x \in L_k$, then $x \in g^{-1}(L_m) = \bigcup_{j \in J} L_j$ so $k \in J$ and $g(L_k) \subseteq L_m$.

□

An immediate consequence of this lemma is that we can simplify our functions and study cycles with a single chain attached to each (some, one) of its points, eliminating all the extra branching.

Lemma 21. *If $g \in S(A, \boldsymbol{f})^*$, then g is increasing in the induced order, i.e., for points x and y not in the cycle, $x \prec y$ implies $g(x) \preceq g(y)$.*

Proof. Consider the topology τ whose base is the sets $\mathcal{U}_n = \bigcup_{i=0}^{n} L_i$. One should observe that if $x \prec y$ and $x \in \mathcal{U}_k$, then $y \in \mathcal{U}_k$.

Also, $\boldsymbol{f}^{-m}(\mathcal{U}_n) = \mathcal{U}_{n+m}$, so $\boldsymbol{f}$ is τ–continuous and if L_m is the level of $g(x)$, then

$$x \in g^{-1}(\mathcal{U}_m) = \bigcup_{j \in J} \mathcal{U}_j = \mathcal{U}_k,$$

for some $k \in \mathbb{N}$.

So if $x \prec y$,

$$y \in \mathcal{U}_k = g^{-1}(\mathcal{U}_m), \quad \text{so} \quad g(y) \in \mathcal{U}_m \quad \text{and} \quad g(x) \preceq g(y).$$

□

Lemma 22. *Let $g \in S(A, \boldsymbol{f})^*$. If there is a point mapped by g onto a point not in the cycle of its orbit, then there exists an $n \in \mathbb{N}$ such that for any point x not in the same orbit, $g(x) = \boldsymbol{f}^n(x)$.*

Proof. Let $a \in \mathcal{O}_1$ be as in the hypothesis. Say $g(a) = \boldsymbol{f}^n(a)$.

For any $x \notin \mathcal{O}_1$, define

$$\mathcal{U} = \{\boldsymbol{f}^n(a), \boldsymbol{f}^n(x)\}.$$

Since a is in a branch, for any k, $\boldsymbol{f}^{-k}(\mathcal{U})$ only contains points that are below $\boldsymbol{f}^n(a)$ in that branch.

If we consider the topology $\tau(\boldsymbol{f}, \mathcal{U})$, all these $\boldsymbol{f}^{-k}(\mathcal{U})$'s are disjoint over $\mathcal{O}_1$ and a belongs only to $\boldsymbol{f}^{-n}(\mathcal{U})$. Of course x also belongs to $\boldsymbol{f}^{-n}(\mathcal{U})$.

Now since g is $\tau(\boldsymbol{f}, \mathcal{U})$–continuous and $a \in g^{-1}(\mathcal{U})$, by the comments in the previous paragraph, $x \in g^{-1}(\mathcal{U})$, so $g(x) \in \mathcal{U}$ and thus $g(x) = \boldsymbol{f}^n(x)$. □

Corollary 23. *Let $g \in S(A, \boldsymbol{f})^*$. If there are two points in two different orbits that are mapped by g onto points not in the cycle of their respective orbits, then there exists an $n \in \mathbb{N}$ such that $g = \boldsymbol{f}^n$.*

Corollary 24. *Let* $g \in S(A, \boldsymbol{f})^*$ *and assume that there are two branches of the same maximal length attached to two different cycles. Then either* g *maps all branches onto points in their respective cycle or there exists an* $n \in \mathbb{N}$ *such that* $g = \boldsymbol{f}^n$ *.*

Proof. Let x_0 and y_0 be the extreme points of the two branches of maximal length, i.e., the points that have no predecessors. If $g(x_0) = \boldsymbol{f}^n(x_0)$ is a point in the branch of x_0, then by Lemma 22, $g(y_0) = \boldsymbol{f}^n(y_0)$, so it is in the branch of y_0, so by Corollary 23, there exists an $n \in \mathbb{N}$ such that $g = \boldsymbol{f}^n$.

If $g(x_0)$ is not in the branch of x_0, then since it is maximal, by Lemma 22, no other point can be mapped into its branch. □

Before proving our main theorem we will study separately a very special case, namely that of a function all of whose cycles are of length 1, that is, fixed points. These functions will be called *length–1–cyclic.*

Theorem 25. *Let* $\boldsymbol{f}$ *be a length–1–cyclic function. Then* $S(A, \boldsymbol{f}) = S_0^c(A, \boldsymbol{f}) =$ *if and only if the lengths of the two longest branches that belong to two different orbits differ at most by 1.*

Observe that this includes the case when all branches belong to the same orbit and have length 1.

