

ON THE ALGEBRAIZABILITY OF ANNOTATED LOGICS

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ABSTRACT. Annotated logics were introduced by V. S. Subrahmanian as logical foundations for computer programming. One of the difficulties of these systems from the logical point of view is that they are not structural, i.e., their consequence relations are not closed under substitutions. In this paper we give systems of annotated logics that are equivalent to those of Subrahmanian in the sense that everything provable in one type of system has a translation that is provable in the other. Moreover these new systems are structural. We prove that these systems are *weakly congruential*, namely, they have an infinite system of congruence 1-formulas. Moreover, we prove that an annotated logic is algebraizable (i.e., it has a finite system of congruence formulas,) if and only if the lattice of annotation constants is finite.

1. INTRODUCTION

Annotated logics were introduced in [9] by V. S. Subrahmanian as logical foundations for computer programming. Blair and Subrahmanian [2] proved that these systems were paraconsistent and showed that they could form the basis of a programming language for reasoning about data bases that contain inconsistent information. Subsequently, numerous applications in Artificial Intelligence, like inheritance networks, object oriented data bases, etc. have been developed.

A complete study of these systems, from the model theoretical and proof theoretical points of view has been done in [1] and [5]. They show that almost all classical basic results in model theory can be adapted to these systems. However, since there are several kinds of axioms (some for complex formulas, others for atomic formulas and still others for arbitrary formulas), these systems are not structural in the sense that their consequence relation is not closed under substitutions. One of

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the difficulties with non-structural systems is that one cannot find its algebraic counterpart.

In Section 2 of this paper we give a summary of the main results about annotated systems $P\tau$. In Section 3, we introduce a structural version $SP\tau$ of annotated logics and we prove that they are equivalent to the original $P\tau$ systems, in the sense that everything provable in a system of one type has a translation that is provable (using some additional premises) in the corresponding system of the other type.

Even though a semantical approach arguably would be the most natural, we have chosen a purely syntactical one. The reason for this is fairly natural, the methods of algebraization used, an application of which was the main motivation for this paper, are best suited to work syntactically.

Finally, in Section 4, we study the algebraizability of these systems using as main framework the theory of algebraization of deductive systems developed by Blok and Pigozzi in [3]. The main result proven here, is that these systems are, using the terminology of [4], *weakly congruential*, namely, they have an infinite system of congruence 1-formulas. Moreover, we characterize the class of all annotated logics that are algebraizable, i.e. that have a finite system of congruence formulas.

One should note that strictly speaking, Blok–Pigozzi’s theory does not apply to these systems since they contain an infinitary rule of inference and thus, there are infinite proofs, in the terminology of [3], they are not standard. Nevertheless, the methods developed there can be extended to arbitrary systems through the intrinsic characterization of algebraizable deductive systems via the Leibniz equality function. See Theorem 4.2 of [3]. It should be noted that Herrman in [8] does this generalization and obtains the corresponding theorems that relate the Leibniz operator with protoalgebraic logics, algebraizable logics and finitely algebraizable logics. The latter is what Blok and Pigozzi call algebraizable logic.

The results in this paper are mainly a contribution to algebraic logic, presenting a natural example of a system that is weakly congruential but not algebraizable. Nevertheless some of its content is relevant to the theory of annotated logics, namely, it provides a straightforward extension of the systems $P\tau$ that is structural and thus more standard. Moreover, in the finitary case, it provides, via Blok-Pigozzi’s or other authors related theories, an algebraic counterpart for these calculi. This algebraic semantics has yet to be developed and studied.

No attempt has been made to explain how these results might be useful for logic programming since this falls well beyond the scope of the paper and the authors' interests.

Several authors have studied this problem in recent years, among them we mention J. Czelakowski,[6], in this context, annotated logics would be equivalential logics. Also, as noted above, Herrmann in [8] and Font–Jansana in [7].

2. THE SYSTEMS $P\tau$

In this section we will present the main features of the systems of annotated logic. The reader is referred to [5] and [1] for proofs and more details.

2.1. Systems $P\tau$. Let τ be a (fixed) complete lattice and $\sim: \tau \mapsto \tau$ a function. Let \perp, \top denote the least and the greatest elements of the lattice.

The function \sim , intended to act as a negation for annotated formulas, is absolutely arbitrary.

The language of $P\tau$ will consist of the logical symbols $\wedge, \vee, \rightarrow, \neg$, a set \mathcal{P} of propositional letters p, q, r, \dots , annotation constants λ, μ, \dots , for each element of the lattice τ and parentheses. The set \mathcal{F} of formulas of $P\tau$ is defined recursively as follows.

- (1) If p is a propositional letter and λ is an annotation constant, p_λ is a formula.
- (2) If A is a formula, then $\neg A$ is a formula.
- (3) If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulas.
- (4) An expression is a formula if and only if it is obtained by the above rules.

The formulas $\neg\neg\dots\neg p_\mu$, where there are k negation symbols, will be denoted by $\neg^k p_\mu$, $k \geq 0$ and will be called *hyperliterals*. A formula that is not a hyperliteral will be called a *complex formula*.

The axioms and inference rules of $P\tau$ are as follows.

- (\rightarrow_1) $A \rightarrow (B \rightarrow A)$,
- (\rightarrow_2) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,
- (\rightarrow_3) $((A \rightarrow B) \rightarrow A) \rightarrow A$,
- (\rightarrow_4) $\frac{A, A \rightarrow B}{B}$,
- (\wedge_1) $(A \wedge B) \rightarrow A$,
- (\wedge_2) $(A \wedge B) \rightarrow B$,
- (\wedge_3) $A \rightarrow (B \rightarrow (A \wedge B))$,

- (\vee_1) $A \rightarrow (A \vee B)$,
- (\vee_2) $B \rightarrow (A \vee B)$,
- (\vee_3) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$.

If F and G are complex formulas,

- (\neg_1) $(F \rightarrow G) \rightarrow ((F \rightarrow \neg G) \rightarrow \neg F)$,
- (\neg_2) $F \rightarrow (\neg F \rightarrow A)$,
- (\neg_3) $F \vee \neg F$,
- (τ_1) p_\perp ,
- (τ_2) $\neg^k p_\mu \leftrightarrow \neg^{k-1} p_{\sim\mu}$, for $k \geq 1$,
- (τ_3) $p_\mu \rightarrow p_\lambda$, for $\mu \geq \lambda$,
- (τ_4) if $A \rightarrow p_{\mu_j}$ for each $j \in J$, then $A \rightarrow p_\mu$, where $\mu = \bigvee_{j \in J} \mu_j$.

Remark 2.1. (1) The systems $P\tau$ are not structural. If σ is a substitution such that $\sigma(p_\perp) = p_\lambda$, for $\lambda \neq \perp$, we have

$$\vdash_{P\tau} p_\perp \quad \text{but} \quad \not\vdash_{P\tau} \sigma(p_\perp).$$

This can be checked using theorem 2.4 below.

- (2) Observe that complex formulas behave classically in the following sense. If $\varphi(x_1, \dots, x_n)$ is a classical tautology and A_1, \dots, A_n are complex formulas, then $\vdash_{P\tau} \varphi(A_1, \dots, A_n)$. The proof of this is straightforward since the axioms for binary connectives are classical and those for negation, when restricted to complex formulas, are also classical.
- (3) In rule τ_4 , if $J = \emptyset$, then $\bigvee_{j \in J} \mu_j = \perp$ and by axioms τ_1 , \rightarrow_1 and modus ponens, the consequence of the rule is a theorem of our system. So we may assume that $J \neq \emptyset$.

Theorem 2.1. $\Sigma, A \vdash_{P\tau} B$ if and only if $\Sigma \vdash_{P\tau} A \rightarrow B$.

