Chang's Ł* Logic

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Abstract

In this paper we study the logic Ł*, introduced by C. C.Chang as a natural extension of Łukasiewicz' logic Ł. This logic has positive and negative truth values in the real number interval $[-1, 1]$.

We study deductive filters, we prove a deduction theorem and give detailed proofs of the soundness and completeness theorems. In the last section, we prove that the tautology problem for the logic Ł* is co–NP.

This paper is to be considered a continuation of the paper MV^*–Algebras, by the same authors and appearing in this same volume. In that paper we study a class of algebras introduced by Chang as what is now known as an equivalent algebraic semantics for the logic Ł. For most definitions and other algebraic concepts, the reader is referred to that paper.

Keywords: many–valued logics, positive and negative truth values, MV^*-algebra

1 Introduction

In [3], Chang introduces a new logic with positive and negative truth values as follows.
"*Consider the following natural extension* Ł* *of* Ł *having the closed interval* $[-1, 1]$ *as the set of truth values and containing one propositional constant 1, two propositional connectives* $\neg$, *and* $\rightarrow$, *and one quantifier* $\exists$, *which are to be interpreted as follows...*"

Even though Chang gives no hint on the interest in considering this logic nor on why this is a "natural" extension of Ł, the logic Ł* is a forerunner in two respects.

First, it is a paraconsistent logic. Since truth is given by positive values, if a sentence is assigned truth value 0, then its negation will also receive truth value 0, that is they are both true. Similarly, a sentence like $p \rightarrow (\neg p \rightarrow q)$ is not an Ł*–tautology and $p, \neg p \vdash_{Ł^*} q$ does not hold. In [9] we study the strictly paraconsistent fragment of Ł*, that is, the logic associated to the truth value 0.

The logic Ł* is also a forerunner in its main feature, that of being a logic with both positive and negative values. In much later developments, in the last decade or so, logics with positive and negative truth values have appeared in different contexts, both theoretical and applied. For instance in [2, 1], comparative logics are introduced to model situations in which propositions are either true or false, but not necessarily in the same way, thus one can admit that one proposition might be "truer" than another, so truth values are many shades of truth and falsehood. It should be noted that the algebraic semantics for comparative logics is the variety of pre–groups. Loosely speaking, a pre–group is a lattice–ordered Abelian group, or ℓ–group, in which the properties of 0 are split into two, i.e. distinguishing 0, who "collects opposites" $(x+(-x)=0)$, from -0, who is the neutral element, $(x+(-0)=x)$. If $-0=0$ then the pre–group is an ℓ–group.

Positive and negative truth values have also appeared in the context of uncertain information processing and inconsistency tolerance in expert systems. In [6], the authors make a theoretical analysis of uncertainty processing in a broad class of compositional expert systems similar to MYCIN and PROSPECTOR. In these, the knowledge of a questionnaire, that is, an assignment of a weight to each question, can be extended to all propositions in such a way that each rule in the Rule Base of the system contributes to the weight of each proposition according to some fixed function. Some natural conditions imposed on the set of weights are that they are a linearly ordered set G, with a largest element $\top$ (*true*,) a least element $\bot$ (*false*) and a distinguished element o (*no preference*,) that is also a neutral element for the combining function (*addition*) $\oplus$. The addition is closed on $G-\{\top,\bot\}$ (uncertainties cannot give certainty), and $G-\{\top,\bot\}$ has a structure of an *ordered Abelian group.* Therefore, G with this operation and order is what is called an *extended ordered Abelian group.*

In the case of PROSPECTOR, its ordered Abelian group of weights is $\mathbf{PP}=\langle(0,1),\oplus,\le\rangle$, defined by the usual order and $x\oplus y=\frac{xy}{xy+(1-x)(1-y)}$. The ordered Abelian group of certainty factors of MYCIN is isomorphic to **PP**. A review of these results and many references on the subject appear in [7].

Finally, in [5] we introduce a system $\mathcal{BAL}$ whose equivalent algebraic semantics is (equivalent to) the variety of ℓ–groups. This system is closely related to Meyer and Slaney's logic A. The logics $\mathcal{BAL}$ and A might be considered versions of what we could call "the logic of ℓ–groups" and are directly related to Ł* through their semantics. This is subject of ongoing research.

