MV^*–Algebras

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Abstract

In this paper we make an algebraic study of the variety of MV^*–algebras introduced by C. C.Chang as an algebraic counterpart for a logic with positive and negative truth values.

We build the algebraic theory of MV^*–algebras within its own limits using a concept of ideal and of prime ideal that are very naturally related to the corresponding concepts in ℓ–groups. The main results are a subdirect representation theorem, a completeness theorem, a study of simple and semisimple algebras, and a characterization of the ideal of infinitesimals as an ℓ–group. In the last section we develop a detailed proof a the one–dimensional theorem of McNaughton, that is, the free MV^*–algebra in one generator is the algebra of McNaughton functions over $[-1, 1]$. In contrast with the rest of the paper, this last result is based on work done for MV–algebras.

Keywords: MV^*–algebra, MV-algebra, ℓ–groups, many-valued logics

1 Introduction

MV^*–algebras were introduced by Chang in [2] in an attempt to model the properties of the real interval $[-1, 1]$ equipped with truncated addition $x \oplus y = \max\{-1, \min\{1, x + y\}\}$ and negation $-x$, paralleling similar work done for MV–algebras. This variety is associated to a logic Ł* which generalizes in the same sense Łukasiewiczs' infinite–valued logic Ł.

In the very active field of many-valued logics and their algebraic counterparts, there is a close connection between these classes of algebras and lattice-ordered Abelian groups, or ℓ–groups for short. The relation between MV^*–algebras and ℓ–groups is, in a sense, more natural than that existing between MV–algebras and ℓ–groups. For instance, every ℓ–group is the ideal of infinitesimals of an MV^*–algebra, see Corollary 10.6. The study of the variety of MV^*–algebras, gives us some insight into the role that the logic associated with ℓ–groups plays in the field of many-valued logic. Furthermore, in the last fifteen or so years, numerous logics with both positive and negative truth values have emerged, see for instance [5] for references. All these have algebraic semantics closely associated with ℓ–groups. This is a very new topic and very little is known about it. In [4], we develop the logic Ł* and prove some of its main metatheorems as well as its relation with MV^*–algebras.

An important class of MV^*–algebras is defined as follows. Let G be an ℓ–group and let p be a positive element of G, then $G(p) = \langle [-p, p]; \oplus, -, 0, p \rangle$, where $x \oplus y = \max\{-p, \min\{p, x + y\}\}$, is an MV^*–algebra. This is simply a generalization of the original case, where $G = \mathbb{R}$ and $p = 1$. In [5] we prove that this defines a categorical equivalence between the categories of MV^*–algebras and the category of ℓ–groups with strong unit.

Among the properties of the variety of MV^*–algebras proven in [2], the most important, at least from the algebraic point of view, is the following representation theorem.

Every MV^*–algebra is a subdirect product of (linear) MV^*–algebras $G(p)$.
Chang proves this theorem by reducing it to a similar problem in MV–algebras, and applying the representation theorem for these. We give an independent proof, see Theorem 5.1.

In this paper we will build the algebraic theory of MV^*–algebras within its own limits using a concept of ideal and of prime ideal (Sections 3 and 4) that are very naturally related to the corresponding concepts in ℓ–groups. The main results are a subdirect representation theorem (Section 5), a completeness theorem (Section 7), a study of simple and semisimple algebras (Section 9), and a characterization of the ideal of infinitesimals as an ℓ–group (Section 10). In the last section we develop a detailed proof of the one–dimensional theorem of McNaughton, that is, the free MV^*–algebra in one generator is the algebra of McNaughton functions over $[-1, 1]$ (Section 10).

2 MV^*–Algebras

An MV^*–algebra is a system $\mathbf{B} = \langle B; \oplus, -, \mathbf{0}, \mathbf{1}\rangle$ satisfying the following equations. We define

$$
\begin{aligned}
x^+ &:= \mathbf{1} \oplus (-\mathbf{1} \oplus x)\,, \\
x^- &:= -\mathbf{1} \oplus (\mathbf{1} \oplus x)\,, \\
x \ominus y &:= x \oplus (-y)\,, \\
x \vee y &:= (\, x^+ \oplus (-(x^+) \oplus y^+)^+ \,) \oplus (\, x^- \oplus (-(x^-) \oplus y^-)^+ \,)\,.
\end{aligned}
$$

(Bx1) $x \oplus y = y \oplus x$
(Bx2) $(\mathbf{1} \oplus x) \oplus (y \oplus (\mathbf{1} \oplus z)) = ((\mathbf{1} \oplus x) \oplus y) \oplus (\mathbf{1} \oplus z)$
(Bx3) $x \oplus -x = \mathbf{0}$
(Bx4) $(x \oplus \mathbf{1}) \oplus \mathbf{1} = \mathbf{1}$
(Bx5) $x \oplus \mathbf{0} = x$
(Bx6) $-(x \oplus y) = -x \oplus -y$
(Bx7) $--x = x$
(Bx8) $x \oplus y = (x^+ \oplus y^+) \oplus (x^- \oplus y^-)$
(Bx9) $(-x \oplus (x \oplus y))^+ = -(x^+) \oplus (x^+ \oplus y^+)$
(Bx10) $x \vee y = y \vee x$
(Bx11) $x \vee (y \vee z) = (x \vee y) \vee z$
(Bx12) $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z)$

Some elementary properties of MV^*–algebras are proved in [2]. The addition $\oplus$ is commutative, has a zero–element and every element has an additive inverse. However if the algebra is not trivial, $\oplus$ is not associative for, if it were, using Axioms (Bx1), (Bx3) , (Bx4) and (Bx5) $1 = 0 \oplus 1 = (-1 \oplus 1) \oplus 1 = -1 \oplus (1 \oplus 1) = -1 \oplus 1 = 0$. It is also true that $x^+ = x \vee \mathbf{0}$, $x^- = x \wedge \mathbf{0}$ and $x \vee y = (x^+ \vee y^+) \oplus (x^- \vee y^-)$.

It follows from (Bx10) and (Bx11) that the prescription $x \leq y$ if and only if $x \vee y = y$ defines an order on B for which the element $x \vee y$ is the least upper bound of x and y. Since $-$ reverses the order by definition of $x \vee y$ and the fact that $(-x)^+ = -(x^-)$, we may see that the greatest lower bound of x and y is $x \wedge y = -(-x \vee -y)$. The addition $\oplus$ distributes over $\vee$ and $\wedge$.

Even though addition is not associative in general, associativity holds in some cases, namely, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if x and z are both positive or both negative. We will refer to this as *restricted associativity* and we will use this fact without further mention.

Lemma 2.1
In any MV^*–algebra,

1. if $x \ominus y = \mathbf{0}$, then $x = y$, and
2. if $x \ominus y \geq \mathbf{0}$, then $x \geq y$.

Proof. Assume $x \ominus y = \mathbf{0}$. We first observe that if x and y are both positive or both negative,

$$y = \mathbf{0} \oplus y = (x \ominus y) \oplus y = x \oplus (-y \oplus y) = x \oplus \mathbf{0} = x \,,$$

by restricted associativity. So let us check the general case. We have

$$x^- \ominus y = (x \wedge \mathbf{0}) \ominus y = (x \ominus y) \wedge (\mathbf{0} \ominus y) = \mathbf{0} \wedge -y = (-y)^-$$

also

$$x^- \ominus y^- = x^- \ominus (y \wedge \mathbf{0}) = (x^- \ominus y) \vee (x^- \ominus \mathbf{0}) = (-y)^- \vee x^- \leq \mathbf{0} \,.$$

Similarly, since $y \ominus x = \mathbf{0}$, $y^- \ominus x^- \leq \mathbf{0}$, so $x^- \ominus y^- = \mathbf{0}$ and since both terms are negative, this yields $x^- = y^-$.

Applying the same argument to $-x \ominus (-y) = \mathbf{0}$, we get $x^+ = y^+$, so we have that $x = x^+ \oplus x^- = y^+ \oplus y^- = y$.

In order to prove the second assertion, assume $x \ominus y \geq \mathbf{0}$. Then $(x \wedge y) \ominus y = (x \ominus y) \wedge (y \ominus y) = \mathbf{0}$, so by the previous proof, $x \wedge y = y$, so $x \geq y$. ∎

Lemma 2.2
In any MV^*–algebra, the following hold.

1. $(x \oplus y^-) \oplus y^+ = x \oplus y$.
2. $(x^+ \oplus y^+) \ominus y^+ = x^+ \wedge (1 \ominus y^+)$, $(x^- \ominus y^-) \oplus y^- = x^- \vee (y^+) \ominus \mathbf{1}$.
3. $(x \oplus y^+) \ominus y^+ \leq x$.
4. $(x \ominus y^+) \oplus y^+ \geq x$.
5. $(x \oplus z^+) \oplus y^- \leq x \oplus (y^- \oplus z^+) \leq (x \oplus y^-) \oplus z^+$.

Proof. (*1.*)

$$\begin{aligned} (x \oplus y^-) \oplus y^+ &= (x \oplus y^-) \oplus (y \vee 0)\,, \\ &= ((x \oplus y^-) \oplus y) \vee (x \oplus y^-)\,, \\ &\leq (x \oplus y) \vee (x \oplus y)\,, \\ &= x \oplus y\,. \end{aligned}$$

Also

$$\begin{aligned}(x \oplus y^-) \oplus y^+ &= (x \oplus (y \wedge 0)) \oplus y^+ \,, \\ &= ((x \oplus y) \wedge x) \ \oplus \ y^+ \,, \\ &= ((x \oplus y) \oplus y^+) \wedge (x \oplus y^+) \,, \\ &\geq (x \oplus y) \wedge (x \oplus y) \,, \\ &= x \oplus y \,.\end{aligned}$$

(*2.*) Let $x = x^+$, $y = y^+$. From the definition of supremum and the fact that $a \wedge b = -(-a \vee -b)$ we have, for positive a and b, $a \wedge b = b \oplus (a \ominus b)^-$. Therefore, $x \wedge (1 \ominus y) = (1 \ominus y) \oplus (((x \ominus (1 \ominus y)) \oplus 1) \ominus 1) = (1 \ominus y) \oplus ((x \oplus y) \ominus 1) = (x \oplus y) \ominus y$, by associativity and from the equality $y = (y \ominus 1) \oplus 1$.