Proof. Assume first that the two longest branches have length $n+1$ and n and that they belong to orbits $\mathcal{O}_1$ or $\mathcal{O}_2$, respectively. Let x_0 and y_0 be the extreme points of each of these branches, c_1 and c_2 the fixed points in their respective orbits, for $r = 1, \ldots, n$,

$$x_r = \boldsymbol{f}^r(x_0),$$

and for $r = 1, \ldots, n-1$,

$$y_r = \boldsymbol{f}^r(y_0).$$

Let $g \in S(A, \boldsymbol{f})^*$ and let $g(x_0) = \boldsymbol{f}^r(x_0)$. We have three cases.

Case 1. $r > n$.

Then by Lemma 22 for any point x outside $\mathcal{O}_1$ and by Lemma 21 for any point x in $\mathcal{O}_1$, $g(x) = \boldsymbol{f}^r(x)$, so $g \in S_0(A, \boldsymbol{f})$.

Case 2. $r < n$.

Then by Lemma 22, $g(y_0) = \boldsymbol{f}^r(y_0) = y_r$ belongs to the branch of orbit $\mathcal{O}_2$, so by corollary 23, $g = \boldsymbol{f}^r \in S_0(A, \boldsymbol{f})$.

Case 3. $r = n$.

In this case, all points outside $\mathcal{O}_1$ are mapped by g into their respective fixed points.

Let $\mathcal{U} = \{c_1, c_2\}$, and consider the topology $\tau(\boldsymbol{f}, \mathcal{U})$. Observe that y_0 belongs to a basic open set $\boldsymbol{f}^{-k}(\mathcal{U})$ if and only if $k \geq n$. So since

$y_0 \in g^{-1}(\mathcal{U})$, for some $k \geq n$

$$\boldsymbol{f}^{-k}(\mathcal{U}) \subseteq g^{-1}(\mathcal{U}),$$

and thus for $r = 1, \ldots, n,\ x_r \in g^{-1}(\mathcal{U})$.

So

$$g(x_1) = \cdots = g(x_n) = g(c_1) = c_1,$$

That is, $g = \boldsymbol{f}^n$ and thus belongs to $S_0(A, \boldsymbol{f})$.

Assume on the contrary that the longest branch is at least two points longer than the next longest branch that belongs to a different orbit. For notational convenience, we will let the lengths of the branches be $n+1$ and $m+1$, respectively. Using the notation of the previous paragraphs, this means that $n+1 \geq m+3$. We will build a function g that belongs to $S(A, \boldsymbol{f})^*$ but not to $S_0(A, \boldsymbol{f})$.

Define

$$g(x) = \begin{cases} x_{n-1} & \text{, if } x \text{ is } x_0, x_1, \ldots, x_{n-m-2} \\ c_1 & \text{, if } x \text{ is } x_{n-m-1}, \ldots, x_n, c_1 \\ c_2 & \text{, if } x \text{ is } y_0, \ldots, y_m, c_2 \end{cases}$$

and all points in other orbits are mapped into their respective fixed points.

Let τ be a topology and $\mathcal{U} \in \tau$.

The main observation is that for any orbit $\mathcal{O}$ different from $\mathcal{O}_1$, if c is its fixed point and $r \geq m+1$,

$$\boldsymbol{f}^{-r}(\mathcal{U}) \cap \mathcal{O} = \begin{cases} \emptyset & \text{, if } c \notin \mathcal{U} \\ \mathcal{O} & \text{, if } c \in \mathcal{U}. \end{cases}$$

but also,

$$g^{-1}(\mathcal{U}) \cap \mathcal{O} = \begin{cases} \emptyset & \text{, if } c \notin \mathcal{U} \\ \mathcal{O} & \text{, if } c \in \mathcal{U}. \end{cases}$$

Using these two facts, we can check that if $c_1 \notin \mathcal{U}$,

$$g^{-1}(\mathcal{U}) = \begin{cases} \boldsymbol{f}^{-(n+1)}(\mathcal{U}) & \text{, if } x_{n-1} \notin \mathcal{U} \\ \bigcup_{i=m+2}^{n} \boldsymbol{f}^{-i}(\mathcal{U}) & \text{, if } x_{n-1},\ x_n \in \mathcal{U} \\ \bigcup_{i=m+1}^{n-1} \boldsymbol{f}^{-i}(\mathcal{U}) & \text{, if } x_{n-1} \in \mathcal{U} \text{ and } x_n \notin \mathcal{U}. \end{cases}$$