In this paper we continue the study of (the propositional fragment of) Ł*. We study deductive filters, we prove a deduction theorem and give detailed proofs of the soundness and completeness theorems. In the last section, we prove that the tautology problem for the logic Ł* is co–NP by generalizing the corresponding proof for MV–algebras that appears in [4].

2 Chang's Logic Ł*

We begin this section recalling part of the contents of [3], some of them have been rephrased to better suit our notation. The soundness and (weak) completeness theorem of Ł* with respect to a semantics of valuations into the interval $[-1,1]$ are the main results in that paper. We will extend these results.

2.1 The Deductive System Ł*

Language

The set $\mathcal{F}_{L^*}(X)$ of the formulas of this logic, the Ł*–terms, is recursively generated from a denumerable set X of propositional variables by a binary operation $\to$, a unary operation $\neg$, and a constant $\mathbf{1}$. As usual, $x \longleftrightarrow y$ stands for $x\to y$ and $y\to x$. We will also use the following abbreviations.

$$x^+ \quad := \quad (x\to\mathbf{1})\to\mathbf{1},$$

$$\begin{aligned} x^- &:= (x \to \neg\mathbf{1}) \to \neg\mathbf{1}, \\ x \vee y &:= (\,(x^+ \to y^+)^+ \to (\neg x)^-\,) \to (\,(y^- \to x^-)^- \to x^-\,). \end{aligned}$$

Axioms of Ł*

(P1) $(x \to y) \longleftrightarrow (\neg y \to \neg x)$,

(P2) $x \longleftrightarrow ((y \to y) \to x)$,

(P3) $\neg(x \to y) \longleftrightarrow (y \to x)$,

(P4) $x \to \mathbf{1}$,

(P5) $\mathbf{1} \longleftrightarrow ((\mathbf{1} \to x) \to \mathbf{1})$,

(P6) $(\,(x \to \mathbf{1}) \to ((y \to \mathbf{1}) \to z)\,) \to (\,(y \to \mathbf{1}) \to ((x \to \mathbf{1}) \to z)\,)$,

(P7) $(x \to y) \longleftrightarrow (\,(y^+ \to x^-) \to (x^+ \to y^-)\,)$,

(P8) $(x \to (\neg x \to y)\,)^+ \longleftrightarrow (x^+ \to (\neg(x^+) \to y^+)\,)$,

(P9) $(x \to (y \vee z)) \longleftrightarrow (\,(x \to y) \vee (x \to z)\,)$,

(P10) $(x \vee (y \vee z)) \longleftrightarrow ((x \vee y) \vee z)$.

Rules of Ł*

(R1) $x\ ,\ x \to y \vdash_{\text{Ł}^*} y$,

(R2) $x \to y\ ,\ u \to v \vdash_{\text{Ł}^*} (y \to u) \to (x \to v)$,

(R3) $x \vdash_{\text{Ł}^*} x^-$.

In the sequel we will write $\Gamma \vdash \varphi$, instead of the more cumbersome $\Gamma \vdash_{\text{Ł}^*} \varphi$; no confusion may arise since Ł* is the only deductive system used in the paper. The concepts of proof from a set Γ of formulas and of deducibility, $\Gamma \vdash \varphi$, etc. are defined as usual. The set of formulas probable from Γ will be denoted $\Gamma^{\vdash}$. The set of *theorems* of Ł* is $\emptyset^{\vdash}$, that is, the set of formulas φ such that $\emptyset \vdash \varphi$, which we simply write $\vdash \varphi$. An Ł*–*theory* is a set Γ of formulas that is closed under deducibility or, equivalently, such that $\Gamma^{\vdash} = \Gamma$.

LEMMA 2.1
$\vdash x \to x$.

Remark 1 Consider the following special case of (R2).

(TR) $x \to y \vdash_{\text{Ł}^*} (y \to u) \to (x \to u)$

By P1 and P3, (TR) is equivalent to

(TR') $x \to y \vdash_{\text{Ł}^*} (t \to x) \to (t \to y)$.