(*3.*) From 2, the inequality holds for $x = x^+$. By (Bx9), $((x \oplus y^+) \ominus y^+)^+ = (x^+ \oplus y^+) \ominus y^+$. Hence, $((x \oplus y^+) \ominus y^+)^+ \leq x^+$ (*).

Also, (Bx9) implies $((a \oplus b) \ominus b)^- = (a^- \oplus b^-) \ominus b^-$ (because $u^- = -(-u)^+$), from where $((x \oplus y^+) \ominus y^+)^- = x^-$ (**).

Therefore, $(x \oplus y^+) \ominus y^+ \leq x$, from (*) and (**).

(*4.*) From (*3.*), changing x to $-x$.

(*5*). We will first prove this inequality for a negative x, say for x^-. We will use the restricted associativity of MV^*, in each case we will underline the positive (or negative) terms involved.

$$\begin{aligned}(***) \quad (x^- \oplus y^-) \oplus z^+ &= [x^- \oplus (\underline{y^-} \oplus (z^+ \underline{\ominus z^+}))] \ \oplus \ z^+ \,, \\ &= [\underline{x^-} \oplus ((y^- \oplus z^+) \underline{\ominus z^+}] \ \oplus \ z^+ \,, \\ &= [(x^- \oplus (y^- \oplus z^+)) \ominus z^+] \ \oplus \ z^+ \,, \\ &\geq x^- \oplus (y^- \oplus z^+) \,,\end{aligned}$$

where the last inequality is obtained by (*4.*)

We will now prove the general case from (Bx8), (***), (*1.*) and restricted associativity.

$$\begin{aligned}(x \oplus y^-) \oplus z^+ &= (\underline{x^+} \oplus (x^- \oplus y^-)) \oplus \underline{z^+} \,, \\ &= x^+ \oplus ((x^- \oplus y^-) \oplus z^+) \,, \\ &\geq x^+ \oplus (x^- \oplus (y^- \oplus z^+)) \,, \\ &= x \oplus (y^- \oplus z^+) \,,\end{aligned}$$

This finishes the proof. ∎

Define the absolute value of x as $\mid x \mid = x^+ \vee -x^-$. It is immediate that $\mid x \mid = x^+ \ominus x^- = x^+ \oplus (-x)^+$.

LEMMA 2.3
In any MV^*–algebra, the following inequality holds.

$$|(x \oplus t) \ominus (y \oplus t)| \leq |x \ominus y| \,.$$

PROOF. We will first prove several special cases. We begin with $x \geq y$ and any t.
a) $(x^+ \oplus t^+) \ominus (y^+ \oplus t^+) \leq x^+ \ominus y^+$,
b) $(x^- \oplus t^-) \ominus (y^- \oplus t^-) \leq x^- \ominus y^-$.

Observe that we have $x^+ \geq y^+$ and $x^- \geq y^-$, so

$$\begin{aligned} x^+ \ominus y^+ &= u^+ , \\ x^+ = (\underline{x^+} \ominus y^+) \oplus \underline{y^+} &= u^+ \oplus y^+ , \\ x^+ \oplus t^+ &= (u^+ \oplus y^+) \oplus t^+ , \\ &= u^+ \oplus (y^+ \oplus t^+) , \\ (x^+ \oplus t^+) \ominus (y^+ \oplus t^+) &= [u^+ \oplus (y^+ \oplus t^+)] \ominus (y^+ \oplus t^+) , \\ (x^+ \oplus t^+) \ominus (y^+ \oplus t^+) &\leq u^+ = x^+ \ominus y^+ . \end{aligned}$$

The last line is obtained from Lemma 2.2, *3*.

Inequality b) is proved similarly.

$$\begin{aligned} (x \oplus t) \ominus (y \oplus t) &= (x \oplus t) \ominus [(y^+ \oplus t^+) \oplus (y^- \oplus t^-)] , \\ &\leq [(x \oplus t) \ominus (y^+ \oplus t^+)] \ominus (y^- \oplus t^-) , \\ &= [((x^+ \oplus t^+) \oplus \underline{(x^- \oplus t^-)}) \ominus \underline{(y^+ \oplus t^+)}] \ominus (y^- \oplus t^-) , \\ &= [\underline{((x^+ \oplus t^+) \ominus (y^+ \oplus t^+))} \oplus (x^- \oplus t^-)] \ominus \underline{(y^- \oplus t^-)} , \\ &= ((x^+ \oplus t^+) \ominus (y^+ \oplus t^+)) \oplus ((x^- \oplus t^-) \ominus (y^- \oplus t^-)) , \\ &\leq \underline{(x^+ \ominus y^+)} \oplus (x^- \ominus \underline{y^-}) , \\ &= [(x^+ \ominus \underline{y^+}) \oplus \underline{x^-}] \ominus y^- , \\ &= [(x^+ \oplus x^-) \ominus y^+] \ominus y^- , \\ &= (x \ominus y^+) \ominus y^- , \\ &= x \ominus y . \end{aligned}$$

The first inequality follows from Lemma 2.2, *5*. For the fourth line, observe that $x^+ \oplus t^+ \geq y^+ \oplus t^+$. In the second inequality we use the positive and negative cases previously proved. The last line is obtained from Lemma 2.2, *1*, with y replaced by $-y$.

We will now prove the general case. We first observe that

$$\begin{aligned} [(x \oplus t) \ominus (y \oplus t)]^+ &= [(x \oplus t) \ominus (y \oplus t)] \vee [(y \oplus t) \ominus (y \oplus t)] , \\ &= [(x \oplus t) \vee (y \oplus t)] \ominus (y \oplus t) , \\ &= [(x \vee y) \oplus t)] \ominus (y \oplus t) , \\ &\leq (x \vee y) \ominus y , \\ &= (x \ominus y) \vee 0 = (x \ominus y)^+ . \end{aligned}$$

where the inequality follows from the previous paragraph since $x \vee y \geq y$. With a similar argument we prove

$$[(x \oplus t) \ominus (y \oplus t)]^- \geq (x \ominus y)^- .$$

using these two inequalities we obtain

$$|(x \oplus t) \ominus (y \oplus t)| \leq |x \ominus y| \, .$$

■

3 MV^*–Ideals

A non–empty subset I of the universe A of an MV^*–algebra is an *ideal* if the following conditions hold.

1. If $x, y \in I$, then $x \ominus y \in I$.
2. If $x \in I$, then $x^+ \in I$.
3. I is *convex*, that is, if $x, y \in I$ and $x \leq z \leq y$, then $z \in I$.

Lemma 3.1
$\{\mathbf{0}\}$ is an ideal.

The intersection of a family of ideals is an ideal, so for any $Z \subseteq A$

$$\langle Z \rangle = \bigcap \{I : Z \subseteq I \text{ and } I \text{ is an ideal of } A\}$$

is the least ideal containing Z. The ideal $\langle Z \rangle$ is the *ideal generated by* Z.

Lemma 3.2
If $Z = \emptyset$, then $\langle Z \rangle = \{\mathbf{0}\}$. If $Z \neq \emptyset$, then

$$\langle Z \rangle = \{x : |x| \leq |z_1| \oplus |z_2| \oplus \cdots \oplus |z_n| \, , \; z_1, \ldots, z_n \in Z\}.$$

Proof. This set contains Z, is closed under differences, under taking positive parts and it is convex, so it is an ideal that contains Z. Clearly it is contained in any ideal that contains Z, so it must be the least such an ideal. ■

An ideal I is *proper* if and only if $I \neq A$. A *maximal ideal* is a proper ideal that is not contained in any proper ideal.

Lemma 3.3
For any ideal I, the following hold.

1. $\mathbf{0} \in I$.
2. I is closed under inverses, that is, if $x \in I$, then $-x \in I$.
3. I is closed under taking negative parts, that is, if $x \in I$, then $x^- \in I$.
4. I is closed under addition.
5. I is closed under taking suprema, that is, if $x \in I$ and $y \in I$, then $x \vee y \in I$.
6. I is closed under following version of Modus Ponens.
   If $x \ominus y \in I$ and $y \in I$, then $x \in I$. (MP)
7. If $x \ominus y \in I$ and $t \in A$, then $(x \oplus t) \ominus (y \oplus t) \in I$. (TR)

8. I is closed under following version of the hypothetical syllogism.
   If $x \ominus y \in I$ and $y \ominus z \in I$, then $x \ominus z \in I$. (HS)
9. If $x \ominus y \in I$ and $u \ominus v \in A$, then $(x \oplus u) \ominus (y \oplus v) \in I$. (R2)

PROOF. Since I is non–empty and closed under differences, $\mathbf{0} \in I$. Also, if $x \in I$, then $-x = \mathbf{0} - x \in I$. As a consequence I is closed under addition because $x \oplus y = x \ominus (-y)$.

Since $x^- = -(-x)^+$, if $x \in I$, then $x^- \in I$.

If $x \in I$ and $y \in I$, since I is closed under addition, inverses, positive and negative parts, by definition, $x \vee y \in I$.

In order to prove (MP), assume $x \ominus y \in I$ and $y \in I$. Then

$$x^+ \ominus y = (x \vee \mathbf{0}) \ominus y = (x \ominus y) \vee -y \in I\,,$$

by 2 and 4 above. But then, since $x^+ \ominus y \geq x^+ \ominus y^+$,

$$(x^+ \ominus y) \oplus y^+ \geq (x^+ \ominus y^+) \oplus y^+ = x^+ \geq \mathbf{0}\,,$$

by (Bx2). But also, by 4, $(x^+ \ominus y) \oplus y^+ \in I$, so by convexity, $x^+ \in I$. Similarly, $x^- \ominus y = (x \ominus y) \wedge -y \in I$ and $(x^- \ominus y) \oplus y^- \leq (x^- \ominus y^-) \oplus y^- = x^- \leq \mathbf{0}$, so by convexity $x^- \in I$. Finally, by 4, $x = x^+ \oplus x^- \in I$.

The rule (TR) follows from Lemma 2.3 and convexity.

For a proof of (HS), assume $x \ominus y \in I$ and $y \ominus z \in I$. Then by (TR), $(x \ominus z) \ominus (y \ominus z) \in I$, so by (MP), $x \ominus z \in I$.

