Paraconsistency in Chang's Logic with Positive and Negative Truth Values

Renato A. Lewin*
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Casilla 306 Correo 22, Santiago, CHILE

Marta Sagastume
Departamento de Matemáticas
Universidad Nacional de La Plata
La Plata, ARGENTINA

Abstract

In [6], C. C. Chang introduced a natural generalization of Łukasiewicz infinite valued propositional logic Ł. In this logic the truth values are extended from the interval [0,1] to the interval [-1,1]. We will call Ł* the logic whose designated values are those greater or equal than 0. (Chang calls this logic $p^*[0]$.)

In this semantics, for a truth assignment v the value of the negation is $v(\neg\varphi) = -v(\varphi)$. This implies that there are sentences for which $v(\varphi) = v(\neg\varphi) = 0$, that is, both sentences are tautologies. Moreover, the sentence $\varphi \to (\neg\varphi \to \psi)$ is not a tautology so Ł* is paraconsistent.

Two are the main results of this paper. First we axiomatize the system Ł$^*_0$, the logic whose only designated truth value is 0, that is, the paraconsistent sentences of Ł*. Then, we prove that the categories $\mathcal{MV}$ and $\mathcal{MV}^*$, whose objects are MV–algebras and MV^*–

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algebras respectively, with their corresponding morphisms, are equivalent. These categories are associated with Łukasiewicz' infinite valued calculus and with Chang's logic Ł*, respectively.

1 Introduction

It is well known that $\mathcal{I} = (\langle[0,1];\rightarrow,\neg\rangle,\{1\})$ where the operations on $[0,1]$ are given by

$$\begin{aligned} x \rightarrow y &= \min(1, 1-x+y), \\ \neg x &= 1-x. \end{aligned}$$

is a characteristic matrix for Łukasiewicz infinite valued propositional logic Ł. In [6], C. C. Chang introduced a natural generalization of Ł whose characteristic matrix is $\mathcal{I}^* = (\langle[-1,1];\rightarrow,\neg,\rangle,[0,1])$ with operations defined by

$$\begin{aligned} x \rightarrow y &= \min(1, \max(-1, y-x), \\ \neg x &= -x. \end{aligned}$$

We will call this logic Ł*.

In a previous paper, [4], Chang had established that Ł had the variety of MV–algebras as what we now call an equivalent algebraic semantics in the sense of [1]. In fact this variety is generated by the algebra $\langle[0,1];\oplus,\neg,1\rangle$, whose operations are

$$\begin{aligned} x \oplus y &= \min(1, x+y), \\ \neg x &= 1-x. \end{aligned}$$

The reader may consult [8, 9] for full information on Ł, MV–algebras and other related topics.

In the same sense that the truth values [-1, 1] extend [0, 1], in [6], Chang introduces MV*–algebras and proves that they correspond to the Lindenbaum–Tarski algebras of Ł*, so they are an equivalent algebraic semantics for Ł*. This variety is generated by the algebra $\langle[-1,1];\oplus,\neg,1\rangle$, where the operations are

$$\begin{aligned} x \oplus y &= \min(1, \max(-1, x+y)), \\ \neg x &= -x, \end{aligned}$$

that is, addition is ordinary real number addition truncated below at -1 and above at 1, and negation is the additive inverse.

Chang gives no intuitive interpretation for negative truth values, nevertheless, in the last decade positive and negative truth values have appeared in different contexts, both theoretical and applied.

For instance in [2, 3], comparative logics are introduced to model situations in which propositions are either true or false, but not necessarily in the same way, thus one can admit that one proposition might be "truer" than another, so truth values are many shades of truth and falsehood. It should be noted that the algebraic semantics for comparative logics is the variety of pre–groups. In a pre–group there are two distinguished elements $\mathbf{0}$ for the least positive (or true) truth value and $-\mathbf{0}$ for the largest negative (or false) truth value. Observe that there might exist intermediate values between $-\mathbf{0}$ and $\mathbf{0}$. If $-\mathbf{0} = \mathbf{0}$ then the pre–group is an Abelian lattice ordered group (an l–group.) Moreover, an l–group is a subpre–group of a pre–group P if and only if P is an l–group.

Positive and negative truth values have also appeared in the context of uncertain information processing and inconsistency tolerance in expert systems. In [16], the authors make a theoretical analysis of uncertainty processing in a broad class of compositional expert systems similar to MYCIN and PROSPECTOR. In these, the knowledge of a questionnaire, that is, an assignment of a weight to each question, can be extended to all propositions in such a way that each rule in the Rule Base of the system contributes to the weight of each proposition according to some fixed function. So given a Rule Base $\Theta = \{R_1, \ldots, R_n\}$ and a weight function (or questionnaire) w, the weight of a proposition p is

$$W_\Theta(p \mid w) = w_1 \oplus \cdots \oplus w_n,$$

obtained from the contributions $w_1, \ldots, w_n$ of each rule through a *combining function* $\oplus$.

Some natural conditions imposed on the set of weights are that they are a linearly ordered set G, with a largest element $\top$ (*true*,) a least element $\bot$ (*false*) and a distinguished element o (*no preference*,) that is also a neutral element for the combining function (*addition*) $\oplus$. The addition is closed on $G - \{\top, \bot\}$ (uncertainties cannot give certainty), and $G - \{\top, \bot\}$ has a structure of an *ordered Abelian group*. Therefore, G with this operation and order is what is called an *extended ordered Abelian group*.