Also, it is easy to see that (TR), (TR') and (MP) imply (R2). So, we can replace (R2) by (TR). Moreover, (TR) implies (R3) by using (P2) and the following instance of (TR)
$(y \to y) \to x \vdash_{\text{Ł}^*} (x \to \neg\mathbf{1}) \to ((y \to y) \to \neg\mathbf{1})$.

2.2 The Lindenbaum–Tarski Algebra

The following two theorems appear in [3], Propositions 3.12 and 3.13, respectively. The reader is referred to [8] for the definition of MV^*–algebras.

THEOREM 2.2
Let $\mathcal{F}_{L^*}(X)$ be the set of Ł* –terms. Let the binary relation $\equiv$ over $\mathcal{F}_{L^*}$ be defined by

$$\varphi \equiv \psi \quad \text{if and only if} \quad \vdash \varphi \longleftrightarrow \psi\,.$$

Then $\equiv$ is a congruence on the absolutely free algebra $\langle \mathcal{F}_{L^*}(X);\, \rightarrow\,,\, \neg\,,\, \mathbf{0}\,\rangle$.

Notation: For every formula φ we shall denote by $[\varphi]$ its $\equiv$–equivalence class.

THEOREM 2.3
The system $\mathbf{F}_{\mathbf{L}^*}(X) = \langle \mathcal{F}_{L^*}(X)/_{\equiv}\,;\, \oplus\,,\, -\,,\, \bot\,, \top\rangle$ is an MV^*–algebra, where $\oplus$, $\ominus$, $\bot$, and $\top$ are defined by

$$[\alpha] \oplus [\beta] = [\neg\alpha \rightarrow \beta]$$
$$-[\alpha] = [\neg\alpha]$$
$$\bot = [\mathbf{1} \rightarrow \mathbf{1}]$$
$$\top = [\mathbf{1}]$$

The algebra $\mathbf{F}_{\mathbf{L}^*}(X)$ is free in the class of MV^*–algebras with the set of free generators $\{[x_1], [x_2], \ldots\}$, where $x_1, x_2, \ldots$ are the propositional variables.

The algebra $\mathbf{F}_{L^*}(X)$ is ordered by $[\alpha] \leq [\beta]$ if and only if $\vdash\ \alpha \rightarrow \beta$.

In what follows we will use $x_1, x_2 \ldots$, instead of $[x_1], [x_2] \ldots$ when no confusion is likely to arise.

3 MV^*–algebras, Deductive Filters and Ł* –Theories

For the sake of brevity we have not included here neither the lengthy definitions of MV^*–algebras, MV^*–ideals, etc, nor the theorems about them. We simply recall from [8] that the variety of MV^*–algebras is the class of all algebras $\mathbf{A} = \langle A; \oplus, -, \mathbf{0}, \mathbf{1}\rangle$ that verify a set of axioms that generalizes the algebra defined on the real number interval $[-1, 1]$ with truncated addition, additive inverse and constants 0 and 1.

An Ł* *–deductive filter*, or simply an Ł* *–filter*, of an MV^*–algebra $\mathbf{A}$ is a non–empty subset F of A such that

(PC) $A^+ \subseteq F$, that is, F contains the positive cone of $\mathbf{A}$.
(MP) If $a \in F$ and $b \ominus a \in F$, then $b \in F$.
(TR) If $a \oplus b \in F$ and $c \in A$, then $(a \oplus c) \oplus (b \ominus c) \in F$.

THEOREM 3.1
Let $\mathbf{A}$ be an MV^*–algebra.

1. If I is an ideal of $\mathbf{A}$, then $I\!\uparrow\, = \{x : x \geq a \ \text{ for some } a \in I\}$ is an Ł* –filter.

2. If F is an Ł*–filter of $\mathbf{A}$, then $F \cap -F$ is an ideal.
3. For every ideal I, $I\uparrow \cap -(I\uparrow) = I$.
4. For every Ł*–filter F, $(F \cap -F)\uparrow = F$.

PROOF. *1.* Let I be an ideal of $\mathbf{A}$. From the definition of $I\uparrow$, it is immediate that since I is closed under addition, so is $I\uparrow$. Since $\mathbf{0} \in I$, then every positive element is in $I\uparrow$, thus (PC) holds.