Proof. This is a consequence of our axioms and rule for implication. \square

The notions of proof, consequence, $\vdash_{P\tau}$ and theorem are defined as usual. A set Δ of formulas is said *trivial* if for every formula $A \in \mathcal{F}$, $\Delta \vdash A$. Δ is said *inconsistent* if there is a formula A such that $\Delta \vdash A$ and $\Delta \vdash \neg A$. Notice that there exist sets of formulas that are both consistent but not trivial.

2.2. Semantics for $P\tau$. An *interpretation* I for the system $P\tau$ is a function

$I : \mathcal{P} \rightarrow \tau$. For each interpretation we can define a *valuation* $v_I : \mathcal{F} \rightarrow \{0, 1\}$ by:

- (1) If $p \in \mathcal{P}$, then $v_I(p_\mu) = 1$ iff $I(p) \geq \mu$.
- (2) (a) $v_I(\neg^k p_\mu) = v_I(\neg^{k-1} p_{\sim\mu})$, for $k \geq 1$.
- (b) If A is complex, then $v_I(\neg A) = 1 - v_I(A)$.

- (3) If $A, B \in \mathcal{F}$, then
- (a) $v_I(A \rightarrow B) = 1$ iff $v_I(A) = 0$ or $v_I(B) = 1$,
 - (b) $v_I(A \vee B) = 1$ iff $v_I(A) = 1$ or $v_I(B) = 1$,
 - (c) $v_I(A \wedge B) = 1$ iff $v_I(A) = 1$ and $v_I(B) = 1$.

Remark 2.2. If we let $I(p) = \top$, then $v_I(p_\lambda \wedge \neg p_\lambda) = 1$, that is, the system is paraconsistent in the sense that there are contradictions that are “true” under certain interpretations and thus $(A \wedge \neg A) \rightarrow B$ does not hold in $P\tau$.

Nevertheless, if F is complex, then under any interpretation I , $v_I(F \wedge \neg F) = 0$. In fact, if $\varphi(x_1, \dots, x_n)$ is a classical contradiction and A_1, \dots, A_n are complex formulas, then $\varphi(A_1, \dots, A_n)$ is false under any interpretation.

The notions of semantical consequence, \models , valid formula, etc., are defined as usual.

Theorem 2.2. *If $\Sigma \vdash_{P\tau} A$, then $\Sigma \models A$.*

Lemma 2.3. *If A is not trivial, then it has a model.*

Even though the systems are not complete, there is a finitary completeness theorem. We will state two forms of this theorem, for a more general result see [5], theorem 16.

Theorem 2.4. (1) *If Σ is finite, then $\Sigma \vdash_{P\tau} A$ iff $\Sigma \models A$.*

(2) *If τ is finite, then $\Sigma \vdash_{P\tau} A$ iff $\Sigma \models A$.*

This latter will be used in the proof of the equivalence between systems $P\tau$ and $SP\tau$.

3. THE SYSTEMS $SP\tau$

In this section we will define new annotated systems $SP\tau$ and we will prove that, for a given lattice τ , the systems $P\tau$ and $SP\tau$ are equivalent in the sense that both systems prove the “same” formulas in the corresponding languages. The advantage of systems $SP\tau$ is that they are structural. We will also give a semantics for $SP\tau$. In contrast with what happens with $P\tau$, $SP\tau$ is not complete with respect to this semantics.

3.1. Systems $SP\tau$.

The language of $SP\tau$

The language will consist of the logical symbols $\wedge, \vee, \rightarrow, \neg, \circ$ and f_λ , for each $\lambda \in \tau$, a set \mathcal{P} of propositional letters p, q, r, \dots , and parentheses. The set \mathcal{F}_S of formulas is defined recursively as follows:

- 1S) Propositional letters are formulas.

- 2S) If A is a formula, then A° is a formula.
- 3S) If A is a formula and λ is an annotation constant, then $f_\lambda A$ is a formula.
- 4S) If A is a formula, then $\neg A$ is a formula.
- 5S) If A and B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are formulas.
- 6S) An expression is a formula if and only if it is obtained by finitely many applications of the above rules.

The formulas of the form $\neg^{k_n} f_{\lambda_n} \neg^{k_{n-1}} f_{\lambda_{n-1}} \cdots \neg^{k_1} f_{\lambda_1} \neg^{k_0} p$, where for $0 \leq i \leq n$, $k_i \in \mathbb{N}$ and $\lambda_i \in \tau$, are called *hyperliterals*. Observe that for $n = 0$, p and $\neg^k p$ are hyperliteral. A formula that is not a hyperliteral will be called a *complex formula*.

The intended meaning for the formula A° is “ A is well behaved”, in particular, with negations.

Remark 3.1. The function \sim is intended to code in the annotation constants the negation for hyperliterals.

When the lattice τ has finite cardinality n , for any $\lambda \in \tau$,

$$\{\sim^m \lambda : m \in \mathbb{N}\}$$

is a finite set. So for any λ , there exists a least integer t_λ such that for some p

$$\sim^{t_\lambda} \lambda = \sim^{t_\lambda + p} \lambda.$$

Of course there will also exist the least such an integer p . Call it p_λ .

Observe that if $r \geq t_\lambda$ and s is a multiple of p_λ , then

$$\sim^r \lambda = \sim^{r+s} \lambda,$$

so defining

$$\begin{aligned} T &= \max\{t_\lambda : \lambda \in \tau\}, \\ P &= \text{m.c.m.}(\{2\} \cup \{p_\lambda : \lambda \in \tau\}). \end{aligned}$$

we have, for any $\lambda \in \tau$,

$$\sim^T \lambda = \sim^{T+P} \lambda.$$

In the definition of P above, we have included 2 in the minimum common multiple for technical reasons which will become clear when we state the fourth axiom of negation.

As we will see, the systems are very different if the lattice τ is infinite or finite. As a matter of fact, in order to obtain our results in the finite case, we will need an extra axiom for negations.

Axioms and Inference Rules of SP_τ

- (1) The axioms for \rightarrow , \wedge and \vee , are the same as those of $P\tau$.
(2) Axioms for negation:

$$\begin{aligned} \neg_{1S}) \quad & (A^\circ \wedge B^\circ) \rightarrow ((A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)), \\ \neg_{2S}) \quad & A^\circ \rightarrow (A \rightarrow (\neg A \rightarrow B)), \\ \neg_{3S}) \quad & A^\circ \rightarrow (A \vee \neg A). \\ \neg_{4S}) \quad & \neg^{T+P} A = \neg^T A, \end{aligned}$$

where T and P are the numbers defined in Remark 3.1. They depend on the lattice and the function \sim .

This axiom is to be included only if τ is finite.

- (3) Axioms for $^\circ$:

$$\begin{aligned} \circ_{1S}) \quad & A^\circ \leftrightarrow (\neg A)^\circ, \\ \circ_{2S}) \quad & A^\circ \leftrightarrow (f_\lambda A)^\circ, \\ \circ_{3S}) \quad & \begin{array}{ll} (A^\circ)^\circ & (A \vee B)^\circ \\ (A \wedge B)^\circ & (A \rightarrow B)^\circ. \end{array} \end{aligned}$$

- (4) The axioms for annotated variables will be replaced by the following axioms:

$$\begin{aligned} \tau_{1S}) \quad & \neg(A^\circ) \rightarrow f_\perp A, \\ \tau_{2S}) \quad & \neg(A^\circ) \rightarrow (\neg^k f_\lambda A \leftrightarrow \neg^{k-1} f_{\sim\lambda} A), \text{ for } k \geq 1, \\ \tau_{3S}) \quad & (f_\mu A \rightarrow f_\lambda A), \text{ for } \mu \geq \lambda, \\ \tau_{4S}) \quad & f_\mu f_\lambda A \leftrightarrow f_\mu A, \\ \tau_{5S}) \quad & \neg(A^\circ) \rightarrow (f_\lambda A \leftrightarrow f_\lambda \neg A), \\ \tau_{6S}) \quad & A^\circ \rightarrow (f_\lambda A \leftrightarrow A). \end{aligned}$$

- (5) Inference Rules

R1) Modus Ponens

$$\frac{A, A \rightarrow B}{B}$$

R2) If $J \neq \emptyset$ and for all $j \in J$, $\lambda_j \in \tau$ and $\lambda = \bigvee_{j \in J} \lambda_j$, then

$$\frac{A \rightarrow f_{\lambda_j} B, \text{ for } j \in J}{A \rightarrow f_\lambda B}.$$

Remark 3.2. (1) The systems $SP\tau$ are structural since all axioms and the two rules of inference are closed under substitutions.