If $c_1 \in \mathcal{U}$ and $x_{n-1} \in \mathcal{U}$,

$$g^{-1}(\mathcal{U}) \cap \mathcal{O} = \mathcal{O} = \boldsymbol{f}^{-n}(\mathcal{U})$$

and by the remarks above,

$$g^{-1}(\mathcal{U}) = \boldsymbol{f}^{-n}(\mathcal{U}).$$

Next we observe that if $c_1 \in \mathcal{U}$ and $x_{n-1} \notin \mathcal{U}$, then

$$g^{-1}(\mathcal{U}) \cap \mathcal{O} = \{x_{n-m-1}, x_{n-m}, \ldots, x_n, c_1\},$$

and also,

$$\mathcal{U} \cap \boldsymbol{f}^{-1}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-n}(\mathcal{U}) \cap \mathcal{O} = \begin{cases} \{c_1, x_n\} & \text{, if } x_n \in \mathcal{U} \\ \{c_1\} & \text{, if } x_n \notin \mathcal{U} \end{cases}$$

Using this fact one easily checks that if $c_1 \in \mathcal{U}$ and $x_{n-1} \notin \mathcal{U}$,

$$g^{-1}(\mathcal{U}) = \begin{cases} \boldsymbol{f}^{-(m+1)}\mathcal{U} \cap \boldsymbol{f}^{-(m+2)}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-(m+n)}(\mathcal{U}) & \text{, if } x_n \in \mathcal{U} \\ \boldsymbol{f}^{-(m+2)}\mathcal{U} \cap \boldsymbol{f}^{-(m+2)}(\mathcal{U}) \cap \cdots \cap \boldsymbol{f}^{-(m+n)}(\mathcal{U}) & \text{, if } x_n \in \mathcal{U} \end{cases}$$

So in any case, $g^{-1}(\mathcal{U}) \in \tau$, so g is τ–continuous. This proves that $g \in S(A, \boldsymbol{f})^*$. Finally, since obviously $g \notin S_0(A, \boldsymbol{f})$, the proof of the theorem is complete. □

Let $\mathcal{O}$ be an orbit of the function $\boldsymbol{f}$ and suppose $\{1, \ldots, l\}$ are the points in its cycle. For $i = 1, \ldots, l$ define

$$\mathcal{U}_i = \{x : \boldsymbol{f}^{k \cdot l}(x) = i, \text{ for some } k \in \mathbb{N}\},$$

that is, all points in the orbit whose distance to the point i is a multiple of l. These l disjoint sets can be visualized as obtained by wrapping the branches around the cycle. They also are a base for a topology for which $\boldsymbol{f}$ is continuous.

Since

$$\boldsymbol{f}(\mathcal{U}_i) = \mathcal{U}_{\boldsymbol{f}(i)} = \mathcal{U}_{i+1},$$

where addition is modulo the length of the appropriate cycle, $\boldsymbol{f}$ induces a function $\boldsymbol{f}_* : A_* \longrightarrow A_*$, where A_* is the set of all elements $\mathcal{U}_i$ of this partition. $\boldsymbol{f}_*$ will have the same number of cycles, of the same length as $\boldsymbol{f}$, only without the branching of $\boldsymbol{f}$.

Lemma 26. *Let $g \in S(A, \boldsymbol{f})^*$. Then g induces a function $g_* : A_* \longrightarrow A_*$ such that $g_* \in S(A_*, \boldsymbol{f}_*)^*$.*

Proof. For any orbit $\mathcal{O}$ of $\boldsymbol{f}$ and $i \in \mathcal{O}$, define

$$\begin{aligned} g_* : A_* &\longrightarrow A_* \\ \mathcal{U}_i &\longmapsto \mathcal{U}_{g(i)} \end{aligned}$$

Observe that

$$g_*(\mathcal{U}_i) = \mathcal{U}_j \text{ if and only if } g(i) \in \mathcal{U}_j.$$