Let us prove (MP). Suppose $a \in I\uparrow$ and $b \ominus a \in I\uparrow$. We first observe that if $a_0 \leq a$ for some $a_0 \in I$, then $a_0^- \leq a^- \leq \mathbf{0}$. So $a^- \in I$ since ideals are convex and closed under negative parts, (see [8], Lemma 6). Thus if $a \in I\uparrow$, then $a^- \in I$. Moreover, $b \ominus a^- \geq b \ominus a$. That is $b \ominus a^- \in I\uparrow$. Since $-a^-$ is positive, $-a^- \in I\uparrow$. By [8] Lemma 2, *3.*, we have that $(b \ominus a^-) \oplus a^- \leq b$, so $b \in I\uparrow$, because $I\uparrow$ is closed under addition.

Now we prove (TR). Let $a \oplus b \in I\uparrow$ and let $c \in A$. Then $(a \oplus b)^- \in I$ and we can write $(a \oplus b)^- = (a \oplus b) \wedge \mathbf{0} = (a \oplus b) \wedge (a \ominus a) = a \oplus (b \wedge -a)$. By [8] Lemma 7, *7.*, ((TR) for ideals) we have that for every c, $(a \oplus c) \oplus ((b \wedge -a) \ominus c) \in I$. But $(a \oplus c) \oplus (b \ominus c) \geq (a \oplus c) \oplus ((b \wedge -a) \ominus c)$, so $(a \oplus c) \oplus (b \ominus c) \in I\uparrow$.

2. Let F be an Ł*–filter. Since $-\mathbf{0} = \mathbf{0}$, $\mathbf{0} \in F \cap -F \neq \emptyset$.

If a, $b \in F \cap -F$, then a, $-b \in F$, so by (TR), $a \ominus b = (a \oplus \mathbf{0}) \oplus (-b \ominus \mathbf{0}) \in F$. Similarly, $b \ominus a \in F$, so $a \ominus b \in F \cap -F$.

If $a \in F \cap -F$, then $a \in F$ and $-a \in F$. Moreover, $a^+ \in F$, because it is positive, and $-(a^+) = (-a)^- = (-a \oplus \mathbf{1}) \ominus \mathbf{1} \in F$, by (TR). So, $a^+ \in F \cap -F$.

To prove the convexity of $F \cap -F$, let a, $b \in F \cap -F$ and $a \leq c \leq b$. Then $c \in F$ since $c \geq a \in F$ and $-c \in F$, since $-c \geq -b \in F$.

The proofs of *3.* and *4.* are straightforward. ∎

COROLLARY 3.2
The lattice of ideals of $\mathbf{A}$ is isomorphic to the lattice of Ł*–filters of $\mathbf{A}$.

COROLLARY 3.3
The intersection of a family of Ł*–filters is an Ł*–filter.

COROLLARY 3.4
If F is nonempty, increasing, closed under $^-$ and closed under addition, then F is an Ł*–filter and conversely.

COROLLARY 3.5
Let Z be a nonempty subset of $\mathbf{A}$. The Ł*–filter generated by Z is the set

$$F(Z) = \{u : \; u \geq z_1^- \oplus z_2^- \oplus \cdots \oplus z_k^-, \;\; z_i \in Z\}.$$

In particular, $F(Z \cup \{a\}) = \{u : \; u \geq z_1^- \oplus z_2^- \oplus \cdots \oplus z_k^- \oplus n.a^-, \;\; z_i \in Z\}$. where $n.x$ is defined recursively by $0.x = \mathbf{0}$ and $(n+1).x = n.x \oplus x$.

Also $F(\emptyset) = A^+$.

THEOREM 3.6
The lattice of ideals of the Lindenbaum algebra $\mathbf{F}_{L^*}(X)$ of the calculus Ł* is isomorphic to the lattice of theories of Ł*.

PROOF. Let Θ be an Ł*–theory. Define the set $[\Theta] = \{[\alpha] \in \mathcal{F}/_{\equiv} \; : \; \alpha \in \Theta\}$. It is easy to verify that $[\Theta]$ is an Ł*–filter.