- (2) The axiomatization of $P\tau$ does not include any axiom that take into account the specific function \sim that we are using. In the finitary case of our axiomatization this is taken care of by axiom τ_{4S} . Since \sim is arbitrary, this axiom takes a rather cumbersome form. Nevertheless it is not complicated in spirit, it codes the way the negation is defined for hyperliteral formulas.

As in the case of $P\tau$, we can prove the following.

Theorem 3.1. $\Sigma, A \vdash_{SP\tau} B$ if and only if $\Sigma \vdash_{SP\tau} A \rightarrow B$.

Theorem 3.2. *Let $\varphi(x_1, x_2, \dots, x_n)$ be a classical tautology. Then*

(1)

$$A_1^\circ, A_2^\circ, \dots, A_n^\circ \vdash_{SP\tau} \varphi(A_1, A_2, \dots, A_n).$$

(2) *If $\varphi(x_1, x_2, \dots, x_n)$ does not contain negations, then*

$$\vdash_{SP\tau} \varphi(A_1, A_2, \dots, A_n).$$

Proof. Assuming $A_1^\circ, A_2^\circ, \dots, A_n^\circ$, by modus ponens, we obtain all axioms for negation, which together with those for \rightarrow , \vee and \wedge , are all we need to prove any classical tautology. If negation is not involved, the axioms for binary connectives suffice. \square

3.2. Semantics for $SP\tau$. The semantics for $SP\tau$ is very similar to that for $P\tau$, for any interpretation $I : \mathcal{P} \rightarrow \tau$, we define a valuation $v_I : \mathcal{F}_S \rightarrow \{0, 1\}$ such that:

- (1) $v_I(p) = 1$ if and only if $I(p) \geq \top$.
- (2) $v_I(\neg^k p) = 1$ if and only if $I(p) \geq \sim^k \top$.
- (3) $v_I(A^\circ) = 1$ if and only if A is not hyperliteral.
- (4) If $A = \neg^{k_1} f_{\lambda_1} \neg^{k_2} f_{\lambda_2} \dots \neg^{k_n} f_{\lambda_n} \neg^{k_{n+1}} p$, then

$$v_I(f_\lambda A) = 1 \text{ iff } I(p) \geq \lambda.$$

(5) If A is complex, then

$$v_I(f_\lambda A) = v_I(A).$$

(6) If A is hyperliteral,

$$v_I(\neg^k f_\lambda A) = v_I(\neg^{k-1} f_{\sim\lambda} A).$$

(7) If A is complex,

$$v_I(\neg A) = 1 - v_I(A).$$

(8) For binary connectives v_I is defined as for $P\tau$.

We now give a semantical version of theorem 3.2.

Theorem 3.3. *Let $\varphi(x_1, x_2, \dots, x_n)$ be a classical tautology. Then*

- (1) *If A_1, A_2, \dots, A_n are complex formulas, then $\models \varphi(A_1, A_2, \dots, A_n)$.*
- (2) *If $\varphi(x_1, x_2, \dots, x_n)$ does not contain negations, then for formulas*

$$A_1, A_2, \dots, A_n, \models \varphi(A_1, A_2, \dots, A_n)$$

Proof. Observe that our definition of valuation for binary connectives is classical. Moreover, if A_1, A_2, \dots, A_n are complex formulas, then our definition of valuation for negation is also classical. \square

Theorem 3.4. *If $\Sigma \vdash_{SP\tau} A$, then $\Sigma \models A$.*

Proof. By theorem 3.3 2), we do not need to verify the axioms for \vee , \wedge and \rightarrow since they are classical tautologies not involving negations.

Moreover, since $v_I(A^\circ) = 1$ if and only if A is complex and the consquents of axioms \neg_{1S} and \neg_{3S} are also classical tautologies, these two axioms will also hold for any valuation.

\neg_{2S}) Let I be an interpretation. If A is hyperliteral, then $v_I(A^\circ) = 0$,
so

$$v_I(A^\circ \rightarrow (A \rightarrow (\neg A \rightarrow B))) = 1.$$

If A is complex, then $v_I(A^\circ) = 1$, so

$$\begin{aligned} v_I(A^\circ \rightarrow (A \rightarrow (\neg A \rightarrow B))) = 0 & \text{ iff } v_I(A \rightarrow (\neg A \rightarrow B)) = 0 \\ & \text{ iff } v_I(A) = 1 \text{ and } v_I(\neg A \rightarrow B) = 0 \\ & \text{ iff } v_I(A) = v_I(\neg A) = 1 \text{ and } v_I(B) = 0 \\ & \text{ iff } v_I(A) = 1, v_I(A) = 0 \text{ and } v_I(B) = 0, \end{aligned}$$

since A is complex, but this is clearly impossible.

So in any case, for any interpretation I ,

$$v_I(A^\circ \rightarrow (A \rightarrow (\neg A \rightarrow B))) = 1.$$

\neg_{4S}) First observe that by remark 3.1, $\sim^{T+P+k_1} \lambda = \sim^{T+k_1} \lambda$,
so if $A = \neg^{k_1} f_{\lambda_1} \neg^{k_2} f_{\lambda_2} \dots \neg^{k_n} f_{\lambda_n} \neg^{k_{n+1}} p$, and I is an interpretation,

$$\neg^{T+P} A = \neg^{T+P} (\neg^{k_1} f_{\lambda_1} \neg^{k_2} f_{\lambda_2} \dots \neg^{k_n} f_{\lambda_n} \neg^{k_{n+1}} p),$$

so

$$\begin{aligned} v_I(\neg^{T+P} A) = 1 & \text{ iff } I(p) \geq \sim^{T+P+k_1} \lambda_1 \\ & \text{ iff } I(p) \geq \sim^{T+k_1} \lambda_1 \\ & \text{ iff } v_I(\neg^T A) = 1. \end{aligned}$$

If A is complex, then since P is a multiple of 2, $v_I(\neg^{T+P} A) = v_I(\neg^T A)$.

So in any case, for any interpretation I ,

$$v_I(\neg^{T+P} A \leftrightarrow \neg^T A) = 1.$$

We now check the axioms for $^\circ$.

\circ_{1S}) $v_I(A^\circ) = 1$ iff A is complex iff $\neg A$ is complex iff $v_I((\neg A)^\circ) = 1$,
so

$$v_I(A^\circ \leftrightarrow (\neg A)^\circ) = 1$$

for any interpretation I .

\circ_{2S}) $v_I(A^\circ) = 1$ iff A is complex iff $f_\lambda A$ is complex iff $v_I((f_\lambda A)^\circ) = 1$,
so

$$v_I(A^\circ \leftrightarrow (f_\lambda A)^\circ) = 1$$

for any interpretation I .

\circ_{3S}) If C is A° , $A \vee B$, $A \wedge B$ or $A \rightarrow B$, since these are all complex,
for any interpretation I , $v_I(C^\circ) = 1$.

Finally, we check the τ_S axioms.

τ_{1S} If $v_I(\neg(A^\circ)) = 1$, then A is hyperliteral and its propositional
letter is, say, p . Then since $I(p) \geq \perp$, $v_I(f_\perp A) = 1$. So for any
valuation I ,

$$v_I(\neg(A^\circ) \rightarrow f_\perp A) = 1.$$

τ_{2S} If $v_I(\neg(A^\circ)) = 1$, then A is hyperliteral so $v_I(\neg^k f_\lambda A) = v_I(\neg^{k-1} f_{\sim\lambda} A)$,
thus for any interpretation I ,

$$v_I(\neg(A^\circ) \rightarrow (\neg^k f_\lambda A \leftrightarrow f_{\sim^k \lambda} A)) = 1.$$

τ_{3S} Let $\mu \geq \lambda$.