Moreover, let τ be the topology generated by the $\mathcal{U}_i$'s. Since $g \in S(A, \boldsymbol{f})^*$ and $\boldsymbol{f}$ is τ–continuous, if $g(i) \in \mathcal{U}_j$, then $i \in g^{-1}(\mathcal{U}_j)$, so $\mathcal{U}_i \subseteq g^{-1}(\mathcal{U}_j)$, that is for any $x \in \mathcal{U}_i$, $g(x) \in \mathcal{U}_j$ so

$$\bigcup g_*^{-1}(\{\mathcal{U}_j\}) = g^{-1}(\mathcal{U}_j),$$

and in particular

$$\bigcup f_*^{-n}(\{\mathcal{U}_j\}) = f^{-n}(\mathcal{U}_j).$$

Let τ be a topology over A_* such that $\boldsymbol{f}_*$ is τ–continuous. Then for any $\mathcal{V} \in \tau$,

$$\begin{aligned}
g_*^{-1}(\mathcal{V}) &= \bigcup_{\mathcal{U}_j \in \mathcal{V}} g_*^{-1}(\{\mathcal{U}_j\}) \\
&= \bigcup_{\mathcal{U}_j \in \mathcal{V}} \{\mathcal{U}_i \in A_* : g_*(\mathcal{U}_i) = \mathcal{U}_j\} \\
&= \bigcup_{\mathcal{U}_j \in \mathcal{V}} \{\mathcal{U}_i \in A_* : i \in g^{-1}(\mathcal{U}_j)\} \\
&= \bigcup_{\mathcal{U}_j \in \mathcal{V}} \{\mathcal{U}_i \in A_* : i \in \bigcup_l \bigcap_k \boldsymbol{f}^{-n_{lk}}(\mathcal{U}_j)\} \\
&= \bigcup_{\mathcal{U}_j \in \mathcal{V}} \left(\bigcup_l \bigcap_k \{\mathcal{U}_i \in A_* : i \in \boldsymbol{f}^{-n_{lk}}(\mathcal{U}_j)\} \right) \\
&= \bigcup_{\mathcal{U}_j \in \mathcal{V}} \left(\bigcup_l \bigcap_k \{\mathcal{U}_i \in A_* : \mathcal{U}_i \in \boldsymbol{f}_*^{-n_{lk}}(\{\mathcal{U}_j\})\} \right) \\
&= \bigcup_{\mathcal{U}_j \in \mathcal{V}} \left(\bigcup_l \bigcap_k \boldsymbol{f}_*^{-n_{lk}}(\{\mathcal{U}_j\}) \right) \\
&= \bigcup_l \bigcap_k \left(\bigcup_{\mathcal{U}_j \in \mathcal{V}} \boldsymbol{f}_*^{-n_{lk}}(\{\mathcal{U}_j\}) \right) \\
&= \bigcup_l \bigcap_k \boldsymbol{f}_*^{-n_{lk}}(\mathcal{V}),
\end{aligned}$$

which completes the proof of the continuity of g_*.

$\square$

We can now prove our main theorem. As we will see, if $\boldsymbol{f}$ has cycles of length greater than one, only the cycles but not the branches have an effect on the semigroup $S(A, \boldsymbol{f})^*$.

Theorem 27. *Let $\boldsymbol{f} : A \longrightarrow A$ be a non–fixed–point function. Then $S(A, \boldsymbol{f}) = S_0^c(A, \boldsymbol{f})$ if and only if for any prime p, if p^k divides the length of one cycle of $\boldsymbol{f}$, it divides the length of another cycle.*

Proof. One should observe that since $\boldsymbol{f}$ and $\boldsymbol{f}_*$ have the same cycles, the above condition is either verified by both of them or by none of them.

Suppose the condition above holds for $\boldsymbol{f}$. Then since it also holds for $\boldsymbol{f}_*$, by Theorem 19, $S(A_*, \boldsymbol{f}_*)^* = S_0(A_*, \boldsymbol{f}_*)$

Now let $g \in S(A, \boldsymbol{f})^*$, by Lemma 26, $g_* = \boldsymbol{f}_*^n$ for some n, so this implies that for all $x \in A$,

$$g(x) = \boldsymbol{f}^{n+kl_i}(x),$$

where l_i is the length of the cycle of the orbit of x.

If g maps all points in the branches of $\boldsymbol{f}$ to points in their respective cycles, then choosing an appropriate multiple m of the least common multiple of the l_i's, by remark 1,

$$g(x) = \boldsymbol{f}^{n+m}(x),$$

so $g \in S_0(A, \boldsymbol{f})$.

If g does not map all branches into their respective cycles, let x_0 be the extreme point of the longest branch. Assume this point belongs to orbit $\mathcal{O}$. Let $g(x_0) = \boldsymbol{f}^n(x_0)$. By Lemma 22, $g(x_0)$ belongs to the branch of $\mathcal{O}$ or else all points would be mapped by g into their cycles. Moreover, by Lemma 22, when restricted to points not in $\mathcal{O}$, $g = \boldsymbol{f}^n$.