In the case of PROSPECTOR, its ordered Abelian group of weights is $\mathbf{PP} = \langle (0,1), \oplus, \leq \rangle$, defined by the usual order and

$$x \oplus y = \frac{xy}{xy + (1-x)(1-y)}.$$

The ordered Abelian group of certainty factors of MYCIN is isomorphic to $\mathbf{PP}$. A review of these results and many references on the subject appear in [17].

Recent work on uninorms should also be mentioned in the context of positive and negative truth values. Uninorms are a generalization of t–norms and t–conorms having a neutral element e. There is a natural association of the interval $(e,1)$ with positive values and of $(0,e)$ with negative values. Uninorms are also related to the combining functions mentioned in the previous paragraphs, for instance, in [14] it is proved that the combining function for MYCIN is a uninorm. In fact, in [16], though they do not use the concept, it is shown that all MYCIN–like expert systems combining functions are (representable) uninorms. For information on uninorms see for example [20, 15, 13, 14, 12]

Even without these interpretations, a system with positive and negative truth values such as $Ł^*$ has a special interest in itself as a non–classical logic. We observe that certain sentences, like for instance $p \rightarrow p$, take value 0 for any valuation $v : Variables \longrightarrow [-1,1]$. Now since 0 is a designated value, $p \rightarrow p$ is a tautology of $Ł^*$. But then, $\neg(p \rightarrow p)$ is also a tautology. Furthermore, we observe that $v(p \rightarrow (\neg p \rightarrow q)) = v(q)$, that is, if q takes a negative value, $p \rightarrow (\neg p \rightarrow q)$ is not a tautology. So this is a non–trivial inconsistent system, that is, $Ł^*$ is paraconsistent.

The paper is organized as follows. After this Introduction, the first section introduces Chang's logic $Ł^*$, its semantics and shows that what is now called its equivalent algebraic semantics, is the class of MV*–algebras introduced by Chang, all this is a rephrasing of the contents of [6]. In the next section we give an axiomatization for the subsystem of all those formulas that take value 0 for any valuation v, that is, the paraconsistent fragment of $Ł^*$, and we prove its algebraizability in the sense of [1]. In the last section we prove that $\mathcal{MV}$ and $\mathcal{MV}^*$, the categories whose objects are MV–algebras and MV^*–algebras respectively, with their corresponding morphisms, are equivalent.

2 The Logic Ł*

In this section we reproduce part of the contents of [6]. Some of them have be rephrased to better suit our purposes.

2.1 Language

The set $\mathcal{F}m$ of the formulas of this logic is recursively generated from a denumerable set Va of propositional variables by a binary operation $\rightarrow$, a unary operation $\neg$, and a constant $\mathbf{1}$. As usual, $x \leftrightarrow y$ stands for $x \rightarrow y$ and $y \rightarrow x$. We will also use the following abbreviations.

$$\begin{aligned} x^+ &:= (x \rightarrow \mathbf{1}) \rightarrow \mathbf{1}, \\ x^- &:= (x \rightarrow \neg\mathbf{1}) \rightarrow \neg\mathbf{1}, \\ x \vee y &:= (\,(x^+ \rightarrow y^+)^+ \rightarrow (\neg x)^-\,) \rightarrow (\,(y^- \rightarrow x^-)^- \rightarrow x^-). \end{aligned}$$

2.2 Axioms

P1 $(x \rightarrow y) \leftrightarrow (\neg y \rightarrow \neg x)$,

P2 $x \leftrightarrow ((y \rightarrow y) \rightarrow x)$,

P3 $\neg(x \rightarrow y) \leftrightarrow (y \rightarrow x)$,

P4 $x \rightarrow \mathbf{1}$,

P5 $\mathbf{1} \leftrightarrow ((\mathbf{1} \rightarrow x) \rightarrow \mathbf{1})$,

P6 $(\,(x \rightarrow \mathbf{1}) \rightarrow ((y \rightarrow \mathbf{1}) \rightarrow z)\,) \rightarrow (\,(y \rightarrow \mathbf{1}) \rightarrow ((x \rightarrow \mathbf{1}) \rightarrow z)\,)$,

P7 $(x \rightarrow y) \leftrightarrow (\,(y^+ \rightarrow x^-) \rightarrow (x^+ \rightarrow y^-)\,)$,

P8 $(x \rightarrow (\neg x \rightarrow y)\,)^+ \leftrightarrow (x^+ \rightarrow (\neg(x^+) \rightarrow y^+)\,)$,

P9 $(x \rightarrow (y \vee z)) \leftrightarrow (\,(x \rightarrow y) \vee (x \rightarrow z)\,)$,

P10 $(x \vee (y \vee z)) \leftrightarrow ((x \vee y) \vee z)$.

2.3 Rules

R1 $x\ ,\ x \to y \vdash_{\text{Ł}^*}\ y$,

R2 $x \to y\ ,\ u \to v \vdash_{\text{Ł}^*}\ (y \to u) \to (x \to v)$,

R3 $x\ \vdash_{\text{Ł}^*}\ x^-$.

2.4 Semantics

We extend recursively any valuation $v : Va \longrightarrow [-1,1]$ to all formulas by defining

$$\overline{v} : \mathcal{F}m \longrightarrow [-1,1]\,,$$

as follows.

1. $\overline{v}(x) = v(x)$,
   for any propositional letter $x \in Va$,
2. $\overline{v}(x \to y) = \min(1, \max(-1, \overline{v}(y) - \overline{v}(x))\,)$,
3. $\overline{v}(-x) = -\overline{v}(x)$.

The following lemma is immediate.