Let Φ be the function defined by

$$\Theta \longmapsto [\Theta] \cap -[\Theta]\,.$$

The inverse Φ^{-1} of this function is given by

$$\Theta_I = \{\alpha,\, :\, [\alpha] \in I\}$$

for an ideal I of $\mathbf{F}_{L^*}(X)$.

It is straightforward to prove that both functions are increasing and thus determine a lattice isomorphism. ∎

LEMMA 3.7
Let Γ be a set of Ł*–terms, $\Gamma^{\vdash}$ the theory generated by Γ and $[\Gamma]$ the set of classes $[\alpha]$, with $\alpha \in \Gamma$. Then the Ł*–filter of the Lindenbaum algebra $\mathbf{F}_{L^*}(X)$ associated with $\Gamma^{\vdash}$ is the filter generated by $[\Gamma]$, that is, $[\Gamma^{\vdash}] = F([\Gamma])$.

PROOF. In the first place, if $[\alpha] \in [\Gamma^{\vdash}]$, that is, $\alpha \in \Gamma^{\vdash}$, then there exists a deduction $\langle \gamma_1, \ldots, \gamma_n \rangle$ of α from Γ. Since $[\Gamma] \subseteq F([\Gamma])$ and $F([\Gamma])$ is closed under the rules (MP)and (TR), it follows that $[\alpha] \in F([\Gamma])$.

Conversely, suppose $[\alpha] \in F([\Gamma])$, where $\Gamma \neq \emptyset$. Then, $[\alpha] \geq [\gamma]^- = [\gamma_1]^- \oplus [\gamma_2]^- \oplus \cdots \oplus [\gamma_k]^-$, for $\gamma_i \in \Gamma$, $i = 1, \ldots, k$. It is clear that $[\gamma]^-$ belongs to $[\Gamma^{\vdash}]$, because $[\Gamma^{\vdash}]$ is closed under $^-$ and under addition (see Corollary 3.4). The inequality $[\alpha] \geq [\gamma]^-$, is equivalent to $\vdash \gamma^- \to \alpha$, so by (MP) $\gamma^- \vdash \alpha$, so $[\alpha] \in [\Gamma^{\vdash}]$.

If $\Gamma = \emptyset$ and $[\alpha] \in F([\Gamma])$, then $\bot \leq [\alpha]$, that is, $\vdash (\mathbf{1} \to \mathbf{1}) \to \alpha$ or $\vdash \alpha$. ∎

THEOREM 3.8
Algebraic Deduction Theorem If $[\beta] \in F([\Gamma] \cup \{[\alpha]\})$, then $[\alpha^- \longrightarrow_m \beta] \in F([\Gamma])$, for some non–negative integer m, where $\varphi \longrightarrow_1 \psi = \varphi \to \psi$ and $\varphi \longrightarrow_{n+1} \psi = \varphi \to (\varphi \longrightarrow_n \psi)$.

COROLLARY 3.9
If $\Gamma, \alpha \vdash \beta$, then there is a positive integer m such that $\Gamma \vdash \alpha^- \longrightarrow_m \beta$.

3.1 Semantics

We extend recursively any valuation $v : X \longrightarrow [-1,1]$ to all formulas by defining $\overline{v} : \mathcal{F}_{L^*}(X) \longrightarrow [-1,1]$, as follows.

1. $\overline{v}(x) = v(x)$, for any propositional letter $x \in Va$,
2. $\overline{v}(\alpha \to \beta) = \min(\max(-1, \overline{v}(\beta) - \overline{v}(\alpha)), 1)$,
3. $\overline{v}(-\alpha) = -\overline{v}(\alpha)$.

The following lemma is immediate.

LEMMA 3.10
For any valuation v,

1. $\overline{v}(\mathbf{0}) = 0$ and $\overline{v}(\mathbf{1}) = 1$.
2. $\overline{v}(\alpha^+) = \max(0, \overline{v}(\alpha))$,

3. $\overline{v}(\alpha^-) = -\max(0, \overline{v}(-\alpha))$,
4. If $\overline{v}(\alpha \to \beta) = 0$, then $\overline{v}(\alpha) = \overline{v}(\beta)$.