If A is hyperliteral, p is its propositional letter and $v_I(f_\mu A) = 1$,
then $I(p) \geq \mu \geq \lambda$ so $v_I(f_\lambda A) = 1$.

If A is complex, so are $f_\lambda A$ and $f_\mu A$ so $v_I(A) = v_I(f_\lambda A) = v_I(f_\mu A)$.

Thus, in both cases, for any interpretation I ,

$$v_I(f_\mu A \rightarrow f_\lambda A) = 1.$$

τ_{4S} If A is hyperliteral, and p is its propositional letter, then $v_I(f_\mu f_\lambda A) = 1$
iff $I(p) \geq \mu$ iff $v_I(f_\mu A) = 1$.

If A is complex, so is $f_\lambda A$, so

$$v_I(f_\mu f_\lambda A) = v_I(f_\lambda A) = v_I(A) = v_I(f_\mu A).$$

Thus, in any case, for any interpretation I ,

$$v_I(f_\mu f_\lambda A \leftrightarrow f_\mu A) = 1.$$

τ_{5S} If $v_I(\neg(A^\circ)) = 1$, then A is hyperliteral and so are $f_\lambda A$ and
 $f_\lambda \neg A$. They have the same propositional letter p and $v_I(f_\lambda A) = 1$
iff $I(p) \geq \lambda$ iff $v_I(f_\lambda \neg A) = 1$, so for any interpretation I ,

$$v_I(\neg(A^\circ) \rightarrow (f_\lambda A \leftrightarrow f_\lambda \neg A)) = 1.$$

τ_{6S} If $v_I(A^\circ) = 1$, then A is complex and so is $f_\lambda A$. So $v_I(A) = v_I(f_\lambda A)$,
and thus, for any interpretation I ,

$$v_I(A^\circ \rightarrow (f_\lambda A \leftrightarrow A)) = 1.$$

We will now check the soundness of our two rules.

R1 Modus Ponens is obviously sound.

R2 Let I be a valuation such that for each $j \in J$, $v_I(A \rightarrow f_{\lambda_j}B) = 1$ and
 $v_I(A \rightarrow f_{\lambda}B) = 0$. This last assumption implies that $v_I(A) = 1$ and
 $v_I(f_{\lambda}B) = 0$.

Suppose B is hyperliteral, let p be its propositional letter. Since $v_I(f_{\lambda}B) = 0$, $I(p) \not\geq \lambda$. But $v_I(f_{\lambda_j}B) = 1$, for all $j \in J$, so $I(p) \geq \lambda_j$, thus $I(p) \geq \bigvee_{j \in J} \lambda_j = \lambda$, which is a contradiction.

If B is complex, since $J \neq \emptyset$ and we are assuming that $v_I(A) = 1$, there is a $j \in J$ such that $v_I(f_{\lambda_j}B) = 1$, so $v_I(f_{\lambda}B) = 1$ and thus $v_I(A \rightarrow f_{\lambda}B) = 1$.

So in any case, if $v_I(A \rightarrow f_{\lambda_j}B) = 1$ for each $j \in J$, then $v_I(A \rightarrow f_{\lambda}B) = 1$, and the rule is sound.

□

Remark 3.3. The system is not complete since $\models_{SP\tau}$ is not structural and \vdash is. In fact, it is easy to check that $\models f_{\perp}p$, but if we let $\sigma(p) = A \wedge \neg A$, where A is complex, then $\not\models f_{\perp}\sigma(p)$. Nevertheless, as we just proved, the system is sound and this is all we will need for the main goal of the next section.

3.3. Interpretations Between $P\tau$ and $SP\tau$. In this section we will prove that there are interpretations from one system into the other so that both systems prove essentially the same formulas.

We will first define interpretations from $SP\tau$ into $P\tau$ by means of appropriate syntactical translations between the two sets of formulas.

Definition 3.1. Define $\theta : \mathcal{F}_S \longrightarrow \mathcal{F}$ by recursion as follows:

- (1) If A is hyperliteral, then $\theta(A) = A$.
- (2) If $A = B^\circ$, then $\theta(A) = \neg(f_{\top}\theta(B) \leftrightarrow f_{\top}\theta(\neg B))$.
- (3) If $A = f_{\lambda}B$, B complex, then $\theta(A) = f_{\lambda}\theta(B)$.
- (4) If $A = \neg B$, B complex, then $\theta(A) = \neg\theta(B)$.
- (5) If $A = B * C$, then $\theta(A) = \theta(B) * \theta(C)$, for any binary connective $*$.

Observe that $\theta(A)$ does not contain $^\circ$ and that if A does not contain $^\circ$, $\theta(A) = A$.

Next we define $\chi : \mathcal{F}_S \longrightarrow \mathcal{F}$ by recursion as follows.

- (1) If $A = p$, then $\chi(A) = p_{\top}$.
- (2) If $A = f_{\lambda} \neg^{k_n} f_{\lambda_n} \cdots \neg^{k_1} f_{\lambda_1} \neg^{k_0} p$, then $\chi(A) = p_{\lambda}$.
- (3) If $A = f_{\lambda}B$, for a complex formula B , then $\chi(A) = \chi(B)$.
- (4) If $A = \neg B$, then $\chi(A) = \neg\chi(B)$.
- (5) If $A = B * C$, then $\chi(A) = \chi(B) * \chi(C)$, for any binary connective $*$.

Finally, define

$$\begin{aligned}\psi(A) : \mathcal{F}_S &\longrightarrow \mathcal{F} \\ A &\longmapsto \chi(\theta(A))\end{aligned}$$

Similarly, define $\varphi : \mathcal{F} \longrightarrow \mathcal{F}_S$ by recursion as follows.

- (1) If $A = p_\lambda$, then $\varphi(A) = f_\lambda p$.
- (2) If $A = \neg B$, then $\varphi(A) = \neg\varphi(B)$.
- (3) If $A = B * C$, for a binary connective $*$, then $\varphi(A) = \varphi(B) * \varphi(C)$.

Theorem 3.5.

- (1) If $\{A_1, \dots, A_n\} \vdash_{P\tau} A$, then

$$\{\varphi(A_1), \dots, \varphi(A_n)\} \cup \{\neg(p^\circ) : p \in \mathcal{P}\} \vdash_{SP\tau} \varphi(A).$$
- (2) If $\{A_1, \dots, A_n\} \vdash_{SP\tau} A$, then $\{\psi(A_1), \dots, \psi(A_n)\} \vdash_{P\tau} \psi(A)$.

Proof.

1) Recall that for an infinite lattice τ , rule τ_4 is infinitary, so our proofs may be infinite. Nonetheless, limit stages of the proof can only come from an application of rule τ_4 .

Let $\langle \sigma_1, \dots, \sigma_\alpha \rangle$, where α is an ordinal, be a proof in $P\tau$ of A from A_1, \dots, A_n . We will turn it into a proof of $\varphi(A)$ in $SP\tau$ by replacing each σ_i by a finite set of formulas, that includes $\varphi(\sigma_i)$, all of which are probable in $SP\tau$ from $\{\varphi(A_1), \dots, \varphi(A_n)\} \cup \{\neg(p^\circ) : p \in \mathcal{P}\}$. This will be done by induction on the length of the proof.

We just have to check the axioms and rules of $P\tau$.

Axioms for $\rightarrow, \vee, \wedge$ are obvious since $\varphi(\sigma_i)$ is an instance of the same axioms in $SP\tau$. So we replace σ_i by $\varphi(\sigma_i)$.