Lemma 1. *For any valuation* v *,*

1. $\overline{v}(x^+) = \max(0, \overline{v}(x))$,
2. $\overline{v}(x^-) = -\max(0, \overline{v}(-x))$,
3. *If* $\overline{v}(\varphi \to \psi) = 0$*, then* $\overline{v}(\varphi) = \overline{v}(\psi)$.

Observe that the value of x^- is not the usual negative part of x but minus the negative part of x.

The soundness and (weak) completeness theorem is the main result in [6].

Theorem 2.

$\vdash_{\text{Ł}^*}\ x$ *if and only if* $\models_{\mathcal{I}^*} \varphi$ *if and only if* $\overline{v}(x) \geq 0$ *for any valuation* v.

An easy induction on the complexity of the proof will show that the strong version of the soundness theorem also holds. It is enough to check that the three rules go from non–negative values to non–negative values.

Theorem 3.

$$\Gamma \vdash_{\text{Ł}^*}\ \varphi \quad \Rightarrow \quad \Gamma \models_{\mathcal{I}^*} \varphi\,.$$

2.5 Algebraization

In [6], Chang introduces MV*–algebras as a generalization of MV–algebras. MV*–algebras can be built from a totally ordered group $\boldsymbol{G}$ and a positive element u of $\boldsymbol{G}$ in the same way as MV–algebras are built from ordered groups and a positive element. Let $G(u) = \{x \in G : -u \leq x \leq u\}$ and define the algebra $\boldsymbol{G}(u) = \langle G(u); \oplus, \neg, \mathbf{1}\rangle$ where

$$\begin{aligned} x \oplus y &= \min(u, \max(-u, x+y)), \\ \neg x &= -x, \\ \mathbf{1} &= u. \end{aligned}$$

Then $\boldsymbol{G}(u)$ is an MV*–algebra. In Theorem 2.21 Chang proves that every MV*–algebra is a subdirect product of algebras $\boldsymbol{G}(u)$.

The most interesting result for our paper is Theorem 3.13, where it is proven that the Lindenbaum–Tarski algebra of Ł* is an MV*–algebra.

We rewrite this in the current terminology of algebraic logic, (see [1].)

Theorem 4. *The deductive system Ł* is algebraizable with one defining equation*

$$x \approx x^+$$

and one equivalence formula

$$x\Delta y := x \leftrightarrow y$$

(strictly speaking these are two formulas.) The equivalent algebraic semantics is the variety of MV*–algebras.

3 The Logic Ł$_0^*$

In this section we introduce an axiomatization for all those sentences of the language of Ł* that take value 0 for any valuation.

3.1 Axioms

Ax1 $(x \to y) \leftrightarrow (\neg y \to \neg x)$,

Ax2 $x \leftrightarrow ((y \to y) \to x)$, Ax3 $\neg(x \to y) \leftrightarrow (y \to x)$,

Ax4 $\mathbf{1} \leftrightarrow ((\mathbf{1} \to x) \to \mathbf{1})$,

Ax5 $((x \to \mathbf{1}) \to ((y \to \mathbf{1}) \to z)) \to ((y \to \mathbf{1}) \to ((x \to \mathbf{1}) \to z))$,

Ax6 $(x \to y) \leftrightarrow ((y^+ \to x^-) \to (x^+ \to y^-))$,

Ax7 $(x \to (\neg x \to y))^+ \leftrightarrow (x^+ \to (\neg(x^+) \to y^+))$,

Ax8 $(x \to (y \vee z)) \leftrightarrow ((x \to y) \vee (x \to z))$,

Ax9 $(x \vee (y \vee z)) \leftrightarrow ((x \vee y) \vee z)$.

3.2 Rules

MP $x\,,\; x \to y \vdash_{Ł_0^*} y$.

TR $x \to y \vdash_{Ł_0^*} (y \to v) \to (x \to v)$.

K* $x \vdash_{Ł_0^*} y \to (x \to y)$.

3.3 Some Results

In this section we summarize several results that will be useful in the sequel. The main axioms, rules and previous results used in the proofs of the first two theorems are indicated.

Theorem 5. *Define* $\mathbf{0} := \mathbf{1} \to \mathbf{1}$. *The following hold in* $Ł_0^*$.

1. *If* $\vdash_{Ł_0^*} x \to y$ *and* $\vdash_{Ł_0^*} y \to z$, *then* $\vdash_{Ł_0^*} x \to z$.
2. $x \to y\,,\; u \to v \vdash_{Ł_0^*} (y \to u) \to (x \to v)$.
3. *If* $\vdash_{Ł_0^*} x \leftrightarrow y$, *then* $\vdash_{Ł_0^*} \neg x \leftrightarrow \neg y$.
4. *If* $\vdash_{Ł_0^*} x \leftrightarrow y$ *and* $\vdash_{Ł_0^*} u \leftrightarrow v$, *then* $\vdash_{Ł_0^*} (x \to u) \leftrightarrow (y \to v)$.
5. *If* $\vdash_{Ł_0^*} x \leftrightarrow y$ *and* $\vdash_{Ł_0^*} y \leftrightarrow z$, *then* $\vdash_{Ł_0^*} x \leftrightarrow z$.
6. *Let* ψ *be obtained from* φ *by replacing some instances of the subformula* σ *of* φ *by the formula* τ. *Assume also that* $\vdash_{Ł_0^*} \sigma \leftrightarrow \tau$. *Then* $\vdash_{Ł_0^*} \varphi \leftrightarrow \psi$.

7. $\vdash_{Ł_0^*} x \to x$.

8. $\vdash_{Ł_0^*} (x \to x) \leftrightarrow (y \to y)$.