We say that a valuation v *satisfies* φ if $\overline{v}(\varphi) \geq 0$. The formula φ is $[-1,1]$–*valid*, or it is a $[-1,1]$–*tautology*, if for any valuation v, $\overline{v}(\varphi) \geq 0$, in which case we write $\models \varphi$. Similarly, given a set Γ of formulas and a formula φ, we say φ is a *semantic consequence* of Γ, or simply a *consequence*, if any valuation that satisfies all formulas of Γ also satisfies φ. In this case write $\Gamma \models \varphi$ and denote $\Gamma^{\models}$ the set of semantic consequences of Γ.

Lemma 3.11
For any valuation $v : X \longrightarrow [-1,1]$, let ν be the canonical projection from the totally free term algebra into $\mathbf{F}_{L^*}(X)$ and define the function $u : [X] \longrightarrow [-1,1]$ such that $u([x]) = v(x)$.

1. The function u can be extended to a unique homomorphism $\overline{u} : \mathbf{F}_{L^*}(X) \longrightarrow [-1,1]$.
2. The set $\overline{u}^{-1}([0,1]) = \overline{u}^{-1}(\{0\})\uparrow = \{[\varphi] : \overline{u}(\varphi)) \geq 0\}$ is a maximal Ł*–filter of $\mathbf{F}_{L^*}(X)$.

Proof. For the second part, let G be an Ł*–filter such that $\overline{u}^{-1}([0,1]) \subsetneq G$ and let $\alpha \in G \setminus \overline{u}^{-1}([0,1])$. Then $\overline{u}(\alpha) < 0$ so there is an n such that $\overline{u}(n.\alpha) = -1$ and since G is closed under addition, $[n.\alpha] \in G$ and thus $G = \mathcal{F}_{L^*}/_{\equiv}$ ∎

3.2 A Semantical Proof of the Deduction Theorem

We recall the definition of a McNaughton* function introduced in [8]

Definition 3.12
Let $n \geq 1$ be an integer. Then a function

$$f : [-1,1]^n \longrightarrow [-1,1]$$

is called a *McNaughton* function over* $[-1,1]^n$ if and only if it satisfies the following conditions:
(i) f is continuous with respect to the natural topology of $[-1,1]^n$,
(ii) there are linear polynomials $p_1, \ldots, p_k$ with integer coefficients, such that for each point $\mathbf{y} = (y_1, \ldots, y_n) \in [-1,1]^n$ there is a $j \in \{1, \ldots, k\}$ with $f(\mathbf{y}) = p_j(\mathbf{y})$.

We observe that each McNaughton*–function partitions $[-1,1]^n$ into a finite number of simplexes. On each of these $f(\mathbf{y}) = p_j(\mathbf{y})$, for some $j = 1, \ldots, k$. The continuity of f implies that the polynomials must coincide on the faces shared by any two simplexes. (For illuminating related work in the context of MV–algebras, see [4], pages 62-64.

We also recall from [8], Theorem 40, that if α is an MV^*–term such that the set of variables appearing in α is contained in $\{x_1, \ldots, x_n\}$, then the interpretation $\alpha^{[-1,1]} : [-1,1]^n \longrightarrow [-1,1]$ of α in the MV^*–algebra $[-1,1]$ is a McNaughton function over $[-1,1]^n$.

Lemma 3.13
Let f, $g : [-1,1]^n \longrightarrow [-1,1]$ be McNaughton*–functions such that for all $\mathbf{x}$

1. $g(\mathbf{x}) \geq 0$,
2. if $f(\mathbf{x}) > 0$, then $g(\mathbf{x}) > 0$,
3. for some $\mathbf{a} \in [-1,1]^n$, $f(\mathbf{a}) = g(\mathbf{a}) = 0$.

Then there is a neighborhood N of $\mathbf{a}$ and a positive integer m such that $h_m(\mathbf{x}) = f(\mathbf{x}) - m\,g(\mathbf{x}) \leq 0$, for all $\mathbf{x} \in N$.