If $\sigma_i = (F \rightarrow G) \rightarrow ((F \rightarrow \neg G) \rightarrow \neg F)$, where F and G are complex. Then

$$\varphi(\sigma_i) = (\varphi(F) \rightarrow \varphi(G)) \rightarrow ((\varphi(F) \rightarrow \neg\varphi(G)) \rightarrow \neg\varphi(F)).$$

But since $\varphi(F), \varphi(G)$ are complex,

$$\vdash_{SP\tau} \varphi(F)^\circ \text{ and } \vdash_{SP\tau} \varphi(G)^\circ,$$

and these together with axiom \neg_{1S} yield

$$\vdash_{SP\tau} \varphi(\sigma_i).$$

So we replace σ_i by the set of formulas

$$\varphi(F)^\circ, \varphi(G)^\circ, (\varphi(F)^\circ \wedge \varphi(G)^\circ) \rightarrow \varphi(\sigma_i) \text{ and } \varphi(\sigma_i),$$

of which the first two are theorems, the third is an axiom of $SP\tau$ and the last one is obtained from the others by modus ponens.

A similar replacement will work if σ_i is an instance of axioms \neg_2 or \neg_3 .

If $\sigma_i = p_\perp$, then $\varphi(\sigma_i) = f_\perp p$. Then we can replace σ_i by the set

$$\neg(p^\circ) \rightarrow f_\perp p, \neg(p^\circ) \text{ and } \varphi(\sigma_i),$$

of which the first is an axiom of $\text{SP}\tau$, the second is in $\{\neg(p^\circ) : p \in \mathcal{P}\}$ and the last one is obtained from the others by modus ponens.

If $\sigma_i = \neg^k p_\mu \leftrightarrow \neg^{k-1} p_{\sim\mu}$, then

$$\varphi(\sigma_i) = \neg^k f_\mu p \leftrightarrow \neg^{k-1} f_{\sim\mu} p.$$

We replace σ_i by

$$\neg(p^\circ) \rightarrow (\neg^k f_\mu p \leftrightarrow \neg^{k-1} f_{\sim\mu} p), \neg(p^\circ) \text{ and } \varphi(\sigma_i),$$

of which the first is an axiom of $\text{SP}\tau$, the second is in $\{\neg(p^\circ) : p \in \mathcal{P}\}$ and the last one is obtained from the others by modus ponens.

If $\sigma_i = p_\mu \rightarrow p_\lambda$, for $\mu \geq \lambda$, then

$$\varphi(\sigma_i) = f_\mu p \rightarrow f_\lambda p.$$

is an axiom of $\text{SP}\tau$.

If $\sigma_i = A \rightarrow p_\mu$, where $\mu = \bigvee_{j \in J} \mu_j$, and σ_i is obtained by rule τ_4 from

$\sigma_j = A \rightarrow p_{\mu_j}$, where for all $j \in J$, $j < i$. Observe that by remark 2.1 (3), we may assume $J \neq \emptyset$. Then

$$\varphi(\sigma_i) = \varphi(A) \rightarrow f_\mu p,$$

and for all $j \in J$,

$$\varphi(\sigma_j) = \varphi(A) \rightarrow f_{\mu_j} p,$$

so applying rule τ_{7S} we obtain $\varphi(\sigma_i)$.

Finally, if σ_i is obtained by modus ponens from a pair of premises, since this is a rule in both systems, $\varphi(\sigma_i)$ is also obtained from the corresponding premises.

This completes the proof of 1).

2) Let $\langle \sigma_1, \dots, \sigma_\alpha \rangle$, where α is an ordinal, be a proof in $\text{SP}\tau$ of A from A_1, \dots, A_n . We will prove by induction on the length of the proof that for all $i \leq \alpha$,

$$\{\psi(A_1), \dots, \psi(A_n)\} \vdash_{P\tau} \psi(\sigma_i).$$

We just have to check that if σ_i is an axiom of $\text{SP}\tau$ or it is obtained from previous σ_j 's by one of the two rules, then $\psi(\sigma_i)$ is a theorem of $\text{P}\tau$.

Since for axioms for \rightarrow , \vee and \wedge , $\psi(\sigma_i)$ is an instance of the same axioms in $\text{P}\tau$, the assertion holds for all these axioms.

The same can be said if σ_i is obtained by modus ponens from previous premises.

Before taking care of the other axioms, we observe that A is complex in $\text{SP}\tau$ if and only if $\theta(A)$ is complex in $\text{SP}\tau$ if and only if $\psi(A)$ is complex in $\text{P}\tau$ and that

$$\begin{aligned}\psi(A^\circ) &= \chi(\neg(f_\top\theta(A) \leftrightarrow f_\top\neg\theta(A))) \\ &= \neg\chi(f_\top\theta(A) \leftrightarrow \chi(f_\top\neg\theta(A))) \\ &= \neg(\chi(\theta(A)) \leftrightarrow \neg\chi(\theta(A))) \\ &= \neg(\psi(A) \leftrightarrow \neg\psi(A)),\end{aligned}$$

and also

$$\begin{aligned}\psi(\neg(A^\circ)) &= \neg(\chi(f_\top\theta(A) \leftrightarrow f_\top\neg\theta(A))) \\ &= \neg(\chi(f_\top\theta(A)) \leftrightarrow \chi(f_\top\neg\theta(A))) \\ &= \neg(\chi(\theta(A)) \leftrightarrow \neg\chi(\theta(A))) \\ &= \neg(\psi(A) \leftrightarrow \neg\psi(A)),\end{aligned}$$

so if A is complex,

$$\vdash_{\text{P}\tau} \neg\psi(A^\circ), \quad (1)$$

and

$$\vdash_{\text{P}\tau} \psi(\neg(A^\circ)). \quad (2)$$

On the other hand, if A is hyperliteral, assuming

$$A = \neg^{k_n} f_{\lambda_n} \neg^{k_{n-1}} f_{\lambda_{n-1}} \cdots \neg^{k_1} f_{\lambda_1} \neg^{k_0} p,$$

then

$$\theta(A) = A,$$

and

$$\begin{aligned}\psi(A^\circ) &= \chi(f_\top\theta(A) \leftrightarrow f_\top\neg\theta(A)) \\ &= \chi(f_\top A) \leftrightarrow \chi(f_\top\neg A) \\ &= p_\top \leftrightarrow p_\top.\end{aligned}$$

and also

$$\begin{aligned}\psi(\neg(A^\circ)) &= \neg(\chi(f_\top\theta(A) \leftrightarrow f_\top\neg\theta(A))) \\ &= \neg(\chi(f_\top\theta(A)) \leftrightarrow \chi(f_\top\neg\theta(A))) \\ &= \neg(p_\top \leftrightarrow p_\top).\end{aligned}$$

So if A is hyperliteral,

$$\vdash_{P\tau} \psi(A^\circ), \quad (3)$$

and

$$\vdash_{P\tau} \neg\psi(\neg(A^\circ)). \quad (4)$$

So let us now assume σ_i is axiom of negation \neg_{1S} ,

$$(A^\circ \wedge B^\circ) \rightarrow ((A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)),$$

then $\psi(\sigma_i)$ equals

$$(\psi(A^\circ) \wedge \psi(B^\circ)) \rightarrow ((\psi(A) \rightarrow \psi(B)) \rightarrow ((\psi(A) \rightarrow \neg\psi(B)) \rightarrow \neg\psi(A))),$$

We have two cases.

If either A or B is hyperliteral, then using (4), the fact that $\psi(\neg(A^\circ))$ and $\psi(\neg(B^\circ))$ are complex in $P\tau$ and a couple of classical tautologies, we have

$$\vdash_{P\tau} \psi(\sigma_i).$$

If both A and B are complex in $SP\tau$, then both $\psi(A)$ and $\psi(B)$ are complex in $P\tau$, so

$$(\psi(A) \rightarrow \psi(B)) \rightarrow ((\psi(A) \rightarrow \neg\psi(B)) \rightarrow \neg\psi(A))$$

is an instance of axiom \neg_1 , and by theorem 2.1 we also have

$$\vdash_{P\tau} \psi(\sigma_i).$$

The next two axioms for negation are treated similarly.