9. $\vdash_{Ł_0^*} \neg(x \to x) \leftrightarrow (x \to x)$.

10. $\vdash_{Ł_0^*} \neg\neg(x \to x) \leftrightarrow (x \to x)$.

11. $\vdash_{Ł_0^*} \neg\neg x \leftrightarrow x$.

12. $\vdash_{Ł_0^*} \neg x \leftrightarrow (x \to \mathbf{0})$.

13. $\vdash_{Ł_0^*} \neg(x \to y) \leftrightarrow (\neg x \to \neg y)$.

14. $x , y \vdash_{Ł_0^*} x + y$.

15. $x , y \vdash_{Ł_0^*} x \to y$.

16. $x \vdash_{Ł_0^*} \neg x$.

Proof. Item *1.* is obtained from (TR) by (MP). The second item is $Ł^*$'s Rule R2, of which the transitivity rule TR is a special case. We have chosen to replace R2 by TR since the latter is simpler. Assertions *3.* through *11.* were proven in [6] for $Ł^*$, without using axiom P4 or rule R3. Given that R2 can be replaced by TR, the same proof works here. Notice that *8.* states that for any x, $\vdash_{Ł_0^*} (x \to x) \leftrightarrow \mathbf{0}$.

12., *13.* and *14.* follow from Ax1, Ax2 and Ax3, while *15.* and *16.* follow from K^* and previous items of this theorem. □

Theorem 6. *The following hold in* $Ł_0^*$.

1. $x \to y \vdash_{Ł_0^*} x^+ \to y^+$.

2. $x \vdash_{Ł_0^*} x^-$ *and* $x \vdash_{Ł_0^*} x^+$.

3. $\vdash_{Ł_0^*} x \leftrightarrow (x^+ + x^-)$.

4. $\vdash_{Ł_0^*} (\neg x)^+ \leftrightarrow \neg x^-$ *and* $\vdash_{Ł_0^*} (\neg x)^- \leftrightarrow \neg x^+$.

5. $\vdash_{Ł_0^*} x^{-+}$ *and* $\vdash_{Ł_0^*} x^{+-}$.

6. $\vdash_{Ł_0^*} x^+ \leftrightarrow x^{++}$ *and* $\vdash_{Ł_0^*} x^- \leftrightarrow x^{--}$.

7. $\vdash_{Ł_0^*} x \vee \mathbf{0} \leftrightarrow x^+$.

Proof. Item *1.* is a consequence of Thm. 5 (2), while *2.*, which is Chang's rule R2, follows from K* . Item *3.* is a special case of Ax6. Item *4.* follows from several items in Thm. 5, while *5.* uses Ax4. Items *6.* and *7.* follow from all previous results. □

The next lemma contains some technical properties related to the order defined within the system. Item *4* is specially useful in the proof of the completeness theorem.

Lemma 7. *The following hold in* $Ł_0^*$.

1. $(x \to y)^- \vdash_{Ł_0^*} y \leftrightarrow (x \vee y)$.
2. $(x \to y)^-, (y \to z)^- \vdash_{Ł_0^*} (x \to z)^-$.
3. $(x \to y)^- \vdash_{Ł_0^*} ((z \to x) \to (z \to y))^-$.
4. $(x \to y)^- \, (u \to v)^- \vdash_{Ł_0^*} ((y \to u) \to (x \to v))^-$.

Proof. Using Ax8 and Thm. 6 (7), we prove that

$$\vdash_{Ł_0^*} (y \to (x \vee y)) \leftrightarrow (y \to x)^+ , \qquad (*)$$

and then by Thm. 6 (4), we obtain *1.*

Item *2* follows from *1*, Ax9 and (*). The proof of item *3* uses *1*, Ax8 and (*).

Next we observe that by Ax1 and Thm. 5 (13), $\vdash_{Ł_0^*} ((u \to x) \to (u \to y)) \leftrightarrow ((y \to u) \to (x \to u))$, which together with *2* and *3* yield *4*.

□

3.4 Soundness and Completeness

Consider the matrix $\mathcal{I}_0^* = (\langle [-1,1]; \to, \neg\, \mathbf{1} \rangle, \{0\})$. We first observe that the system is sound.

Theorem 8. *If* $\Gamma \vdash_{Ł_0^*} \varphi$, *then* $\Gamma \vDash_{\mathcal{I}_0^*} \varphi$.

Proof. Simply check that for any valuation all axioms take value 0 and the rules go from value 0 to value 0. □

We will now prove a weak completeness theorem.

Theorem 9. *If* $\vDash_{\mathcal{I}_0^*} \varphi$ *, then* $\vdash_{\text{Ł}_0^*} \varphi$*.*

Proof. Assume that φ takes value 0 for any valuation. Then $\neg\varphi$ also takes value 0 for any valuation.

By the Completeness Theorem 2, in Ł* there are proofs $\langle \sigma_1, \sigma_2, \ldots, \sigma_n \rangle$ of φ and $\langle \tau_1, \tau_2, \ldots, \tau_m \rangle$ of $\neg\varphi$.

We claim that for $1 \leq i \leq n$ and $1 \leq j \leq m$ $\vdash_{\text{Ł}_0^*} \sigma_i^-$ and $\vdash_{\text{Ł}_0^*} \tau_j^-$. This is a straightforward induction on the complexity of the proof.

<u>Case 1</u>. If σ_i is an axiom different from Axiom P4, then by Thm. 6 (2), $\vdash_{\text{Ł}_0^*} \sigma_i^-$.