The rule (R2) is proven as follows. Assume that both $x \ominus y \in I$ and $u \ominus v \in I$. Then applying (TR) we have $(x \oplus u) \ominus (y \oplus u) \in I$ and $(u \oplus y) \ominus (v \oplus y) \in I$. Then by (HS), $(x \oplus u) \ominus (y \oplus v) \in I$.

∎

THEOREM 3.4
Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a homomorphism and let I be an ideal of $\mathbf{B}$. Then $f^{-1}(I)$ is an ideal of $\mathbf{A}$.

PROOF. $f(\mathbf{0}) = \mathbf{0}$, so $f^{-1}(I) \neq \emptyset$.

Assume x and $y \in f^{-1}(\mathcal{I})$. Then $f(x)$ and $f(y) \in \mathcal{I}$, so $f(x \ominus y) = f(x) - f(y) \in \mathcal{I}$, since $\mathcal{I}$ is an ideal, so $x \ominus y \in f^{-1}(\mathcal{I})$.

Similarly, if $x \in f^{-1}(I)$. Then $f(x) \in I$, so $f(x^+) = (f(x))^+ \in I$ so $x^+ \in f^{-1}(I)$.

Finally, if x and $y \in f^{-1}(I)$ and $x \leq z \leq y$, $f(x) \leq f(z) \leq f(y)$, so by convexity, $f(z) \in I$ and thus $z \in f^{-1}(I)$. ∎

Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a homomorphism of MV^*–algebras. Define the *kernel* of f as $\ker(f) = \{x \in A : f(x) = \mathbf{0}\}$.

From Lemma 3.1 and Theorem 3.4 it is immediate that $\ker(f) = f^{-1}(\{\mathbf{0}\})$ is an ideal.

We observe that in Universal Algebra the kernel of a homomorphism is the congruence relation defined by $\{(x, y) : f(x) = f(y)\}$. The next lemma shows that the two concepts are naturally related in a familiar way.

LEMMA 3.5
Let $f : \mathbf{A} \longrightarrow \mathbf{B}$ be a homomorphism of MV^*–algebras. Then $f(x) = f(y)$ if and only if $x \ominus y \in \ker(f)$.

PROOF. $f(x) = f(y)$ if and only if $f(x) \ominus f(y) = \mathbf{0}$ (observe the use of Lemma 2.1 for the *only if* part) if and only if $f(x \ominus y) = \mathbf{0}$ if and only if $x \ominus y \in \ker(f)$. ∎

THEOREM 3.6
Let $\mathbf{A}$ be an MV^*–algebra and let I be an ideal. Define

$$x \sim_I y \quad \text{if and only if} \quad y \ominus x \in I.$$

Then $\sim_I$ is a congruence relation over $\mathbf{A}$.

PROOF. Since $x \ominus x = \mathbf{0} \in I$ for all $x \in A$, $\sim_I$ is reflexive and since ideals are closed under inverses, $\sim_I$ is symmetric. The rule (HS) asserts the transitivity of $\sim_I$.

In order to prove that $\sim_I$ is a congruence, observe that if $x \ominus y \in I$, by symmetry, $-x \ominus (-y) = y \ominus x \in I$. The rule (R2) asserts that $\sim_I$ is compatible with addition. ∎

Let $\mathcal{I}(\mathbf{A})$ be the set of all ideals of $\mathbf{A}$. Then $\langle \mathcal{I}(\mathbf{A}), \subseteq \rangle$, that is, the ideals of $\mathbf{A}$ ordered by inclusion, is a complete lattice. The operations are

$$\begin{aligned} \bigwedge_{j \in J} I_j &= \bigcap_{j \in J} I_j \\ \bigvee_{j \in J} I_j &= \bigcap \{J \in \mathcal{I}(\mathbf{A}) : \bigcup_{j \in J} I_j \subseteq J\} \end{aligned}$$

This will be a corollary to the next theorem.

THEOREM 3.7
Let $Con(\mathbf{A})$ be the set of all congruences of $\mathbf{A}$. Define

$$\begin{aligned} \Phi : \mathcal{I}(\mathbf{A}) &\longrightarrow Con(\mathbf{A}) \\ I &\longmapsto \sim_I . \end{aligned}$$

Then Φ is a lattice isomorphism.

PROOF. The mapping is clearly well defined and one–to–one.
In order to prove that it is also onto, let $\sim$ be a congruence. Then it is immediate that $[\mathbf{0}]_\sim$, the equivalence class of $\mathbf{0}$, is the ideal I. Now let π be the projection from $\mathbf{A}$ onto the quotient $\mathbf{A}/\sim$. Then $I = \pi^{-1}(\{[\mathbf{0}]_\sim\})$, is the inverse image of an ideal, so itself is an ideal. Now we simply have to observe that $\sim_I = \sim$.

It is also immediate that if $I \subseteq J$, then $\Phi(I) \subseteq \Phi(J)$.
Let $\sim_1 \subseteq \sim_2$ be two congruences. Then $[\mathbf{0}]_{\sim_1} \subseteq [\mathbf{0}]_{\sim_2}$, so $\Phi^{-1}([\mathbf{0}]_{\sim_1}) \subseteq \Phi^{-1}([\mathbf{0}]_{\sim_2})$. Since both Φ and Φ^{-1} are monotone, Φ is an order isomorphism and since $\mathbf{Con}(\mathbf{A})$ is a lattice, so is $\mathcal{I}(\mathbf{A})$, and Φ is a lattice isomorphism. ∎

4 Prime Ideals

An ideal I is *prime* if and only if it is proper and for any $x \in A$, either $x^+ \in I$ or $x^- \in I$.

THEOREM 4.1
An ideal I is prime if and only if the quotient algebra $\mathbf{A}/I$ is totally ordered.

PROOF. The quotient algebra $\mathbf{A}/I$ is totally ordered if and only if for any two classes $[a]_I$ and $[b]_I$, either $[a]_I \leq [b]_I$ or $[b]_I \leq [a]_I$ if and only if $[a \wedge b]_I = [a]_I$ or $[a \wedge b]_I = [b]_I$ if and only if $(a \wedge b) \ominus a \in I$ or $(a \wedge b) \ominus b \in I$ if and only if $(a \ominus a) \wedge (b \ominus a) \in I$ or $(a \ominus b) \wedge (b \ominus b) \in I$ if and only if $(b \ominus a)^- \in I$ or $(a \ominus b)^- \in I$ if and only if

$$(a \ominus b)^+ \in I \quad \text{or} \quad (b \ominus a)^+ \in I\,. \qquad (*)$$

Now assume that $\mathbf{A}/I$ is totally ordered and let $x \in A$. Then (*) applied to x and $\mathbf{0}$, implies that, either

$$x^+ = (x \ominus \mathbf{0})^+ \in I \quad \text{or} \quad (-x)^+ = (\mathbf{0} \ominus x)^+ \in I\,,$$

and since $x^- = -(-x)^+$, I is prime.

On the other hand, if I is prime, for any $x, y \in A$

$$(x \ominus y)^+ \in I \quad \text{or} \quad (x \ominus y)^- \in I\,,$$

so (*) holds and thus $\mathbf{A}/I$ is totally ordered. ∎

THEOREM 4.2

In every MV^*–algebra the following properties hold.

1. Every proper ideal that contains a prime ideal is prime.
2. The set of all ideals that contain a given prime ideal P is ordered by inclusion.
3. Every prime ideal is contained in a maximal ideal.

Define $n.x$ recursively by $0.x = \mathbf{0}$ and $(n+1).x = n.x \oplus x$. Observe that by axiom (Bx6), $n.(-x) = -(n.x)$ and that if x is either positive or negative, this iterated addition is associative, so the parentheses are not necessary.

LEMMA 4.3

For any two positive integers m and n, and any x, $n.x^+ \wedge m.(-x)^+ = 0$.

PROOF. We first prove the simplest case, $x^+ \wedge (-x)^+ = 0$.

$$\begin{aligned} 0 \leq x^+ \wedge (-x)^+ &\leq ((x^+ \wedge (-x)^+) \ominus (-x)^+) \oplus (-x)^+\,,\\ &= ((x^+ \ominus (-x)^+) \wedge 0) \oplus (-x)^+\,,\\ &= x^- \oplus (-x)^+ = 0\,. \end{aligned}$$

The inequality follows from Lemma 2.2, *4*.

Next we prove by induction on n that $n.x^+ \wedge (-x)^+ = 0$, of which the previous is the initial step. So assume that for some n, $n.x^+ \wedge (-x)^+ = 0$. Then

$$\begin{aligned} x^+ &= (n.x^+ \wedge (-x)^+) \oplus x^+\,,\\ &= (n+1).x^+ \wedge ((-x)^+ \oplus x^+)\,,\\ x^+ \wedge (-x)^+ &= [(n+1).x^+ \wedge ((-x)^+ \oplus x^+)] \wedge (-x)^+\,,\\ &= (n+1).x^+ \wedge (-x)^+\,. \end{aligned}$$

Using similar arguments, an induction on m will prove the theorem. ∎

THEOREM 4.4
Let $I \in \mathcal{I}(\mathbf{A})$ and let $a \notin I$. Then there exists a prime ideal J such that $I \subseteq J$ and $a \notin J$.

PROOF. An application of Zorn's lemma shows that there exists an ideal J, such that $I \subseteq J$ and is maximal with respect to the property $a \notin J$.