In the finitary case, axiom \neg_{4S} needs to be handled more carefully since it has no analog in $P\tau$.

If σ_i is an instance of \neg_{4S} , $\neg^{T+P}A \leftrightarrow \neg^T A$, then one easily checks that

$$\psi(\sigma_i) = \neg^{T+P}\psi(A) \leftrightarrow \neg^T\psi(A).$$

Let $I : \mathcal{P} \rightarrow \tau$ be an interpretation.

If $A = \neg^{k_1} f_{\lambda_1} \neg^{k_2} f_{\lambda_2} \dots \neg^{k_n} f_{\lambda_n} \neg^{k_{n+1}} p$, then $\psi(A) = \neg^{k_n} p_{\lambda_n}$

Now by Remark 3.1, $\sim^{T+P+k_n} \lambda_n = \sim^{T+k_n} \lambda_n$, so

$$v_I(\neg^{T+P}\psi(A)) = 1 \text{ iff } v_I(\neg^T\psi(A)) = 1.$$

Now apply theorem 2.4, (2).

If A is complex, then $v_I(\neg^{T+P}\psi(A)) = v_I(\neg^T\psi(A))$, since P is a multiple of 2.

In any case, $\models_{P\tau} \psi(\sigma_i)$ and so by theorem 2.4, (2),

$$\vdash_{P\tau} \psi(\sigma_i).$$

If σ_i is one of axioms \circ_{1S} or \circ_{2S} , we observe that for any formula A , $\psi(A)$ is complex if and only if $\psi(f_\lambda A)$ is complex if and only if $\psi(\neg A)$ is complex, so by (1) and (3) above, $\psi(\sigma_i)$ is a theorem of $P\tau$.

If σ_i is one of the instances of axiom \circ_{3S} , by (2), $\psi(\sigma_i)$ is also a theorem of $P\tau$.

So in any case, if σ_i is an axiom for \circ ,

$$\vdash_{P\tau} \psi(\sigma_i).$$

We now check the axioms τ_S . If σ_i is τ_{1S} , $A^\circ \rightarrow f_\perp A$, and A is hyperliteral, then

$$\psi(\sigma_i) = \psi(A^\circ) \rightarrow p_\perp.$$

But p_\perp is an axiom of $P\tau$, so by theorem 2.1, $\psi(\sigma_i)$ is a theorem of $P\tau$.

If A is complex,

$$\psi(\sigma_i) = \psi(A^\circ) \rightarrow \psi(A),$$

so by (1) and the fact that $\psi(A^\circ)$ is complex, after using a couple classical tautologies, $\psi(\sigma_i)$ is also a theorem of $P\tau$.

If σ_i is τ_{2S} , $A^\circ \rightarrow (\neg^k f_\lambda A \leftrightarrow f_{\sim^k \lambda} A)$.

If A is hyperliteral with propositional letter p , observe that

$$\psi(\neg^k f_\lambda A \leftrightarrow f_{\sim^k \lambda} A) = \neg^k p_\lambda \leftrightarrow p_{\sim^k \lambda},$$

and the latter is a theorem of $P\tau$, so by theorem 2.1, $\psi(\sigma_i)$ is also a theorem.

If A is complex, then

$$\psi(\sigma_i) = \psi(A^\circ) \rightarrow \psi(\neg^k f_\lambda A \leftrightarrow f_{\sim^k \lambda} A).$$

So noticing that both the antecedent and consequent are complex, using (4) and several classical tautologies, we get that $\psi(\sigma_i)$ is a theorem of $P\tau$.

Similarly, if σ_i is τ_{3S} , $f_\mu A \rightarrow f_\lambda A$, for $\mu \geq \lambda$.

If A is complex, then

$$\psi(\sigma_i) = \psi(A) \rightarrow \psi(A).$$

If A is hyperliteral, then

$$\psi(\sigma_i) = p_\mu \rightarrow p_\lambda.$$

So in any case

$$\vdash_{P\tau} \psi(\sigma_i).$$

If σ_i is τ_{4S} , $f_\mu f_\lambda A \leftrightarrow f_\mu A$, and A is complex, then

$$\psi(\sigma_i) = \psi(A) \leftrightarrow \psi(A).$$

If A is hyperliteral, then

$$\psi(\sigma_i) = p_\mu \leftrightarrow p_\mu.$$

So in any case

$$\vdash_{P\tau} \psi(\sigma_i).$$

Axioms τ_{5S} and τ_{6S} are handled in the same way.

Finally, assume $\sigma_i = A \rightarrow f_\lambda B$ is obtained by an application of rule τ_{7S} from $A \rightarrow f_{\lambda_j} B$, $j < i$ for all $j \in J$, where $J \neq \emptyset$ and $\lambda = \bigvee_{j \in J} \lambda_j$.

If B is complex, then for all $j \in J$,

$$\psi(\sigma_i) = \psi(A \rightarrow f_\lambda B) = \psi(A) \rightarrow \psi(B) = \psi(A \rightarrow f_{\lambda_j} B),$$

so by inductive hypothesis $\psi(\sigma_i)$ is a theorem of $P\tau$.

If B is hyperliteral,

$$\psi(A \rightarrow f_{\lambda_j} B) = \psi(A) \rightarrow p_{\lambda_j},$$

and thus

$$\psi(A \rightarrow f_\lambda B) = \psi(A) \rightarrow p_\lambda,$$

is obtained by an application of rule τ_4 .

This ends the proof of the theorem. □

4. ON THE ALGEBRAIZATION OF SYSTEMS $SP\tau$

In this section we prove our main result, namely, that a system $SP\tau$ is algebraizable if and only if τ is finite. When τ is infinite, one should recall that since in $SP\tau$ there is an infinitary rule, technically, the system cannot be algebraizable in the sense of Blok and Pigozzi. Nevertheless, the methods of [3] can be generalized to include systems like $SP\tau$. This was done by Herrmann in [8]. In particular, the characterization of algebraizable systems through properties of the Leibniz equality function, Theorem 4.2 in [3] is specially suited for such a generalization.

Here and in the rest of this section we will write \vdash instead of $\vdash_{SP\tau}$.

Following [3], theorem 4.7, in order to prove that the system $SP\tau$ is algebraizable, we need to find unary terms δ_i, ε_i , $i \in I$ and binary terms Δ_j , $j \in J$, where I and J are finite, such that

- i) $\vdash A\Delta A$
- ii) $A\Delta B \vdash B\Delta A$
- iii) $A\Delta B, B\Delta A \vdash A\Delta A$
- iv) a) $A\Delta B \vdash A^\circ \Delta B^\circ$
 b) $A\Delta B \vdash \neg A \Delta \neg B$
 c) For each $\mu \in \tau$, $A\Delta B \vdash f_\mu A \Delta f_\mu B$

- d) $A_1 \Delta B_1, A_2 \Delta B_2 \vdash A_1 \wedge A_2 \Delta B_1 \wedge B_2$
- e) $A_1 \Delta B_1, A_2 \Delta B_2 \vdash A_1 \vee A_2 \Delta B_1 \vee B_2$
- f) $A_1 \Delta B_1, A_2 \Delta B_2 \vdash A_1 \rightarrow A_2 \Delta B_1 \rightarrow B_2$
- v) $A \dashv\vdash \delta(A) \Delta \varepsilon(A)$

Where $\vdash A \Delta B$ means that for all $j \in J$, $\vdash \Delta_j(A, B)$ and similarly for δ and ε .

Δ is called *system of equivalence formulas for S* ; $\delta \approx \varepsilon$ are called *defining equations*.

The following theorem proves that a certain set of formulas of $\text{SP}\tau$ define a congruence on \mathcal{F}_S that verifies the above conditions, that is, $\text{SP}\tau$ is weakly congruential. In general, $\text{SP}\tau$ is not algebraizable since this set might be infinite.