If σ_i is Axiom P4, then $\sigma_i^- = (\tau \to \mathbf{1})^-$. But then

$\vdash_{\text{Ł}_0^*} \mathbf{1} \to ((\mathbf{1} \to \tau) \to \mathbf{1})$,	Ax4,
$\vdash_{\text{Ł}_0^*} \mathbf{1} \to (\neg(\tau \to \mathbf{1}) \to \neg\neg\mathbf{1})$,	Ax3, Thm. 5 (8),
$\vdash_{\text{Ł}_0^*} \mathbf{1} \to \neg((\tau \to \mathbf{1}) \to \neg\mathbf{1})$,	Thm. 5 (13),
$\vdash_{\text{Ł}_0^*} ((\tau \to \mathbf{1}) \to \neg\mathbf{1}) \to \neg\mathbf{1}$,	Ax1,
$\vdash_{\text{Ł}_0^*} (\tau \to \mathbf{1})^-$,	Def.

So for any axiom σ_i, $\Gamma \vdash_{\text{Ł}_0^*} \sigma_i{}^-$

<u>Case 2</u>. If σ_i is obtained in Ł* by Rule R1, that is, for some $j,\ k < i$, $\sigma_k = \sigma_j \to \sigma_i$, so

$\vdash_{\text{Ł}_0^*} (\sigma_j \to \sigma_i)^-$,	Induction Hyp.,
$\vdash_{\text{Ł}_0^*} \sigma_j^-$,	Induction Hyp.,
$\vdash_{\text{Ł}_0^*} (\mathbf{0} \to \sigma_j)^-$,	Ax2,
$\vdash_{\text{Ł}_0^*} ((\sigma_j \to \sigma_j) \to (\mathbf{0} \to \sigma_i))^-$,	Lem. 7 (4),
$\vdash_{\text{Ł}_0^*} (\mathbf{0} \to \sigma_i)^-$,	Ax2,
$\vdash_{\text{Ł}_0^*} \sigma_i^-$,	Ax2.

<u>Case 3</u>. If σ_i is obtained by Rule R2, $\vdash_{\text{Ł}_0^*} \sigma_i^-$ is obtained by Lem. 7 (4).

<u>Case 4</u>. If σ_i is obtained by Rule R3, then $\sigma_i = \sigma_k^-$, for some $k < i$, and thus, $\sigma_i^- = \sigma_k{}^{--}$. Since by inductive hypothesis, $\vdash_{\text{Ł}_0^*} \sigma_k^-$, by Theorem 6 (6), $\vdash_{\text{Ł}_0^*} \sigma_i^-$.

The proof for the τ_j's is similar. So we have proven that both $\vdash_{\text{Ł}_0^*} \varphi^-$ and $\vdash_{\text{Ł}_0^*} (\neg\varphi)^-$. We know that the latter is equivalent to, $\vdash_{\text{Ł}_0^*} \neg(\varphi^+)$, so

$\neg(\varphi^+), \varphi^- \vdash_{\text{Ł}_0^*} \neg(\varphi^+) \to \varphi^-$,	Thm. 5 (15),
$\neg(\varphi^+), \varphi^- \vdash_{\text{Ł}_0^*} \varphi$,	Thm. 6 (3), Def.

This completes the proof of $\vdash_{\text{Ł}_0^*} \varphi$ and thus, of our theorem. □

4 Algebraizability

Theorem 10. *The deductive system* $Ł_0^*$ *is algebraizable with one defining equation*

$$x \approx \neg x$$

and one equivalence formula

$$x\Delta y := x \leftrightarrow y\,.$$

Proof. The proof that $x\Delta y$ defines a congruence on the algebra of terms follows immediately from the theorems in Section 3.2.

$x \vdash_{Ł_0^*} x \leftrightarrow \neg x$ follows from Thm. 5 (15) and (16).

$x \leftrightarrow \neg x \vdash_{Ł_0^*} x^+ \leftrightarrow (\neg x)^+\,,$	Thm. 6 (1),
$x \leftrightarrow \neg x \vdash_{Ł_0^*} x^+ \leftrightarrow \neg x^-\,,$	Thm. 6 (4),
$x \leftrightarrow \neg x \vdash_{Ł_0^*} (\neg x^- \to x^+) \leftrightarrow (\neg x^- \to \neg x^-)\,,$	Thm. 5 (2),
$x \leftrightarrow \neg x \vdash_{Ł_0^*} (x^- + x^+) \leftrightarrow \mathbf{0}\,,$	Thm. 5 (8), Def.
$x \leftrightarrow \neg x \vdash_{Ł_0^*} x\,,$	Thm. 6 (3).

This proves that $x \dashv\vdash_{Ł_0^*} \delta(x)\Delta\varepsilon(x)$, so by [1], Thm. 4.7, $Ł_0^*$ is algebraizable. □

5 Equivalence of the categories $\mathcal{MV}$ and $\mathcal{MV}^*$

In this section we give a precise relation between the categories $\mathcal{MV}$ and $\mathcal{MV}^*$ of MV–algebras and MV^*–algebras, namely they are categorically equivalent.

An MV–algebra (Chang, [4] Mangani, [18]) is a system $\boldsymbol{A} = \langle A; \oplus, \neg, \mathbf{0}\rangle$ satisfying the following equations.