In order to prove that J is prime, let $x \in A$ and assume neither $x^+ \in J$ nor $x^- \in J$. Define

$$\begin{aligned} J_1 &= \{z \in A : |z| \leq p^+ \oplus n.x^+ \text{, for some } p \in J \text{ and } n \in \mathbb{N}\}, \\ J_2 &= \{z \in A : |z| \leq p^+ \oplus n.(-x)^+ \text{, for some } p \in J, n \in \mathbb{N}\}. \end{aligned}$$

We prove that J_1 is an ideal. J_1 is non–empty since it contains J and also $x \in J_1$. If z and $y \in J_1$ then there exists p, $q \in J$ and non–negative integers n and m such that

$$\begin{aligned} -p^+ \ominus n.x^+ &\leq z \leq p^+ \oplus n.x^+ \qquad (**)\\ -q^+ \ominus m.x^+ &\leq y \leq q^+ \oplus m.x^+ \end{aligned}$$

then also

$$-q^+ \ominus m.x^+ \leq -y \leq q^+ \oplus m.x^+$$

and thus

$$\begin{aligned} (-p^+ \ominus n.x^+) \oplus (-q^+ \ominus m.x^+) &\leq z - y \leq (p^+ \oplus n.x^+) \oplus (q^+ \oplus m.x^+), \\ (-p^+ \oplus -q^+) \ominus (n+m).x^+ &\leq z - y \leq (p^+ \oplus q^+) \oplus (n+m).x^+), \end{aligned}$$

where the last equation is obtained from the previous one since in one side all terms are negative and in the other all terms are positive, so restricted associativity holds for them. This proves that J_1 is closed under addition.

Next, if z, $y \in J_1$ are as above and $z \leq u \leq y$, it is immediate that

$$-(p^+ \vee q^+) \ominus (n+m).x^+ \leq u \leq (p^+ \vee q^+) \oplus (m+n).x^+$$

so J_1 is convex and thus it is an ideal.

Finally, taking positive part in (**)

$$\begin{aligned} (-p^+ \ominus n.x^+)^+ &\leq z^+ \leq (p^+ \oplus n.x^+)^+ \\ \mathbf{0} &\leq z^+ \leq p^+ \oplus n.x^+ \end{aligned}$$

since the first term is negative and the second term is positive. By convexity, J_1 is closed under taking positive part.

A similar proof yields that J_2 is also an ideal that contains J and such that $-x \in J_2$.

Since J is maximal with respect to the property $a \notin J$, $a \in J_1 \cap J_2$, that is, there exist p, $q \in J$ and non–negative integers n and m such that

$$\begin{aligned} -p^+ \ominus n.x^+ &\leq a \leq p^+ \oplus n.x^+ \\ -q^+ \ominus m.(-x)^+ &\leq a \leq q^+ \oplus m.(-x)^+. \end{aligned}$$

Letting $r = p^+ \vee q^+ \in J$,

$$\begin{aligned} -r^+ \ominus n.x^+ &\leq a \leq r^+ \oplus n.x^+ \\ -r^+ \ominus m.(-x)^+ &\leq a \leq r^+ \oplus m.(-x)^+, \end{aligned}$$

so

$$\begin{array}{rcl} (-r^+ \ominus n.x^+) \vee (-r^+ \ominus m.(-x)^+) & \leq a \leq & (r^+ \oplus n.x^+) \wedge (r^+ \oplus m.(-x)^+)\,, \\ -r^+ \ominus (\, n.x^+ \wedge m.(-x)^+) & \leq a \leq & r^+ \oplus (\, n.x^+ \wedge m.(-x)^+\,)\,, \\ -r^+ & \leq a \leq & r^+\,, \end{array}$$

by Lemma 4.3. But then, by convexity, $a \in J$ contradicting our hypothesis. This implies that the assumption that neither $x^+ \in J$ nor $x^- \in J$ cannot hold and thus J is prime. ∎

COROLLARY 4.5
Every proper ideal of an MV^*–algebra is an intersection of prime ideals.

COROLLARY 4.6
For every $a \in A$, $a \neq \mathbf{0}$, there exists a prime ideal P_a such that $a \notin P_a$.

PROOF. Apply Theorem 4.4 to a and the ideal $\{\mathbf{0}\}$. ∎

COROLLARY 4.7
Every maximal ideal is prime.

PROOF. Let M be a maximal ideal of an MV^*–algebra $\mathbf{A}$ and assume it is not prime. Then there exists an $a \in A$ such that $a^+ \notin M$ and $a^- \notin M$. Then by Lemma 4.4 there is a prime ideal P such that $M \subseteq P$ and $a^+ \notin P$, but then $a^- \in P$, contradicting the maximality of M. ∎

LEMMA 4.8
If $I_j \in \mathcal{I}(\mathbf{A})$ for $j \in J$ and $\bigcap_{j\in J} I_j = \{\mathbf{0}\}$, then the natural morphism

$$\nu : A \longrightarrow \prod_{j\in J} \mathbf{A}/I_j$$

defined by $\nu(a)(j) = [a]_{i_j}$ is a subdirect embedding.

PROOF. The fact that ν is a homomorphism is straightforward. Since $\bigcap_{j\in J} I_j = \{\mathbf{0}\}$, ν is one–to–one. Finally, since the projections $\pi_j \circ \nu$ are onto $\mathbf{A}/I_j$ for each $j \in J$, ν is subdirect. ∎

5 Subdirect Representation Theorem

The following special case of Birkhoff's Theorem was proved by C. C. Chang in [2]. The following proof does not depend on results about MV–algebras.

THEOREM 5.1
(C. C. Chang) Every non–trivial MV^*–algebra isomorphic to a subdirect product of linearly ordered MV^*–algebras.

PROOF. Consider the family of prime ideals $\{P_a : a \in A - \{\mathbf{0}\}\ \}$ such that $a \notin P_a$ built in Theorem 4.4. Then $\bigcap\{P_a : a \in A - \{\mathbf{0}\}\ \} = \{\mathbf{0}\}$, so by Lemma 4.8 the injection ν is a subdirect embedding. Since each P_a is prime, by Theorem 4.1, for each a, $\mathbf{A}/P_a$ is linear. ∎

COROLLARY 5.2
If an MV^*–algebra is subdirectly irreducible then it is a linear MV^*–algebra.

6 MV^*–chains and the Product

Let **A** be an MV^*–algebra. For a, $b \in A$ define their *product* as follows.

$$a \odot b = (\mathbf{1} \oplus [(a \ominus \mathbf{1}) \oplus (b \ominus \mathbf{1})]) \;\oplus\; ([(a \oplus \mathbf{1}) \oplus (b \oplus \mathbf{1})] \ominus \mathbf{1})\,.$$

In order to understand the meaning of this operation, let us evaluate it in the interval algebra $[-1, 1]$. For x, $y \in [-1, 1]$,

$$x \odot y = \begin{cases} x + y - 1 & \text{if } x + y > 1\,, \\ 0 & \text{if } 1 \geq x + y \geq -1\,, \\ x + y + 1 & \text{if } -1 > x + y\,, \end{cases}$$

where addition is in the reals. Thus, intuitively, the product is how much is cut from the regular addition when one trims it between -1 and 1, as one does in $\Gamma(\mathbb{R}, 1)$.

For any MV^*–chain.

$$a \odot b = \begin{cases} ((a \ominus \mathbf{1}) \oplus (b \ominus \mathbf{1})]\,) \oplus \mathbf{1} & \text{if both } a \text{ and } b \text{ are positive,} \\ (\,(a \oplus \mathbf{1}) \oplus (b \oplus \mathbf{1})\,) \ominus \mathbf{1} & \text{if both } a \text{ and } b \text{ are negative,} \\ 0 & \text{o.w.} \end{cases}$$

The following lemma states some immediate consequences of the definition of the product.

Lemma 6.1
Let **A** be an MV^*–chain.

1. $\mathbf{0} \odot a = \mathbf{0}$
2. $\mathbf{1} \odot a = a$
3. $a \odot b = b \odot a$
4. $a \odot (b \odot c) = (a \odot b) \odot c$
5. $-(a \odot b) = (-a) \odot (-b)$
6. If a, $b \geq \mathbf{0}$, then

$$\begin{aligned} a \oplus b &= [(a \ominus \mathbf{1}) \odot (b \ominus \mathbf{1})] \oplus \mathbf{1} \\ a \odot b &= [(a \ominus \mathbf{1}) \oplus (b \ominus \mathbf{1})] \oplus \mathbf{1} \end{aligned}$$

Lemma 6.2
Let **A** be an MV^*–chain.

1. If a, b, $c \geq \mathbf{0}$ and $a \oplus b = b \oplus c = \mathbf{1}$, then $[(a \ominus \mathbf{1}) \oplus (b \ominus \mathbf{1})] \oplus c = a \oplus [(b \ominus \mathbf{1}) \oplus (c \ominus \mathbf{1})]$.
2. If b, $c \geq \mathbf{0}$ and $b \oplus c < \mathbf{1}$, then $(b \oplus c) \ominus b = c$.
3. If b, $c \geq \mathbf{0}$ and $b \oplus c < \mathbf{1}$, then $(b \oplus c) \ominus \mathbf{1} = (b \ominus \mathbf{1}) \oplus c$.
4. If a, b, $c \geq \mathbf{0}$, $a \oplus b = \mathbf{1}$ and $b \oplus c < \mathbf{1}$, then $[(a \ominus \mathbf{1}) \oplus (b \ominus \mathbf{1})] \oplus c = (a \ominus \mathbf{1}) \oplus ((b \oplus c) \ominus \mathbf{1})]$.
5. If a, b, $c \geq \mathbf{0}$, $a \oplus b < \mathbf{1}$ and $b \oplus c < \mathbf{1}$, then $(a \oplus b) \odot c = a \odot (b \oplus c)$.
6. If $a < \mathbf{0}$, b, $c \geq \mathbf{0}$, $b \oplus c = \mathbf{1}$ then $a \oplus ((b \odot c) \ominus \mathbf{1}) = -\mathbf{1}$ if and only if either $(a \oplus b) \oplus c < \mathbf{1}$, or $a \oplus b = \mathbf{1} \ominus c$.
   Also, $[a \odot ((b \odot c) \ominus \mathbf{1})] \oplus \mathbf{1} = (a \oplus b) \oplus c$.
7. If b and c are both positive or both negative, and $|b \oplus c| < \mathbf{1}$, then $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

Proof. *1.* Let a, b, $c \geq \mathbf{0}$ and $a \oplus b = b \oplus c = \mathbf{1}$.
Since $a \oplus b = \mathbf{1}$, $\mathbf{0} = (a \oplus b) \ominus \mathbf{1} \leq a \oplus (b \ominus \mathbf{1})$, by Lemma 2.2,*5*. Then

$$\begin{aligned}
[(a \ominus \mathbf{1}) \oplus (b \ominus \mathbf{1})] \oplus c &= [(a \oplus \underline{-\mathbf{1}}) \oplus (\underline{b \ominus \mathbf{1}})] \oplus c \\
&= [(\underline{a \oplus (b \ominus \mathbf{1})}) \ominus \mathbf{1}] \oplus \underline{c} \\
&= (a \oplus (b \ominus \mathbf{1})) \oplus (c \ominus \mathbf{1}) \\
&\leq a \oplus [(b \ominus \mathbf{1}) \oplus (c \ominus \mathbf{1})],
\end{aligned}$$

by Lemma 2.2,*5*. A similar argument proves the inequality in the other direction.