Theorem 4.1. *For any lattice τ , let*

$$\begin{aligned}\delta(A) &= A \wedge A, \\ \varepsilon(A) &= A \rightarrow A.\end{aligned}$$

Define equivalence formulas of three types,

a)

$$\Delta_{\circ}(A, B) = A^{\circ} \leftrightarrow B^{\circ}.$$

b)

$$\begin{aligned}\Delta_0(A, B) &= A \leftrightarrow B, \\ &\vdots \\ \Delta_k(A, B) &= \neg^k A \leftrightarrow \neg^k B, \\ &\vdots\end{aligned}$$

c)

$$\Delta_{\lambda}(A, B) = f_{\lambda}A \leftrightarrow f_{\lambda}B,$$

for each $\lambda \in \tau$.

Then $\Delta = \{\Delta_j : j \in \{\circ\} \cup \omega \cup \tau\}$, and the single defining equation $\delta \approx \varepsilon$ (i.e. $|I| = 1$), verify conditions i) through v) above.

Proof. The proofs of i) , ii) and iii) above are immediate.

For a proof of iv)a, by axiom \circ_{3S} ,

$$\vdash (A^{\circ})^{\circ} \text{ and } \vdash (B^{\circ})^{\circ},$$

so

$$\vdash (A^{\circ})^{\circ} \leftrightarrow (B^{\circ})^{\circ},$$

so

$$\vdash (A^{\circ})^{\circ} \leftrightarrow (B^{\circ})^{\circ},$$

that is,

$$\vdash \Delta_o(A^\circ, B^\circ).$$

Next observe that by theorem 3.2, for all n ,

$$(A^\circ)^\circ, (B^\circ)^\circ \vdash (\neg^n(A^\circ) \leftrightarrow \neg^n(B^\circ)) \leftrightarrow (A^\circ \leftrightarrow B^\circ),$$

since the latter is a classical tautology. So

$$\Delta_o(A, B) \vdash \Delta_n(A^\circ, B^\circ).$$

Finally, by axiom τ_{6S} ,

$$\begin{aligned} (A^\circ)^\circ &\vdash f_\lambda(A^\circ) \leftrightarrow A^\circ \\ (B^\circ)^\circ &\vdash f_\lambda(B^\circ) \leftrightarrow B^\circ \end{aligned}$$

so we have

$$\Delta_o(A, B) \vdash \Delta_\lambda(A^\circ, B^\circ).$$

This completes the proof of

$$A\Delta B \vdash A^\circ\Delta B^\circ.$$

In order to prove iv)b, by axiom \circ_{1S}

$$A^\circ \leftrightarrow B^\circ \vdash (\neg A)^\circ \leftrightarrow (\neg B)^\circ.$$

So

$$A\Delta B \vdash \Delta_o(\neg A, \neg B).$$

It is immediate that

$$A\Delta B \vdash \Delta_n(\neg A, \neg B).$$

By axiom τ_{5S} ,

$$\begin{aligned} \neg(A^\circ) &\vdash f_\lambda A \leftrightarrow f_\lambda \neg A \\ \neg(B^\circ) &\vdash f_\lambda B \leftrightarrow f_\lambda \neg B, \text{ so} \\ \Delta_\lambda(A, B), \neg(A^\circ), \neg(B^\circ) &\vdash f_\lambda \neg A \leftrightarrow f_\lambda \neg B. \end{aligned} \quad (1)$$

Also, by axiom τ_{6S} ,

$$\begin{aligned} A^\circ &\vdash f_\lambda \neg A \leftrightarrow \neg A \\ B^\circ &\vdash f_\lambda \neg B \leftrightarrow \neg B, \text{ so} \\ \Delta_1(A, B), A^\circ, B^\circ &\vdash f_\lambda \neg A \leftrightarrow f_\lambda \neg B. \end{aligned} \quad (2)$$

So from (1) and (2) using a classical argument, we prove

$$A\Delta B \vdash f_\lambda \neg A \leftrightarrow f_\lambda \neg B.$$

This ends the proof of iv)b.

We now prove iv)c. Let $\mu \in \tau$. By axiom \circ_{2S}

$$A^\circ \leftrightarrow B^\circ \vdash (f_\mu A)^\circ \leftrightarrow (f_\mu B)^\circ.$$

So

$$A\Delta B \vdash \Delta_{\circ}(f_{\mu}A, f_{\mu}B).$$

To check the conditions for negation we will use the same method as in iv)b.

By axiom τ_{2S} ,

$$\begin{aligned} \neg(A^{\circ}) &\vdash \neg^n f_{\mu}A \leftrightarrow f_{\sim^n \mu}A \\ \neg(B^{\circ}) &\vdash \neg^n f_{\mu}B \leftrightarrow f_{\sim^n \mu}B \\ \Delta_{\sim^n \mu}(A, B), \neg(A^{\circ}), \neg(B^{\circ}) &\vdash \neg^n f_{\mu}A \leftrightarrow \neg^n f_{\mu}B. \end{aligned} \quad (3)$$

On the other hand, by axiom τ_{6S} , and a couple of classical tautologies,

$$\begin{aligned} A^{\circ} &\vdash \neg^n f_{\mu}A \leftrightarrow \neg^n A \\ B^{\circ} &\vdash \neg^n f_{\mu}B \leftrightarrow \neg^n B \\ \Delta_n(A, B), A^{\circ}, B^{\circ} &\vdash \neg^n f_{\mu}A \leftrightarrow \neg^n f_{\mu}\neg B. \end{aligned} \quad (4)$$

So by (3) and (4),

$$A\Delta B \vdash \Delta_n(f_{\mu}A, f_{\mu}B).$$

Next, by axiom τ_{4S} ,

$$\Delta_{\lambda}(A, B) \vdash f_{\lambda}f_{\mu}A \leftrightarrow f_{\lambda}f_{\mu}B,$$

which completes the proof that

$$A\Delta B \dashv\vdash f_{\mu}A\Delta f_{\mu}B.$$

The proofs of iv)d, iv)e and iv)f), follow from the fact that if A and B are complex, they behave classically and it is straightforward to check that

$$A \leftrightarrow B \vdash A\Delta B.$$

Now if $*$ is a binary connective, then $A_1 * A_2$ and $B_1 * B_2$ are complex so it is enough to prove that

$$A_1\Delta B_1, A_2\Delta B_2 \vdash (A_1 * A_2) \leftrightarrow (B_1 * B_2),$$

which in all three cases can be obtained using Modus Ponens and a classical tautology that does not contain a negation.

For a proof of v), by the remark just made, since $\delta(A)$ and $\varepsilon(A)$ are complex, it is enough to prove

$$A \vdash \delta(A) \leftrightarrow \varepsilon(A).$$

Since $\vdash A \rightarrow A$, we have, $\vdash (A \wedge A) \rightarrow (A \rightarrow A)$, on the other hand, $A \vdash A \wedge A$, so $A \vdash (A \rightarrow A) \rightarrow A \wedge A$, that is, $A \vdash (A \wedge A) \leftrightarrow (A \rightarrow A)$.

Also, $(A \wedge A)\Delta(A \rightarrow A) \vdash (A \rightarrow A) \rightarrow (A \wedge A)$, but since $\vdash A \rightarrow A$, and $A \wedge A \vdash A$, $(A \wedge A)\Delta(A \rightarrow A) \vdash A$. \square

As we will see, if τ is finite, then $SP\tau$ is algebraizable, however this is not true in general. We will now prove that the systems $SP\tau$ are not algebraizable for an infinite lattice τ . This is accomplished by showing that, for a given lattice τ , the Leibniz equality function Ω defined in [3] does not preserve unions of directed subsets of $\text{Th}(SP\tau)$.

For general definitions and properties of Ω the reader should consult [3], we will prove here some lemmas that will be useful in the sequel.

The first of these lemmas is a generalization of Theorem 1.6, pp.11,12 of [3].