MV1 $\quad x \oplus (y \oplus z) = (x \oplus y) \oplus z$

MV2 $\quad x \oplus y = y \oplus x$

MV3 $\quad x \oplus \mathbf{0} = x$

MV4 $\quad \neg\neg x = x$

MV5 $x \oplus \neg\mathbf{0} = \neg\mathbf{0}$

MV6 $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

In every MV–algebra we can define the constant $\mathbf{1}$ and the binary operator $\rightarrow$ by the formulas:

$$\begin{aligned} \mathbf{1} &:= \neg\mathbf{0} \\ x \rightarrow y &:= \neg x \oplus y \end{aligned}$$

The follo wing operations define a structure of MV–algebra $\langle [0,1]; \oplus, \neg, 0\rangle$ over the unit interval.

$$\begin{aligned} x \oplus y &= \min(1, x+y), \\ \neg x &= 1 - x. \end{aligned}$$

Theorem 11. *(Algebraic completeness.) An equation is satisfied in every MV–algebra if and only if it holds in the MV–algebra* $[0,1]$.

This theorem was first proved by C. C. Chang in [4] and [5]. Other proofs have been given by R. Cignoli in [7], Panti in [19]. A very elementary proof appears in R. Cignoli and D. Mundici [10].

Lemma 12. *Let $\boldsymbol{A} = \langle A; \oplus, \neg, 0\rangle$ be an MV–algebra. Then, the following statements hold.*

a) If $x, y \in A$, $x \wedge y = 0$, then $x = x \odot \neg y$.

b) If $x, y \in A$, then $x \vee y = (x \odot \neg y) \oplus y$, $\quad x \wedge y = (x \oplus \neg y) \odot y$.

An MV^*–algebra (Chang, [6]) is a system $\boldsymbol{B} = \langle B; +, C, \mathbf{0}, \mathbf{1}\rangle$ satisfying the following equations. We define

$$\begin{aligned} -\mathbf{1} &:= C\mathbf{1}, \\ x^+ &:= \mathbf{1} + (-\mathbf{1} + x), \\ x^- &:= -\mathbf{1} + (\mathbf{1} + x), \\ x \vee y &:= [x^+ + (C(x^+) + y^+)^+] + [x^- + (C(x^-) + y^-)^+]. \end{aligned}$$

Bx1 $x + y = y + x$

Bx2 $(\mathbf{1} + x) + (y + (\mathbf{1} + z)) = ((\mathbf{1} + x) + y) + (\mathbf{1} + z)$

Bx3 $x + Cx = \mathbf{0}$

Bx4 $(x + \mathbf{1}) + \mathbf{1} = \mathbf{1}$

Bx5 $x + \mathbf{0} = x$

Bx6 $C(x + y) = Cx + Cy$

Bx7 $CCx = x$

Bx8 $x + y = (x^+ + y^+) + (x^- + y^-)$

Bx9 $(Cx + (x + y))^+ = C(x^+) + (x^+ + y^+)$

Bx10 $x \vee y = y \vee x$

Bx11 $x \vee (y \vee z) = (x \vee y) \vee z$

Bx12 $x + (y \vee z) = (x + y) \vee (x + z)$

Theorem 13. ([6], Thms. (2.15) and (2.16).) *Let $\boldsymbol{B} = \langle B; +, C, \mathbf{0}, \mathbf{1}\rangle$ be an MV^*–algebra. Define $B^+ = \{x^+ : x \in B\}$ and $B^- = \{x^- : x \in B\}$.*

(a) $\boldsymbol{B}^+ = \langle B^+; +, \widehat{\ }, \mathbf{0}\rangle$ is an MV–algebra, where $\widehat{\ }$ is defined by $\widehat{x} = 1 + Cx$.

(b) $\boldsymbol{B}^- = \langle B^-; +, \widetilde{\ }, \mathbf{0}\rangle$ is an MV–algebra, where $\widetilde{\ }$ is defined by $\widetilde{x} = C(1 + x)$.

We note that $(\widetilde{\widetilde{x} + y}) + y = x \vee y$ and that $(\widehat{\widehat{x} + y}) + y = x \wedge y$.

Theorem 14. *Let $\boldsymbol{A} = \langle A; \oplus, \neg, \mathbf{0}\rangle$ be an MV–algebra and let*

$$A^* = \{(a, b) \in A \times A : a \wedge b = \mathbf{0}\}.$$

Define the following operations over A^*.

$$\begin{aligned} (a, b) + (c, d) &:= (((a \oplus c) \odot \neg(b \oplus d)), ((b \oplus d) \odot \neg(a \oplus c)), \\ C(a, b) &:= (b, a), \\ \mathbf{0} &:= (\mathbf{0}, \mathbf{0}), \\ \mathbf{1} &:= (\mathbf{1}, \mathbf{0}). \end{aligned}$$

Then $\boldsymbol{A}^* = \langle A^*; +, C, \mathbf{0}, \mathbf{1}\rangle$ is an MV^*–algebra.

Proof. Axiom Bx1 holds by commutativity of $\oplus$. Axiom Bx3 states that $(a,b)+(b,a) = (\mathbf{0},\mathbf{0})$ which is true because in every MV–algebra $x \odot \neg x = \mathbf{0}$. Next we observe that $(a,b) + (\mathbf{1},\mathbf{0}) = (\neg b,\mathbf{0})$ and that $(\neg b,\mathbf{0}) + (\mathbf{1},\mathbf{0}) = (\mathbf{1},\mathbf{0})$, and thus, axiom Bx4 holds. From the identity $x \odot \neg x = \mathbf{0}$ we deduce $(a,b) + (b,a) = (\mathbf{0},\mathbf{0})$, so Bx5 holds. It is immediate that Bx6 and Bx7 also hold.