2. Let b, $c \geq \mathbf{0}$, $b \oplus c < \mathbf{1}$. If $c \geq \mathbf{1} \ominus b$, then $b \oplus c \geq b \oplus (\mathbf{1} \ominus b) = \mathbf{1}$, a contradiction, so $c < \mathbf{1} \ominus b$. So by Lemma 2.2,*2*, $(b \oplus c) \ominus b = c \wedge (\mathbf{1} \ominus b) = c$.

3. Let b, $c \geq \mathbf{0}$, $b \oplus c < \mathbf{1}$. We observe that $b \oplus c = \mathbf{1} \oplus d^-$, so by *2.*, $c = (b \oplus c) \ominus b = (\mathbf{1} \oplus d^-) \ominus b$, and thus

$$(b \ominus \mathbf{1}) \oplus c = (b \ominus \mathbf{1}) \oplus ((\mathbf{1} \oplus \underline{d^-}) \oplus \underline{-b}) = \underline{(b \ominus \mathbf{1})} \oplus ((\mathbf{1} \ominus b) \oplus \underline{d^-}) = d^- .$$

On the other hand, $(b \oplus c) \ominus \mathbf{1} = (\mathbf{1} \oplus \underline{d^-}) \oplus (\underline{-\mathbf{1}}) = d^-$, so the identity holds.

4. Let a, b, $c \geq \mathbf{0}$, $a \oplus b = \mathbf{1}$ and $b \oplus c < \mathbf{1}$.
Observe that by Lemma 2.2,*1*, $c = (c \oplus (b \ominus \mathbf{1})) \oplus (\mathbf{1} \ominus b)$. Also, since $a \oplus b = \mathbf{1}$, $a \ominus \mathbf{0} \geq -b$.
Now by Lemma 2.2,*1* and *2.* above,

$$\begin{aligned}
[(a \ominus \mathbf{1}) \oplus (b \ominus \mathbf{1})] \oplus c &= [\underline{(a \ominus \mathbf{1}) \oplus (b \ominus \mathbf{1})}] \oplus [\underline{(c \oplus (b \ominus \mathbf{1}))} \oplus (\mathbf{1} \ominus b)] \\
&= [((a \ominus \mathbf{1}) \oplus (b \ominus \mathbf{1})) \oplus (\mathbf{1} \ominus b)] \oplus ((b \oplus c) \ominus \mathbf{1}) \\
&= [(a \ominus \mathbf{1}) \vee ((\mathbf{1} \ominus b) \ominus \mathbf{1})] \oplus ((b \oplus c) \ominus \mathbf{1}) \\
&= (a \ominus \mathbf{1}) \oplus ((b \oplus c) \ominus \mathbf{1}).
\end{aligned}$$

5. Let a, b, $c \geq \mathbf{0}$, $a \oplus b < \mathbf{1}$ and $b \oplus c < \mathbf{1}$. Then by *3.*, $(a \oplus b) \odot c = a \odot (b \oplus c)$.

$$\begin{aligned}
(a \oplus b) \odot c &= [((a \oplus b) \ominus \mathbf{1}) \oplus (c \ominus \mathbf{1})] \oplus \mathbf{1} \\
&= [((\underline{a \ominus \mathbf{1}}) \oplus b) \oplus (\underline{c \ominus \mathbf{1}})] \oplus \mathbf{1} \\
&= [(a \ominus \mathbf{1}) \oplus (b \oplus (c \ominus \mathbf{1}))] \oplus \mathbf{1} \\
&= [(a \ominus \mathbf{1}) \oplus ((b \oplus c) \ominus \mathbf{1})] \oplus \mathbf{1} \\
&= a \odot ((b \oplus c) \ominus \mathbf{1})
\end{aligned}$$

6. Let $a < \mathbf{0}$, b, $c \geq \mathbf{0}$, $b \oplus c = \mathbf{1}$.

$$\begin{aligned}
-\mathbf{1} = a \oplus ((b \odot c) \ominus \mathbf{1}) &= a \oplus ((b \ominus \mathbf{1}) \oplus (c \ominus \mathbf{1})) \\
&= (\underline{a} \oplus (b \oplus \underline{-\mathbf{1}})) \oplus (c \ominus \mathbf{1}) \\
&= ((a \oplus b) \ominus \mathbf{1}) \oplus (c \ominus \mathbf{1}) \\
&= ((a \oplus b) \odot c) \ominus \mathbf{1}
\end{aligned}$$

So $(a \oplus b) \odot c = \mathbf{0}$, and this is equivalent to $(a \oplus b) \oplus c < \mathbf{1}$ or $(a \oplus b) = \mathbf{1} \ominus c$.

Observe that in this case,

$$\begin{array}{rcl} a \odot ((b \odot c) \ominus \mathbf{1}) & = & a \odot ((\underline{b \ominus \mathbf{1}}) \oplus (c \ominus \underline{\mathbf{1}})) \\ & = & a \odot [((b \ominus \mathbf{1}) \oplus c) \ominus \mathbf{1}] \\ & = & [(\underline{a \oplus \mathbf{1}}) \oplus ((b \ominus \mathbf{1}) \oplus \underline{c})] \ominus \mathbf{1} \\ & = & [((\underline{a} \oplus \mathbf{1}) \oplus (\underline{b \ominus \mathbf{1}})) \oplus c] \ominus \mathbf{1} \\ & = & [(a \oplus (\mathbf{1} \oplus (b \ominus \mathbf{1}))) \oplus c] \ominus \mathbf{1} \\ & = & ((a \oplus b) \oplus c) \ominus \mathbf{1} \end{array}$$

so $[a \odot ((b \odot c) \ominus \mathbf{1})] \oplus \mathbf{1} = (a \oplus b) \oplus c$.

The rest of the assertions are proven likewise.

7. Let b, $c \geq \mathbf{0}$, $b \oplus c < \mathbf{1}$.

$$\begin{array}{rcl} a \oplus b & = & a \oplus ((b \oplus c) \ominus c) \\ & = & \underline{a} \oplus ((b \oplus c) \oplus \underline{-c}) \\ & = & (a \oplus (b \oplus c)) \ominus c \end{array}$$

so $(a \oplus b) \oplus c = [(a \oplus (b \oplus c)) \ominus c] \oplus c$.

If $a \oplus (b \oplus c) \geq \mathbf{0}$, then by restricted associativity, $[(a \oplus (b \oplus c)) \ominus c] \oplus c = a \oplus (b \oplus c)$.

If If $(a \oplus (b \oplus c) < \mathbf{0}$, then $[(a \oplus (b \oplus c)) \ominus c] \oplus c = (a \oplus (b \oplus c)) \vee (c \ominus \mathbf{1})$. But if $a \oplus (b \oplus c) < c \ominus \mathbf{1}$, then $(a \oplus (b \oplus c)\,) \oplus \mathbf{1} < (c \ominus \mathbf{1}) \oplus \mathbf{1} = c$, so $(a \oplus \mathbf{1}) \oplus (b \oplus c) < c$, a contradiction. ∎

7 Completeness Theorem

Let $\mathbf{A}$ be an MV^*–chain. We will define an ℓ–group $G(\mathbf{A})$ such that $\mathbf{A}$ is isomorphic to $G(\mathbf{A})(\mathbf{u})$, for some positive $\mathbf{u} \in G(\mathbf{A})$.

Consider the set $G(A)$ of all pairs $(n, a) \in \mathbb{Z} \times A$, where $(n+1, -\mathbf{1})$ and $(n, \mathbf{1})$ have been identified.

$$(n, x) + (m, y) = \begin{cases} (n+m+1\,,\, (a \odot b) \ominus \mathbf{1}) & \text{if } \ a \oplus b = \mathbf{1}\,, \\ (n+m\,,\, a \oplus b) & \text{if } \ \mathbf{1} > a \oplus b > -\mathbf{1}\,, \\ (n+m-1\,,\, (a \odot b) \oplus \mathbf{1}) & \text{if } \ a \oplus b = -\mathbf{1}\,, \end{cases}$$

We observe that for any (m, a),

$$(m, a) + (n, \mathbf{1}) = \begin{cases} (n+m+1\,,\, (a \odot \mathbf{1}) \ominus \mathbf{1}) = (n+m+1\,,\, a \ominus \mathbf{1}) & \text{if } \ a \geq \mathbf{0}\,, \\ (n+m\,,\, a \oplus \mathbf{1}) & \text{if } \ a < \mathbf{0}\,. \end{cases}$$

On the other hand,

$$(m, a) + (n+1,\, -\mathbf{1}) = \begin{cases} (n+m+1\,,\, a \oplus (-\mathbf{1})) = (n+m+1\,,\, a \ominus \mathbf{1}) & \text{if } a \geq \mathbf{0}\,, \\ (n+m\,,\, (a \odot (-\mathbf{1})) \oplus \mathbf{1}) = (n+m\,,\, a \oplus \mathbf{1}) & \text{if } a < \mathbf{0}\,, \end{cases}$$

so this addition is well defined.

We observe that this addition is commutative, $(0, \mathbf{0})$ is a zero and $(-n, -a)$ is the additive inverse of (n, a). In order to prove that $G(\mathbf{A}) = \langle A, +, -, (n, \mathbf{0}) \rangle$ is a commutative group we must prove that $+$ is associative.