Let $x_r = \boldsymbol{f}^r(x_0)$ be the first element in the branch of x_0 that is mapped into the branch and such that $g(x_r) = \boldsymbol{f}^{n+m}(x_r) \neq \boldsymbol{f}^n(x_r)$, where $m \neq 0$ is a multiple of the length of the cycle of $\mathcal{O}$.

Let us consider the topology generated by

$$\mathcal{U} = \{x_0, x_1, \ldots, x_{r+n}\} \cup \{\boldsymbol{f}^{n+1}(c)\},$$

where c belongs to a cycle of length greater than 1 not in orbit $\mathcal{O}$. Observe that such an orbit must exist by hypothesis.

For x_i in the branch of x_0,

$$x_i \in \boldsymbol{f}^{-m}(\mathcal{U}) \text{ if and only if } 0 \leq i \leq r + n - m.$$

Now since

$$x_r \notin g^{-1}(\mathcal{U}) \text{ and } x_{r-1} \in g^{-1}(\mathcal{U}),$$

$$\boldsymbol{f}^{-(n+1)}(\mathcal{U}) \subseteq g^{-1}(\mathcal{U}),$$

but then $c \in g^{-1}(\mathcal{U})$ which contradicts the fact that $g(c) = \boldsymbol{f}^n(c) \neq \boldsymbol{f}^{n+1}(c)$.

This implies that for any point x in the branch of x_0 that is mapped into that branch, $g(x) = \boldsymbol{f}^n(x)$.

Let us assume now that the condition on the cycles is not satisfied. Then by Theorem 19, there exists a function $\hat{g} \in S(A_*, \boldsymbol{f}_*)^* - S_0(A_*, \boldsymbol{f}_*)$.

Let $\hat{g}(\mathcal{U}_i) = \mathcal{U}_j$. Then $j = \boldsymbol{f}^{n_i}(i)$, for some n_i. As a matter of fact, if K is a multiple of the length of the cycle of the orbit that contains $\mathcal{U}_i$, then $j = \boldsymbol{f}^{n_i+K}(i)$. We can choose K large enough so that it is a multiple of the lengths of all the cycles of $\boldsymbol{f}$ and bigger than the length of all branches in $\boldsymbol{f}$.

Define

$$g : A \longrightarrow A$$

as follows. For $x \in \mathcal{U}_i$, $g(x) = \boldsymbol{f}^{n_i+K}(x)$. It is immediate that

$$g_* = \hat{g}.$$

Let $\boldsymbol{f}$ be τ–continuous and $\mathcal{V} \in \tau$. Let $\mathcal{V}_c$ be the set of points in $\mathcal{V}$ that belong to the cycles of $\boldsymbol{f}$. Then

$$\begin{aligned} g^{-1}(\mathcal{V}) &= g^{-1}(\mathcal{V}_c) = \bigcup_{i \in \mathcal{V}_c} g^{-1}(\{i\}) \\ &= \bigcup_{i \in \mathcal{V}_c} \bigcup g_*^{-1}(\{\mathcal{U}_i\}) \\ &= \bigcup g_*^{-1}(\mathcal{V}_*) \\ &= \bigcup \hat{g}^{-1}(\mathcal{V}_*), \end{aligned}$$

where $\mathcal{V}_* = \{\mathcal{U}_i : i \in \mathcal{V}_c\}$.

By hypothesis, $\hat{g}$ is $\tau(\boldsymbol{f}_*, \mathcal{V}_*)$–continuous, so

$$
\begin{aligned}
g^{-1}(\mathcal{V}) &= \bigcup\left(\bigcup_l\bigcap_k \boldsymbol{f}_*^{-n_{lk}}(\mathcal{V}_*)\right)\\
&= \bigcup\left(\bigcup_l\bigcap_k\bigcup_{i\in\mathcal{V}_c} \boldsymbol{f}_*^{-n_{lk}}(\{\mathcal{U}_i\})\right)\\
&= \bigcup_l\bigcap_k\bigcup_{i\in\mathcal{V}_c}\left(\bigcup \boldsymbol{f}_*^{-n_{lk}+K}(\{\mathcal{U}_i\})\right)\\
&= \bigcup_l\bigcap_k\bigcup_{i\in\mathcal{V}_c} \boldsymbol{f}^{-n_{lk}+K}(\{i\})\\
&= \bigcup_l\bigcap_k \boldsymbol{f}^{-n_{lk}+K}(\mathcal{V}_c)\\
&= \bigcup_l\bigcap_k \boldsymbol{f}^{-n_{lk}+K}(\mathcal{V})
\end{aligned}
$$

so g is τ–continuous. □

References

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