Lemma 4.2. *Let \mathbf{A} be an algebra and $F \subseteq A$. Let Θ be a binary relation on A that is definable by a set of formulas over the matrix $\langle \mathbf{A}, F \rangle$ with parameters and without equality.*

- (i) *If Θ is reflexive, then $\Omega_A F \subseteq \Theta$;*
- (ii) *if, in addition, Θ is a congruence on \mathbf{A} that is compatible with F , then $\Omega_A F = \Theta$.*

Proof. Let Θ be defined by an arbitrary set Γ of formulas. Then since Θ is reflexive, for any $\gamma(p, q, r_0, \dots, r_{k-1}) \in \Gamma$, with parameters c_0, \dots, c_{k-1} , in the defining set Γ , and any $a \in A$,

$$\langle \mathbf{A}, F \rangle \models \gamma[a, a, c_0, \dots, c_{k-1}].$$

Now let $(a, b) \in \Omega_A F$, then by the definition of the Leibniz relation, for any such γ and set of parameters,

$$\langle \mathbf{A}, F \rangle \models \gamma[a, b, c_0, \dots, c_{k-1}],$$

and thus

$$\Omega_A F \subseteq \Theta.$$

For a proof of part (ii) simply recall that $\Omega_A F$ is the largest congruence of A that is compatible with F , so $\Theta \subseteq \Omega_A F$. \square

Lemma 4.3. *Let T be a theory in $SP\tau$ and define*

$$\Theta = \{(\varphi, \psi) : \varphi \Delta \psi \in T\}.$$

Then $\Omega T = \Theta$.

Proof. Recall that by [3], theorem 1.5, ΩT is the largest congruence over \mathcal{F}_S compatible with T .

By theorem 4.1, Θ is a congruence.

It is obvious that Θ is compatible with T , for if $(\varphi, \psi) \in \Theta$ and $\varphi \in T$, then $\varphi \Delta \psi \in T$, so in particular, $\varphi \leftrightarrow \psi \in T$ and by modus ponens, $\psi \in T$.

Finally, since Θ is reflexive and defined by a set of formulas, by lemma 4.2,

$$\Omega T = \Theta. \quad \square$$

Lemma 4.4. *Let $\Gamma \subseteq \tau$ be such that $\perp \in \Gamma$. Then*

$$\{f_\gamma p \leftrightarrow f_\gamma q : \gamma \in \Gamma\} \vdash f_\beta p \leftrightarrow f_\beta q$$

if and only if there is a subset Λ of Γ such that

$$\beta = \bigvee_{\lambda \in \Lambda} \lambda.$$

Proof. Let

$$\beta = \bigvee_{\lambda \in \Lambda} \lambda.$$

If $\Lambda = \emptyset$, then $\beta = \perp \in \Gamma$ and the theorem holds.

So we can assume $\Lambda \neq \emptyset$. Now, since $\beta \geq \lambda$, for each $\lambda \in \Lambda$, by axiom τ_{3S} , $\vdash f_\beta p \rightarrow f_\lambda p$, so

$$\{f_\gamma p \leftrightarrow f_\gamma q : \gamma \in \Gamma\} \vdash f_\beta p \rightarrow f_\lambda q,$$

for each $\lambda \in \Lambda$. So by rule τ_{7S} ,

$$\{f_\gamma p \leftrightarrow f_\gamma q : \gamma \in \Gamma\} \vdash f_\beta p \rightarrow f_\beta q.$$

Similarly, we get the implication in the other direction, so

$$\{f_\gamma p \leftrightarrow f_\gamma q : \gamma \in \Gamma\} \vdash f_\beta p \leftrightarrow f_\beta q.$$

Assume now that β is not the supremum of any subset of Γ . Let

$$D = \{\alpha \in \Gamma : \alpha \leq \beta\}.$$

Let $\delta = \bigvee D$. Then

$$\delta < \beta.$$

It is also clear that for $\gamma \in \Gamma$,

$$\gamma < \beta \text{ if and only if } \gamma \leq \delta.$$

Let $I(p) = \delta$ and $I(q) = \beta$. Then for all $\gamma \in \Gamma$,

$$I(f_\gamma p) = 1 \text{ if and only if } I(f_\gamma q) = 1,$$

and thus,

$$I(f_\gamma p \leftrightarrow f_\gamma q) = 1.$$

Nevertheless, $I(f_\beta p) = 0$ and $I(f_\beta q) = 1$, so by theorem 3.4, the proof is complete. \square

Theorem 4.5. *An annotated logic $SP\tau$ is algebraizable if and only if τ is finite.*

Proof. If τ is finite, then the Δ_λ 's are finite. Also, by axiom \neg_{4S} , it is enough to consider Δ_i for $0 \leq i < T + P$, since if $n \geq T + P$, then $n = T + m + kP$, for some $m < P$ and k so

$$\vdash \neg^n A \leftrightarrow \neg^{T+m} A$$

and thus

$$\Delta_{T+m}(A, B) \vdash \Delta_n(A, B).$$

So we have finitely many equivalence formulas and $\text{SP}\tau$ is algebraizable by theorem 4.1.

If τ is infinite, we will say that a subset X *generates* τ if all elements of τ are supremums of elements of X .

Let Λ be a subset that generates τ of minimal cardinality, in the sense that there is no subset of τ of less cardinality than that of Λ that generates τ . Assume further that

$$\Lambda = \{\lambda_i : i < \kappa\}.$$

For $\gamma < \kappa$, let

$$T_\gamma = \text{Cn}\{f_{\lambda_i}p \leftrightarrow f_{\lambda_i}q : i < \gamma\} \cup \{\neg(p^\circ), \neg(q^\circ)\}$$

and

$$T = \text{Cn} \bigcup_{\gamma < \kappa} T_\gamma.$$

Clearly

$$T \vdash \Delta_\circ(p, q),$$

so by axiom \circ_{2S} ,

$$T \vdash \Delta_\circ(f_\top p, f_\top q).$$

Since Λ generates τ , by theorem 4.4, for all $\lambda \in \tau$,

$$T \vdash f_\lambda p \leftrightarrow f_\lambda q,$$

so by axiom τ_{4S} ,

$$T \vdash f_\lambda(f_\top p) \leftrightarrow f_\lambda(f_\top q),$$

that is,

$$T \vdash \Delta_\lambda(f_\top p, f_\top q).$$

Also, since

$$\neg((f_\top p)^\circ) \vdash \neg^n(f_\top p) \leftrightarrow f_{\sim^n \top}(f_\top p)$$

and

$$\neg((f_\top q)^\circ) \vdash \neg^n(f_\top q) \leftrightarrow f_{\sim^n \top}(f_\top q),$$

we have

$$T \vdash \Delta_n(f_\top p, f_\top q),$$

for all $n \geq 0$.

Thus we have proved that

$$T \vdash f_{\top} p \Delta f_{\top} q,$$

so by lemma 4.3

$$(f_{\top} p, f_{\top} q) \in \Omega T.$$

Suppose now that

$$(f_{\top} p, f_{\top} q) \in \bigcup_{\gamma < \kappa} \Omega T_{\gamma}.$$

Then for some $\gamma < \kappa$

$$(f_{\top} p, f_{\top} q) \in \Omega T_{\gamma}.$$

But $\{\lambda_i : i < \gamma\}$ has cardinality less than κ , so it does not generate τ , so by lemma 4.4,

$$T_{\gamma} \not\vdash f_{\lambda} p \leftrightarrow f_{\lambda} q$$

for some $\lambda \in \tau$. But then

$$T_{\gamma} \not\vdash f_{\lambda}(f_{\top} p) \leftrightarrow f_{\lambda}(f_{\top} q),$$

so

$$T_{\gamma} \not\vdash f_{\top} p \Delta f_{\top} q.$$

Thus we get a contradiction.

This proves that Ω does not preserve directed sets of theories, so $\text{SP}\tau$ is not algebraizable for infinite τ . \square

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