Lemma 7.1
For any $n,\, m,\, k \in \mathbb{Z}$, and any $a,\, b,\, c \in A$,

$$(n,a)+((m,b)+(k,c)) = ((n,a)+(m,b))+(k,c)\,.$$

Proof. We must check several cases.
Case I $a,\, b,\, c \geq \mathbf{0}$ and $a \oplus b = b \oplus c = \mathbf{1}$. Then

$$(n,a)+((m,b)+(k,c)) = (n,a)+(m+k+1,(b\odot c)\ominus \mathbf{1}) = (n+m+k+1, a\oplus((b\odot c)\ominus\mathbf{1}))\,,$$

whereas

$$((n,a)+(m,b))+(k,c) = (n+m+1,(a\odot b)\ominus\mathbf{1})+(k,c) = (n+m+k+1,((a\odot b)\ominus\mathbf{1})\oplus c)\,.$$

But by Lemma 6.2,*1.*,

$$(a\odot b)\ominus\mathbf{1})\oplus c = [(a\ominus\mathbf{1})\oplus(b\ominus\mathbf{1})]\oplus c = a\oplus[(b\ominus\mathbf{1})\oplus(c\ominus\mathbf{1})] = a\oplus((b\odot c)\ominus\mathbf{1})\,.$$

Case II $a,\, b,\, c \geq \mathbf{0}$, $a\oplus b = \mathbf{1}$ and $b\oplus c < \mathbf{1}$. Then

$$(n,a)+((m,b)+(k,c)) = (n,a)+(m+k,b\oplus c) = (n+m+k+1,(a\odot(b\oplus c))\ominus\mathbf{1})\,,$$

whereas

$$((n,a)+(m,b))+(k,c) = (n+m+1,(a\odot b)\ominus\mathbf{1})+(k,c) = (n+m+k+1,((a\odot b)\ominus\mathbf{1})\oplus c)\,.$$

But by Lemma 6.2,*4.*,

$$((a\odot b)\ominus\mathbf{1})\oplus c = [(a\ominus\mathbf{1})\oplus(b\ominus\mathbf{1})]\oplus c = (a\ominus\mathbf{1})\oplus((b\oplus c)\ominus\mathbf{1})] = (a\odot(b\oplus c))\ominus\mathbf{1}\,.$$

Case III $a,\, b,\, c \geq \mathbf{0}$, $a\oplus b < \mathbf{1}$ and $b\oplus c < \mathbf{1}$. There are two subcases.
Subcase a) $a\oplus(b\oplus c) < \mathbf{1}$. Then since all elements are positive,
$$(n,a)+((m,b)+(k,c)) = (n+m+k, a\oplus(b\oplus c)) = (n+m+k,(a\oplus b)\oplus c) = ((n,a)+(m,b))+(k,c)\,.$$
Subcase b) $a\oplus(b\oplus c) = \mathbf{1}$. Then

$$(n,a)+((m,b)+(k,c)) = (n+m+k+1,(a\odot(b\oplus c))\ominus\mathbf{1})\,,$$

whereas

$$((n,a)+(m,b))+(k,c) = (n+m+k+1,((a\oplus b)\odot c)\ominus\mathbf{1})\,.$$

But then by Lemma 6.2,*5.*, associativity holds.
Case IV $a,\, c \geq \mathbf{0}$, $b < \mathbf{0}$. Then $a\oplus b < \mathbf{1}$ and $b\oplus c < \mathbf{1}$. Also, $(a\oplus b)\oplus c = a\oplus(b\oplus c)$. There are three subcases.
Subcase a) $(a\oplus b)\oplus c = a\oplus(b\oplus c) = \mathbf{1}$. Then

$$(n,a)+((m,b)+(k,c)) = (n+m+k+1,(a\odot(b\oplus c))\ominus\mathbf{1})\,,$$

whereas

$$((n,a)+(m,b))+(k,c) = (n+m+k+1,((a\oplus b)\odot c)\ominus\mathbf{1})\,,$$

but now

$$\begin{aligned}((a\oplus b)\odot c)\ominus\mathbf{1} &= ((a\oplus b)\ominus\mathbf{1})\oplus(c\ominus\mathbf{1})\\ &= ((a\oplus \underline{b})\oplus\underline{-\mathbf{1}})\oplus(c\ominus\mathbf{1})\\ &= ((\underline{a\ominus\mathbf{1}})\oplus b)\oplus(\underline{c\ominus\mathbf{1}})\\ &= (a\ominus\mathbf{1})\oplus(\underline{b}\oplus(c\oplus\underline{-\mathbf{1}}))\\ &= (a\ominus\mathbf{1})\oplus((b\oplus c)\ominus\mathbf{1})\\ &= (a\odot(b\oplus c))\ominus\mathbf{1}\end{aligned}$$

Subcase b) $-\mathbf{1}<(a\oplus b)\oplus c=a\oplus(b\oplus c)<\mathbf{1}$. then

$$(n,a)+((m,b)+(k,c))=(n+m+k,a\oplus(b\oplus c))\,,$$

and

$$((n,a)+(m,b))+(k,c)=(n+m+k,(a\oplus b)\oplus c)\,,$$

which are equal by restricted associativity.

Subcase c) $(a\oplus b)\oplus c=a\oplus(b\oplus c)=-\mathbf{1}$. Then $a=c=\mathbf{0}$ and $b=-\mathbf{1}$, and the result is trivial.

Case V $a<\mathbf{0}$, b, $c\geq\mathbf{0}$ and $b\oplus c=\mathbf{1}$. There are two subcases.

Subcase a) $a\oplus((b\odot c)\ominus\mathbf{1})>-\mathbf{1}$. Then $a\odot((b\odot c)\ominus\mathbf{1})=\mathbf{0}$, so by Lemma 6.2,*6.*, $(a\oplus b)\oplus c=\mathbf{1}$, so

$$(n,a)+((m,b)+(k,c))=(n,a)+(m+k+1,(b\odot c)\ominus\mathbf{1})=(n+m+k+1,a\oplus((b\odot c)\ominus\mathbf{1}))\,,$$

and

$$((n,a)+(m,b))+(k,c)=(n+m,a\oplus b)+(k,c)=(n+m+k+1,((a\oplus b)\odot c)\ominus\mathbf{1})\,.$$

Now

$$\begin{aligned}a\oplus((b\odot c)\ominus\mathbf{1}) &= a\oplus((b\ominus\mathbf{1})\oplus(c\ominus\mathbf{1}))\\ &= (a\oplus(b\ominus\mathbf{1}))\oplus(c\ominus\mathbf{1})\\ &= (\underline{a}\oplus(b\oplus\underline{-\mathbf{1}}))\oplus(c\ominus\mathbf{1})\\ &= ((a\oplus b)\ominus\mathbf{1})\oplus(c\ominus\mathbf{1})\\ &= ((a\oplus b)\odot c)\ominus\mathbf{1}\end{aligned}$$

and associativity holds.

Subcase b) $a\oplus((b\odot c)\ominus\mathbf{1})=-\mathbf{1}$. Then by Lemma 6.2,*6.*, $(a\oplus b)\oplus c<\mathbf{1}$, or $a\oplus b=\mathbf{1}\ominus c$. Either way, by Lemma 6.2,*6.*,
$(n,a)+((m,b)+(k,c))=(n,a)+(m+k+1,(b\odot c)\ominus\mathbf{1})=$
$(n+m+k,[a\odot((b\odot c)\ominus\mathbf{1})]\oplus\mathbf{1})=(n+m+k,(a\oplus b)\oplus c)\,.$

If $(a\oplus b)\oplus c<\mathbf{1}$,

$$((n,a)+(m,b))+(k,c)=(n+m,a\oplus b)+(k,c)=(n+m+k,(a\oplus b)\oplus c)$$

If $a\oplus b=\mathbf{1}\ominus c$, (and $(a\oplus b)\oplus c=\mathbf{1}$
$((n,a)+(m,b))+(k,c)=(n+m,a\oplus b)+(k,c)=$

$$(n+m+k+1, ((\mathbf{1} \ominus c) \odot c) \ominus \mathbf{1} = (n+m+k+1, -\mathbf{1}),$$

but in $(n+m+k, \mathbf{1}) = (n+m+k+1, -\mathbf{1})$, so associativity holds.

Case VI $a < \mathbf{0}$, b, $c \geq \mathbf{0}$ and $b \oplus c < \mathbf{1}$. Then

$$(n,a) + ((m,b) + (k,c)) = (n,a) + (m+k, b \oplus c) = (n+m+k, a \oplus (b \oplus c)),$$

whereas

$$((n,a) + (m,b)) + (k,c) = (n+m, (a \oplus b)) + (k,c) = (n+m+k, (a \oplus b) \oplus c).$$

By Lemma 6.2,*9.*, associativity holds.

The case a, $b \geq \mathbf{0}$ and $c < \mathbf{0}$, is the same as cases V and VI with the roles of a and c interchanged. All the remaining cases can be reduced to one of I to VI, by replacing a, b, $c \leq \mathbf{0}$ by $-a$, $-b$, and $-c$ respectively, so in any case associativity holds. ∎

THEOREM 7.2
Let $\mathbf{A}$ be an MV^*–chain. Then $G(\mathbf{A})$ ordered lexicographically is an ℓ–group. Moreover, $\mathbf{A}$ is isomorphic to $G(\mathbf{A})((0,\mathbf{1}))$.

PROOF. Lemma 7.1 and the remarks that preceded it prove that $G(\mathbf{A})$ is an Abelian group.
In order to check $G(\mathbf{A})$ is an ℓ–group, it is enough to check that if $(n,a) \leq (m,b)$, then $(n,a) + (k,c) \leq (m,b) + (k,c) \qquad (*)$.
Let $(n,a) \leq (m,b)$ and $n < m$, then there are several cases.
Case I If $b \oplus c = \mathbf{1}$, then $(m,b) + (k,c) = (m+k+1, (b \odot c) \ominus \mathbf{1})$, and since $(n,a) + (k,c) \leq (n+k+1, \mathbf{1})$, we have that $(*)$ holds.
Case II If $b \oplus c < \mathbf{1}$, then $(m,b) + (k,c) = (m+k, b \oplus c)$. If $n > m+1$ or $n = m+1$ and $a \oplus c < \mathbf{1}$ then $(*)$ holds. So let us assume that $n = m+1$ and $a \oplus c = \mathbf{1}$. Observe that in this case a, $c \geq \mathbf{0}$, so $(n,a) + (k,c) = (n+k+1, (a \odot c) \ominus \mathbf{1}) = (m+k, (a \odot c) \ominus \mathbf{1})$ and $(m,b) + (k,c) = (m+k, b \oplus c)$, but then

$$(a \odot c) \ominus \mathbf{1} = (a \ominus \mathbf{1}) \oplus (c \ominus \mathbf{1}) \leq c \ominus \mathbf{1} \leq c \oplus b,$$

so $(*)$ holds.
Case III If $b \oplus c = -\mathbf{1}$, an argument similar to the one in the previous paragraph shows that $(*)$ holds.