Proof. We may assume without loss of generality that in every neighborhood of $\mathbf{a}$ there is a point $\mathbf{x}$ such that $f(\mathbf{x}) > 0$, for if not, since g is non–negative, the result follows trivially for any m. For any such a point, by hypothesis 2, $g(\mathbf{x}) > 0$.

Note that since f is a McNaughton* function there is a finite set of n–dimensional simplexes $\mathcal{S}_1, \ldots, \mathcal{S}_k$ such that f and g are different linear polynomials with integer coefficients over each simplex $\mathcal{S}_i$. These simplexes are obtained by refinement of the corresponding decompositions of f and g. Observe also that if $\mathbf{a} \in \mathcal{S}_i$, by hypothesis 3 above, in a neighborhood N of $\mathbf{a}$ these are homogeneous polynomials of degree one in $\mathbf{x} - \mathbf{a}$. This implies that for each i, $f(\mathbf{x}) = \mathbf{b}_i \cdot (\mathbf{x} - \mathbf{a})$ and $g(\mathbf{x}) = \mathbf{c}_i \cdot (\mathbf{x} - \mathbf{a})$ on $\mathcal{S}_i \cap N$.

If $g(\mathbf{x}) = 0$ for some point in the interior of $\mathcal{S}_i \cap N$, then both $f(\mathbf{x}) = 0$ and $g(\mathbf{x}) = 0$ throughout $\mathcal{S}_i$, so $h_m(\mathbf{x}) \leq 0$ on $\mathcal{S}_i \cap N$. We observe that for each m and each simplex $\mathcal{S}_i$, the neighborhood N might be different because of truncation.

If $g(\mathbf{x}) = 0$ for some point on one face of $\mathcal{S}_i$, then both $f(\mathbf{x}) = 0$ and $g(\mathbf{x}) = 0$ throughout that face and thus $\mathbf{b}_i = \lambda \mathbf{c}_i$, for some constant λ, which is positive since both f and g are positive. This implies that the quotient

$$\frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\mathbf{b}_i \cdot (\mathbf{x} - \mathbf{a})}{\mathbf{c}_i \cdot (\mathbf{x} - \mathbf{a})}$$

is a constant not only on that face, but on the whole simplex, and thus $h_m(\mathbf{x}) \leq 0$ on $\mathcal{S}_i \cap N$. We observe that simplexes that share a face will have different constants. Finally, since there are finitely many simplexes, $h_m(\mathbf{x}) \leq 0$ on a neighborhood N of $\mathbf{a}$, for some positive integer m. ∎

Theorem 3.14
Let $f, g : [-1,1]^n \longrightarrow [-1,1]$ be McNaughton*–functions such that for all $\mathbf{x}$,

1. $g(\mathbf{x}) \geq 0$,
2. if $f(\mathbf{x}) > 0$, then $g(\mathbf{x}) > 0$.

Then there is a a positive integer m such that $h_m(\mathbf{x}) = f(\mathbf{x}) - m\,g(\mathbf{x}) \leq 0$, for all $\mathbf{x} \in N$.

Proof. First observe that if $f(\mathbf{x}) < 0$, the result is obvious, so we may assume $f(\mathbf{x}) \geq 0$. If $g(\mathbf{a}) > 0$, by the continuity of f and of g, we can find a closed neighborhood of $\mathbf{a}$ where $f(\mathbf{x}) \leq R$ and $g(\mathbf{x}) \geq r > 0$. We can now choose a positive integer m such that $R \leq m\,r$, so $f(\mathbf{x}) - m\,g(\mathbf{x}) \leq R - m\,r \leq 0$. Then there is an m such that in the interior of this neighborhood, $h_m(\mathbf{x}) \leq 0$.

If $g(\mathbf{a}) = 0$, then by hypothesis, $f(\mathbf{a}) \leq 0$. As said we are only intersted in the case $f(\mathbf{a}) = 0$, so by the previous lemma, there is an open neighborhood of $\mathbf{a}$ and a positive integer m on which $h_m(\mathbf{x}) \leq 0$.