We now prove the axiom of restricted associativity Bx2. Let $x = (a,b)$, $y = (c,d)$ and $z = (e,f)$. Then, by direct calculations using Lem. 12, $\mathbf{1} + x = (\neg b,\mathbf{0})$ and $(\mathbf{1} + x) + y = ((\neg b \oplus c) \odot \neg d, b \odot d)$, so

$$((\mathbf{1} + x) + y) + (\mathbf{1} + z) = (A,B)$$

where

$$A = (((\neg b \oplus c) \odot \neg d) \oplus \neg f) \odot (\neg b \oplus \neg d)$$
$$B = (b \odot d) \odot \neg(((\neg b \oplus c) \odot \neg d) \oplus \neg f)$$

Similarly $(\mathbf{1} + x) + (y + (\mathbf{1} + z)) = (C,D)$, where

$$C = (((\neg f \oplus c) \odot \neg d) \oplus \neg b) \odot (\neg f \oplus \neg d)$$
$$D = (f \odot d) \odot \neg(((\neg f \oplus c) \odot \neg d) \oplus \neg b)$$

By the completeness theorem (Thm. 11,) it suffices to prove $A = C$, $B = D$ in $[0,1]$. In this case, $c \wedge d = 0$ implies $c = 0$ or $d = 0$. If $d = 0$, then $A = C = (\neg b \oplus c) \oplus \neg f$ and $B = D = 0$. If $c = 0$, we have $\neg A = ((b \oplus d) \odot f) \oplus (b \odot d)$ and $\neg C = ((f \oplus d) \odot b) \oplus (f \odot d)$. The equality follows from the analysis of the six cases that arise in the comparison of b, f and $\neg d$.

Following Chang we define

$$\begin{aligned} -\mathbf{1} &:= C\mathbf{1} \\ x^+ &:= \mathbf{1} + (-\mathbf{1} + x) \\ x^- &:= -\mathbf{1} + (\mathbf{1} + x) \\ x \vee y &:= (x^+ + (C(x^+) + y^+)^+) \; + \; (x^- + (C(x^-) + y^-)^+) \end{aligned}$$

Evaluating these terms in our case we get.

$$\begin{aligned} -\mathbf{1} &= (\mathbf{0},\mathbf{1}) \\ x^+ &= (a,\mathbf{0}) \\ x^- &= (\mathbf{0},b) \\ x \vee y &= (a \vee c, b \wedge d) \end{aligned}$$

We prove the last equality for $x = (a, b)$, $y = (c, d)$.

$$\begin{aligned} x^+ + (C(x^+) + y^+)^+ &= (a, \mathbf{0}) + ((\mathbf{0}, a) + (c, \mathbf{0}))^+ \\ &= (a, \mathbf{0}) + (c \odot \neg a, \mathbf{0}) \\ &= (a \oplus (c \odot \neg a), \mathbf{0}) \\ &= (a \vee c, \mathbf{0}) \end{aligned}$$

Similarly $x^- + (C(x^-) + y^-)^+ = (\mathbf{0}, b \wedge d)$.

To conclude this proof we observe that if $t \wedge s = \mathbf{0}$, then $(t, \mathbf{0}) + (\mathbf{0}, s) = (t, s)$, so $(a \vee c, \mathbf{0}) + (\mathbf{0}, b \wedge d) = (a \vee c, b \wedge d)$. So Bx10 and Bx11, commutativity and associativity respectively, hold.

To prove Bx8, we observe that $x^+ + y^+ = (a \oplus c, \mathbf{0})$ and $x^- + y^- = (\mathbf{0}, b \oplus d)$.

The computation of the first member of Bx9 gives

$$(Cx + (x + y))^+ = ((b \oplus [(a \oplus c) \odot \neg(b \oplus d)]) \odot \neg(a \oplus [(b \oplus d) \odot \neg(a \oplus c)]), \mathbf{0})$$

The second member is $Cx^+ + (x^+ + y^+) = ((a \oplus c) \odot \neg a, \mathbf{0})$, that is $Cx^+ + (x^+ + y^+) = (\neg a \wedge c, \mathbf{0})$.

To prove the identity $((b \oplus [(a \oplus c) \odot \neg(b \oplus d)]) \odot \neg(a \oplus [(b \oplus d) \odot \neg(a \oplus c)]) = \neg a \wedge c$ in [0,1] we can proceed by cases. Recall that $a \wedge b = 0$ and $c \wedge d = 0$ imply that $a = 0$ or $b = 0$, and $c = 0$ or $d = 0$.

The computation of the l.h.s member of Bx12 gives:

$$(a, b) + (c \vee e, d \wedge f) = ((a \oplus (c \vee e)) \odot \neg(b \oplus (d \wedge f)), (b \oplus (d \wedge f)) \odot \neg(a \oplus (c \vee e))),$$

and that of the r.h.s. member is

$$((a \oplus c) \odot \neg(b \oplus d) \vee (a \oplus e) \odot \neg(b \oplus f), (b \oplus d) \odot \neg(a \oplus c) \vee (b \oplus f) \odot \neg(a \oplus e))$$

We can prove the equalities

$$(a \oplus (c \vee e)) \odot \neg(b \oplus (d \wedge f)) = (a \oplus c) \odot \neg(b \oplus d) \vee (a \oplus e) \odot \neg(b \oplus f)$$

and

$$(b \oplus (d \wedge f)) \odot \neg(a \oplus (c \vee e)) = (b \oplus d) \odot \neg(a \oplus c) \vee (b \oplus f) \odot \neg(a \oplus e)$$

in [0,1] by the analysis of the eight possible cases. □

Theorem 15. *Let $\mathcal{MV}$ and $\mathcal{MV}^*$ be the categories whose objects are MV–algebras and MV*–algebras respectively, with their corresponding morphisms. Let $\mathcal{F}$ be the map defined for an MV–algebra $\boldsymbol{A}$ by $\mathcal{F}(\boldsymbol{A}) = \boldsymbol{A}^*$, where $\boldsymbol{A}^*$ is the algebra defined in Theorem 14 and for a MV–morphism $f : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ by $\mathcal{F}(f) = (f \times f) \restriction_{\boldsymbol{A}^*}$. Then $\mathcal{F}$ is a functor that defines a categorical equivalence.*

Proof. It is easy to see that $\mathcal{F}$ is well defined in the sense that $\mathcal{F}(f)$ is an MV^*–morphism that takes A^* into B^*. Functoriality is also straightforward.