The assertion about $G(\mathbf{A})((0,\mathbf{1}))$ is immediate. ∎

COROLLARY 7.3
Let $\mathbf{A}$ be a simple MV^*–chain. Then $G(\mathbf{A})$ is a simple ℓ–group.

PROOF. Let $a \in A$ be such that $a > \mathbf{0}$. Then the ideal generated by a is $\langle a \rangle = \{x : |x| \leq n.a,\ n \in \mathbb{N}\}$ is not trivial, so $\langle a \rangle = A$ and thus $\mathbf{1} = n.a$, for some $n \in \mathbb{N}$.
We will now prove that $G(\mathbf{A})$ is Archimedean. Let (m,a), $(k,b) \in G(A)$, with $(m,a) > (0,\mathbf{0})$. We may assume that $k \geq 0$. There are two cases, $m = 0$ and $a > \mathbf{0}$ or $m > 0$.
In the first case, let n be the least positive integer such that $n.a = \mathbf{1}$. Then $((k+1)n).(0,a) = (k+1).(n.(0,a)) = (k+1).(1, a \odot ((n-1).a) \ominus \mathbf{1}) \geq (k+1, -\mathbf{1}) \geq (k,b)$.
In the second case, simply take any n such that $n\,m > k$ then $n.(m,a) \geq (n\,m, c) > (k,b)$, for some $c \in A$.
These two cases prove that $G(\mathbf{A})$ is Archimedean and by [1], Theorem 2.3., $G(\mathbf{A})$ is simple. ∎

Let $\tau(x_1,\ldots,x_n)$ be an MV^*–term. Define recursivelly the ℓ–group term $\widehat{\tau}(x_1,\ldots,x_n,y)$ as follows.

$$\begin{array}{rcll} \widehat{x_i} & = & x_i & \text{if } x_i \text{ is a variable,} \\ \widehat{-\tau} & = & -\widehat{\tau} & \\ \widehat{\tau\oplus\sigma} & = & -y\vee(y\wedge(\widehat{\tau}\oplus\widehat{\sigma})) & \end{array}$$

Lemma 7.4
Let $\mathbf{A}$ be an MV^*–chain, $\tau(x_1,\ldots,x_n)$ an MV^*–term and $a_1,\ldots,a_n\in A$. Then

$$(0,\tau^{\mathbf{A}}(a_1,\ldots,a_n)\,)=\widehat{\tau}^{\,G(\mathbf{A})}((0,a_1),\ldots,(0,a_n),(0,\mathbf{1})\,)\,.$$

Proof. The proof is an easy induction on the complexity of τ. ∎

Theorem 7.5
Completeness Theorem An equation holds in all MV^*–algebras if and only if it holds in the real number interval MV^*–algebra $[-1,1]$.

Proof. Assume an equation $\sigma(x_1,\ldots,x_n)=\tau(x_1,\ldots,x_n)$ does not hold in an MV^*–algebra. Then by Theorem 5.1, the equation does not hold in an MV^*–chain $\mathbf{A}$. So there are elements $a_1,\ldots,a_n\in A$ such that $\sigma^{\mathbf{A}}(a_1,\ldots,a_n)\neq\tau^{\mathbf{A}}(a_1,\ldots,a_n)$, and by the previous lemma,

$$\sigma^{G(\mathbf{A})}((0,a_1),\ldots,(0,a_n),(0,\mathbf{1})\,)\neq\tau^{G(\mathbf{A})}((0,a_1),\ldots,(0,a_n),(0,\mathbf{1})\,)\,. \qquad (*)$$

By a well known completeness result in the variety of ℓ–groups, this implies that the equation $(*)$ does not hold in the ℓ–group $\mathbb{R}$ of the real numbers.

Finally, it is immediate that $\mathbb{R}$ is isomorphic to $G([-1,1])$ and that the interval algebra $[-1,1]$ is isomorphic to $\mathbb{R}(1)$, so using the previous lemma again, but this time in the opposite direction, we get that the equation does not hold in $[-1,1]$.

The implication in the other direction is immediate. ∎

8 Free MV^*–algebras

For a given set X of variables we denote by $\mathbf{Term}(X)$ the absolutely free MV^*–term algebra, and for any MV^*–algebra $\mathbf{A}$, we denote by $\mathbf{Term}^{\mathbf{A}}(X)$ the corresponding MV^*–algebra of term functions over A^κ, where κ is the the cardinality of the set X.

Theorem 8.1
For a set X of variables, $\mathbf{Term}^{[-1,1]}(X)$ is the free MV^*–algebra over X.

Proof. Let $\mathbf{B}$ be an MV^*–algebra and $f: X\longrightarrow B$ be a function. For any $\rho(x_1,\ldots,x_n)\in Term(X)$ define

$$\widehat{f}(\rho)=\rho^{\mathbf{B}}(f(x_1),\ldots,f(x_n))\,.$$

By the completeness Theorem 7.5, for any two MV^*–terms, ρ and σ, if $\rho^{[-1,1]}=\sigma^{[-1,1]}$, then $\rho^{\mathbf{B}}=\sigma^{\mathbf{B}}$, so $\widehat{f}: Term(X)\longrightarrow B$ is well defined and extends f. An easy induction proves that $\widehat{f}$ is a homomorphism and that it is unique. ∎

9 Simple and Semisimple MV^*–algebras

An MV^*–algebra is *simple* if and only if it has exactly two ideals. For a given algebra $\mathbf{A}$ there are at least two ideals, $\{\mathbf{0}\}$ and A, so $\mathbf{A}$ is simple if and only if it is non–trivial and $\{\mathbf{0}\}$ is a maximal ideal.

THEOREM 9.1
Every simple MV^*–algebra is a chain.

PROOF. Let $\mathbf{A}$ be simple. Since $\{\mathbf{0}\}$ is a maximal ideal, it is prime, so for any $a \in A$ either $a^+ = \mathbf{0}$ or $a^- = \mathbf{0}$ and thus $\mathbf{A}$ is linearly ordered. ∎

THEOREM 9.2
An MV^*–algebra is simple if and only if it is isomorphic to a subalgebra of $[-1, 1]$.

PROOF. Let $\mathbf{A}$ be simple and thus linearly ordered. We first observe that by Corollary 7.3, $G(\mathbf{A})$ is simple. By [1], Theorem 2.3., an ℓ–group is simple if and only if it is isomorphic to a subgroup of the real numbers. This implies that the interval algebra $G(\mathbf{A})(\,(0, \mathbf{1})\,)$ is isomorphic to an interval algebra $\mathbb{R}(\mathbf{u})$, for some positive real number $\mathbf{u}$, but it is obvious that any interval algebra is isomorphic to the algebra $[-1, 1]$.

In the other direction, simply notice that all subalgebras of $[-1, 1]$ are simple since $\{0\}$ is their only maximal ideal. ∎

An algebra is *semisimple* if it is isomorphic to a subdirect product of simple algebras.

THEOREM 9.3
An MV^*–algebra is semisimple if and only if it is a subdirect product of subalgebras of $[-1, 1]$.

COROLLARY 9.4
Every free MV^*–algebra is semisimple.

PROOF. By Theorem 8.1. ∎

10 Infinitesimals and ℓ–Groups

In this section we characterize the ideal of infinitesimals of an MV^*–algebra as ℓ–groups.

Let $\mathbf{A}$ be an MV^*–algebra and let $a \in A$. We say that a is an *infinitesimal* if $a \neq \mathbf{0}$ and for all $n \in \mathbb{N}$, $n.|a| \leq \mathbf{1} - |a|$. Let $\mathcal{I}n(\mathbf{A})$ be the set of all infinitesimals of $\mathbf{A}$. An ideal that is a subset of $\mathcal{I}n(\mathbf{A})$ is an *ideal of infinitesimals*.

THEOREM 10.1
Let $\mathbf{A}$ be an MV^*–algebra. Then $\mathcal{I}n(\mathbf{A}) \cup \{\mathbf{0}\} = \bigcap\{M : M \text{ is a maximal ideal of } \mathbf{A}\}$.

PROOF. The proof is the same as the proof of a similar result proven in [3], Thm 3.6.4, for MV–algebras. We just have to replace a by $|a|$ and $\neg a$ by $\mathbf{1} \ominus a$. ∎

COROLLARY 10.2
The set $\mathcal{I}n(\mathbf{A})$ is an ideal of $\mathbf{A}$.

THEOREM 10.3
Let $\mathbf{A}$ be an MV^*–algebra and let I be an ideal of infinitesimals. Then $\langle I, \oplus, -, \mathbf{0}, \vee, \wedge\rangle$ is an ℓ–group.

Proof. The ideal is closed under the operations. Notice also that, except for associativity of the sum, all axioms of an ℓ–group are easily verified by I.

Let $\mathbf{A}$ be the interval algebra $[-1,1]$ of some ℓ–group $\mathbf{G}$ with a strong unit $\mathbf{1}$. Since a finite sum of infinitesimals is an infinitesimal, it cannot be $\mathbf{1}$. Recalling the definition of $\oplus$ in an interval algebra of a group, we have that $\oplus$ restricted to infinitesimals is associative.

Let $\mathbf{A}$ be an MV^*–algebra. Then by Theorem 5.1, $\mathbf{A}$ is a subdirect product of interval algebras $\mathbf{A}_i$. Since the operations are defined coordinatewise, by the previous case, in order to prove associativity for infinitesimals, it is enough to check that if $a \in \mathcal{I}n(\mathbf{A})$, then for each i, $a_i \in \mathcal{I}n(\mathbf{A}_i)$. This is straightforward. ∎

Theorem 10.4
Let $\mathbf{A}$ be an MV^*–algebra and let G be a subset of A. Then G is a convex ℓ–group if and only if G is an ideal of infinitesimals.

Proof. Assume G is an ℓ–group and let $g \in G$, $g \neq \mathbf{0}$. Suppose $g \notin \mathcal{I}n(\mathbf{A})$. We may also assume that g is positive , so $g = |g|$. Then for some $m \in \mathbb{N}$, $m.g > \mathbf{1} \ominus g.$, and this implies that $\mathbf{1} \leq m.g \oplus g \leq \mathbf{1}$, that is $(m+1).g = \mathbf{1} = (m+2).g$ and since G is a group, we can subtract $(m+1).g$ and get $g = \mathbf{0}$, a contradiction. So every element of G must be an infinitesimal.

On the other hand, if G is an ideal of infinitesimals, by Theorem 10.3 G is a convex ℓ–group. ∎

Corollary 10.5
Let $\mathbf{A}$ be an MV^*–algebra and let G be a subset of A. If G is an ℓ–group, then G is a subgroup of an ideal of infinitesimals.

Corollary 10.6
Any ℓ–group is (isomorphic to) an ℓ–group of infinitesimals of an MV^*–algebra.

Proof. Let $\mathbf{G}$ be an ℓ–group. Consider the lexicographic product $\mathbb{Z} \otimes \mathbf{G}$ and its associated interval MV^*–algebra $\mathbf{A} = \Gamma^*(\mathbb{Z} \otimes \mathbf{G}, (1,0)\,)$. Then $\mathcal{I}n(\mathbf{A}) = \{0\} \times \mathbf{G} \cong \mathbf{G}$. ∎

11 McNaughton's theorem for MV^*–algebras

In the theory of MV–algebras it is well known that for a McNaughton function, $f : [0,1]^n \to [0,1]$ there exists an n–ary MV–term such that its interpretation in the MV–algebra $[0,1]$ coincides with f. In this section we consider a generalization of the notion of McNaughton functions to functions over $[-1,1]$. We prove the one dimensional case of an analogue of the theorem mentioned above for these McNaughton* functions over $[-1,1]$ and MV^*–terms. The proof reduces the MV^* problem to the analogue MV problem by means of a *translation* between the two languages. This is the only section of this paper where we use results from the theory of MV–algebras to MV^*–algebras.

Definition 11.1
Let $n \geq 1$ be an integer. Then a function

$$f : [-1,1]^n \longrightarrow [-1,1]$$

is called a *McNaughton* function over* $[-1,1]^n$ if and only if it satisfies the following conditions:

(i) f is continuous with respect to the natural topology of $[-1,1]^n$,
(ii) there are linear polynomials $p_1,\ldots,p_k$ with integer coefficients,

$$p_i(x_1,\ldots,x_n)=b_i+m_{i1}x_1+\ldots+m_{in}x_n\,,$$

such that for each point $\mathbf{y}=(y_1,\ldots,y_n)\in[-1,1]^n$ there is a $j\in\{1,\ldots,k\}$ with $f(\mathbf{y})=p_j(\mathbf{y})$.

THEOREM 11.2
Let α be an MV^*–term such that $Var(\alpha)$, the set of variables of α is contained in $\{x_1,\ldots,x_n\}$. Then the interpretation $\alpha^{[-1,1]}:[-1,1]^n\longrightarrow[-1,1]$ of α in the MV^*–algebra $[-1,1]$ is a McNaughton function over $[-1,1]^n$.

PROOF. The interpretations of the variables (projections) and of the constant function taking the value 1 over $[-1,1]^n$ are McNaughton functions . It is easily proved that the set of McNaughton functions is closed under pointwise application of the operation $\oplus$ and $-$ defined on $[-1,1]$. The theorem follows by induction on the complexity of the formula α. ∎

LEMMA 11.3
Let α be an MV–term such that $Var(\alpha)\subseteq\{x_1,\ldots,x_n\}$. Then there exists an MV^*–term α^* such that $Var(\alpha)=Var(\alpha^*)$ and

$$(\alpha^*)^{[-1,1]}\restriction_{[0,1]^n}=\alpha^{[0,1]}\,.$$

PROOF. Let α be an MV–term. We consider a translation

$$(\cdot)^*:\,Term_{MV}(x_1,\ldots,x_n)\longrightarrow Term_{MV^*}(x_1,\ldots,x_n)$$

defined recursively as follows.
(i) $x_i^*=x_i\,,$
(ii) $(\neg\alpha)^*=\mathbf{1}\ominus\alpha^*\,,$
(iii) $(\alpha\oplus\beta)^*=\alpha^*\oplus\beta^*\,.$
An argument by induction shows that α^* satisfies the required condition. ∎

LEMMA 11.4
Let f_n, $g_n:[-1,1]\longrightarrow[-1,1]$ be the functions

$$f_n(x)=(n.x\vee 0)\wedge 1\qquad\text{and}\qquad g_n(x)=((1-n.x)\vee 0)\wedge 1\,.$$

Then there exist MV^*–terms α_n and β_n such that $\alpha_n^{[-1,1]}=f_n$ and $\beta_n^{[-1,1]}=g_n$.

PROOF. Define recursively

$$\begin{array}{rclcrcl}\alpha_1&=&x^+\,,&&\beta_1&=&(\mathbf{1}\ominus x)^+\,,\\ \alpha_{n+1}&=&(x^+\oplus\alpha_n)^+\,,&\text{and}&\beta_{n+1}&=&(\beta_n\ominus x^+)^+\,.\end{array}$$

Then the lemma follows by induction. ∎

THEOREM 11.5
Let $f:[-1,1]\longrightarrow[-1,1]$ be a McNaughton* function. Then there exists an MV^*–term α in n variables such that $f=\alpha^{[-1,1]}$.

PROOF. Let f be a McNaughton* function over $[-1,1]$. Then there exist positive McNaughton* functions over $[-1,1]$, f^+ and $-f^-$ such that $f = f^+ + f^-$, where $f^+(x) = (f(x))^+$, $f^-(x) = -(-f(x))^+$ are the positive and negative parts of f, respectively . If α_1, α_2 are MV^*–terms such that $f^+ = \alpha_1^{[-1,1]}$ and $-f^- = \alpha_2^{[-1,1]}$, then $f = (\alpha_1 \ominus \alpha_2)^{[-1,1]}$. This shows that we can restrict ourselves to positive McNaughton* functions over $[-1,1]$.

Let h be a (positive) McNaughton* function over $[-1,1]$. Then the restriction $h_1 = h \upharpoonright_{[0,1]}$ is a McNaughton function over the MV–algebra $[0,1]$, so by the one–dimensional case of McNaughton's Theorem for MV–algebras, see [3], Corollary 3.2.8, there exists an MV–term α such that $h_1 = \alpha^{[0,1]}$. Then by Lemma 11.3, there exists an MV^*–term α^* such that $h_1 = \alpha^{[0,1]} = (\alpha^*)^{[-1,1]} \upharpoonright_{[0,1]}$. (Note that we have no knowledge about $(\alpha^*)^{[-1,1]}$ over $[-1,0]$). Then $(\alpha^*)^{[-1,1]}(0) \in \{0,1\}$ since it is a McNaughton* function over $[-1,1]$. There are two cases.

Case (i). $(\alpha^*)^{[-1,1]}(0) = 0$. Since $(\alpha)^{[0,1]}$ is piecewise linear, there exist $t \in (0,1)$, $b \in \mathbb{N}$ such that $(\alpha)^{[0,1]}|_{[0,t]}(x) = bx$. Let $n \in \mathbb{N}$ be such that $1/n \leq t$, $b \leq n$, then by Lemma 11.4, there is an MV–term α_n such that $\alpha_n^{[-1,1]}(x) = (nx \vee 0) \wedge 1$. Then by construction we get:

$$((\alpha^* \wedge \alpha_n)^+)^{[-1,1]}(x) = \begin{cases} 0 & \text{if } x \in [-1,0], \\ \alpha^{[0,1]}(x) & \text{if } x \in [0,1]. \end{cases}$$

Let $h' : [0,1] \longrightarrow [0,1]$ be such that $h'(x) = h(-x)$. Since $h'(0) = 0$ and h' is a McNaughton function over $[0,1]$, by the previous argument, there is an Ł*–formula β such that

$$\beta^{[-1,1]}(x) = \begin{cases} 0 & \text{if } x \in [-1,0], \\ h'(x) & \text{if } x \in [0,1]. \end{cases}$$

Let $\gamma = (\alpha^* \wedge \alpha_n)^+$ and $\beta' = \overline{\sigma}(\beta)$, where $\overline{\sigma} : \mathbf{Term}_{MV^*}(X) \longrightarrow \mathbf{Term}_{MV^*}(X)$ is the unique morphism extending the substitution $x_1 \mapsto -x_1$, $x_j \mapsto x_j$, $j > 1$.

Claim:

$$h = (\gamma \oplus \beta')^{[-1,1]}.$$

This follows by induction.

Case (ii). $(\alpha^*)^{[-1,1]}(0) = 1$. Since $(\alpha)^{[0,1]}$ is piecewise linear there exist $t \in (0,1)$, $b \in \mathbb{N}$ such that $(\alpha)^{[0,1]}|_{[0,t]}(x) = 1 - bx$. Let $n \in \mathbb{N}$ such that $1/n \leq t$, $b \leq n$, then by Lemma 11.4 there is an MV^*–term β_n such that $\beta_n^{[-1,1]}(x) = (1 - nx \vee 0) \wedge 1$. Then by construction we get

$$(\alpha^* \vee \beta_n)^{[-1,1]}(x) = \begin{cases} 1 & \text{if } x \in [-1,0], \\ \alpha^{[0,1]}(x) & \text{if } x \in [0,1]. \end{cases}$$

Let $h' : [0,1] \longrightarrow [0,1]$ be such that $h'(x) = h(-x)$. Since $h'(0) = 1$ and h' is a McNaughton function over $[0,1]$ there is an MV^*–term β such that:

$$\beta^{[-1,1]}(x) = \begin{cases} 1 & \text{if } x \in [-1,0], \\ h'(x) & \text{if } x \in [0,1]. \end{cases}$$

Let $\gamma = \alpha^* \vee \beta_n$ and $\beta' = \overline{\sigma}(\beta)$, where $\overline{\sigma}$ is as in case (i).

Claim

$$h = (\gamma \wedge (\beta'))^{[-1,1]}.$$

The result follows by induction.

∎

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