Let $\mathcal{G}$ be the map defined for an MV^*–algebra B by $\mathcal{G}(B) = B^+$ and for a MV^*–morphism $h : B \longrightarrow C$ by $\mathcal{G}(h) = h \mid_{B^+}$. Then, $\mathcal{G}$ is well defined and is a functor. We now define the natural equivalences s and t as follows, $s : \mathcal{G} \circ \mathcal{F} \approx \mathbf{1}_{\mathcal{MV}}$, $t : \mathcal{F} \circ \mathcal{G} \approx \mathbf{1}_{\mathcal{MV}^*}$. In what follows we denote * and $^+$ the functors $\mathcal{F}$ and $\mathcal{G}$ respectively.

For an MV–algebra A, let $s_A : (A^*)^+ \longrightarrow A$ be defined by $s_A = pr_1$.

For an MV^*–algebra B, let $t_B : (B^+)^* \longrightarrow B$ be defined by $t_B((a, b)) = a + Cb$.

It is easy to see that for each A, s_A is bijective (note that $(a, b) \in (A^*)^+$ if and only if $b = \mathbf{0}$) and that it preserves addition and $\mathbf{0}$. Also $s_A(\widehat{(a, \mathbf{0})}) = s_A((\mathbf{1}, \mathbf{0}) + C(a, \mathbf{0})) = s_A((\neg a, \mathbf{0})) = \neg a = \neg s_A((a, \mathbf{0}))$. So, s_A is an MV–isomorphism.

To prove that for each B, t_B is injective, we first note that $a \wedge b = \mathbf{0}$ implies that $(a + Cb) \vee \mathbf{0} = a$, $(a + Cb) \wedge \mathbf{0} = Cb$. Surjectivity follows by Bx8, since $x = x^* + x^- = t_B((x^+, Cx^-))$. Also it is trivial to prove that t_B preserves $\mathbf{0}$, $\mathbf{1}$ and C. We now prove that addition is also preserved.

Indeed, $t_B((a, b) + (c, d)) = t_B(X, Y)$, where $X = \mathbf{1} + [(-\mathbf{1} + (a + c)) + C(b+d)]$, $CY = -\mathbf{1} + [(\mathbf{1} + C(b+d)) + (a+c)]$. By associativity (see [6], Thm. (2.5),) we have that $X = U^+$ and $CY = U^-$, were $U = (a + c) + C(b + d)$, so $t_B((a, b) + (c, d)) = (a + c) + C(b + d) = (a + Cb) + (c + Cd) = t_B((a, b)) + t_B((c, d))$, by Bx 8.

It is straightforward to prove that s and t are natural. □

We describe briefly some relationships between the categorical equivalence $\mathcal{F}$ defined by theorem 15 and known categorical equivalences Γ of Chang–Mundici and Σ of Di Nola–Lettieri (originally called $\mathcal{G}$ in [11]). The functors Γ and Σ are treated in Ch. 2 and 7 of the book [9].

Let $\mathcal{LGU}$ be the category of l–groups with strong order unit and l–group unital homomorphisms (see [9].) The functor Γ assigns to every lattice–ordered group $\boldsymbol{G}$ with unit u the MV–algebra given by the interval $[0,u]$ of $\boldsymbol{G}$ with truncated sum and negation $\neg x = u - x$ and to every unital l–group homomorphism its restriction to $[0,u]$. We can obtain a categorical equivalence between $\mathcal{LGU}$ and $\mathcal{MV}^*$ by composition of Γ and the functor $\mathcal{F}$.

An MV–algebra $\boldsymbol{A}$ is called *perfect* if for every element x, either x or $\neg x$ is of finite order. Let $\mathcal{P}$ be the full subcategory of $\mathcal{MV}$ whose objects are the perfect MV–algebras. The functor Σ establishes a categorical equivalence between the category $\mathcal{LG}$ of l–groups and $\mathcal{P}$. For any l–group $\boldsymbol{G}$, $\Sigma(\boldsymbol{G}) = \Gamma(\mathbb{Z} \otimes \boldsymbol{G}, (1,0))$, where $\mathbb{Z} \otimes \boldsymbol{G}$ is the *lexicographic product* of $\mathbb{Z}$ and $\boldsymbol{G}$ (that is the product group with the lexicographic order) and $(1,0)$ is the unit.

Let $\mathcal{P}^*$ be the full subcategory of $\mathcal{MV}^*$ whose objects are images of the objects of $\mathcal{P}$ by the functor $\mathcal{F}$. Otherwise, an algebra $\boldsymbol{B}$ is an object of $\mathcal{P}^*$ if and only if $\boldsymbol{B}^+$ is perfect. Therefore, the composition of Σ and $\mathcal{F}$ gives a categorical equivalence between $\mathcal{LG}$ and $\mathcal{P}^*$.

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