Finally we observe that the set of open neighborhoods defined in the previous paragraphs is an open covering of $[-1,1]^n$. This covering has a finite subcovering which defines positive integers $m_1, \ldots, m_k$. By letting $m = \max\{m_1, \ldots, m_k\}$, for any $\mathbf{x} \in [-1,1]^n$, $h_m(\mathbf{x}) = f(\mathbf{x}) - m\,g(\mathbf{x}) \leq 0$. ∎

We get a similar result if we assume that

1'. $g(\mathbf{x}) \leq 0$,
2'. if $f(\mathbf{x}) < 0$, then $g(\mathbf{x}) < 0$,
namely, there exists a positive integer m such that $h_m(\mathbf{x}) = f(\mathbf{x}) - m\, g(\mathbf{x}) \geq 0$.

Theorem 3.15
Let α and β be Ł*–terms. Then

$$\alpha \models \beta \qquad \text{if and only if} \qquad \models \alpha^- \longrightarrow_m \beta\,,$$

for some positive integer m.

Proof. Suppose $\alpha \models \beta$ and let v be a valuation. Let us consider $f_{\alpha^-}(\mathbf{x}) = \overline{v}(\alpha^-)(\mathbf{x})$ and $f_\beta(\mathbf{x}) = \overline{v}(\beta)(\mathbf{x})$ be the McNaughton* functions associated to α^- and β respectively.

We observe that for all $\mathbf{x}$,
1.) $f_{\alpha^-}(\mathbf{x}) \leq 0$
2.) if $f_\beta(\mathbf{x}) < 0$, then $f_{\alpha^-}(\mathbf{x}) < 0$, by hypothesis.

So by Theorem 3.14, and the remark following it, there is a positive integer m such that $f_\beta(\mathbf{x}) - m\, f_{\alpha^-}(\mathbf{x}) \geq 0$.

We now observe that $\overline{v}(\alpha^- \longrightarrow_m \beta) = f_\beta - m\, f_{\alpha^-}$, and this completes the proof of $\models \alpha^- \longrightarrow_m \beta$.

The proof of the converse is an application of Modus Ponens. ∎

Corollary 3.16
Let $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta$ be Ł*–terms. Then

$$\alpha_1, \alpha_2, \ldots, \alpha_k \models \beta \qquad \text{iff} \qquad \models \alpha_1^- \longrightarrow_{m_1} (\alpha_2^- \longrightarrow_{m_2} (\cdots (\alpha_k^- \longrightarrow_{m_k} \beta)\ldots))\,,$$

for positive integers $m_1, \ldots, m_k$.

3.3 Soundness and Completeness

Theorem 3.17
Let Θ be a set of formulas of Ł*, $\Theta^\vdash$ and $\Theta^\models$ the syntactic and semantic consequences of Θ, respectively.

1. $\emptyset^\vdash = \emptyset^\models$.
2. If Θ is finite, then $\Theta^\vdash = \Theta^\models$.
3. $\Theta^\vdash \subseteq \Theta^\models$. The inclusion may be proper.
4. $\Theta^\vdash = \Theta^\models$ if and only if $[\Theta^\vdash]$ is an intersection of maximal filters of the Lindenbaum algebra.

Proof. *1.* The set of theorems of Ł* coincides with the set of tautologies, this is the only completeness result whose proof appears in [3]. We will prove it in our context.

The inductive proof of $\emptyset^\vdash \subset \emptyset^\models$ is straightforward considering that axioms are tautologies and that the deduction rules when applied to non–negative truth values lead to non–negative truth values.

Suppose $\alpha \notin \emptyset^\vdash$. Then $F = \{[\beta] : \vdash \beta\}$ is an Ł*–filter of $\mathbf{F}_{L^*}(X)$ such that $[\alpha] \notin F$. By Zorn's lemma, we have a maximal Ł*–filter G such that $[\alpha] \notin G$ and $F \subseteq G$. By the correspondence established in Corollary 3.2, $M = G \cap -G$ is a maximal ideal of $\mathbf{F}_{L^*}(X)$. This implies that $\mathbf{F}_{L^*}(X)/M$ is simple and by [8], Theorem 30, there is an isomorphism $f : \mathbf{F}_{L^*}(X)/M \longrightarrow \mathbf{A}$, where $\mathbf{A}$ is a subalgebra of the interval algebra $[-1, 1]$. We observe that $G = I\uparrow = f^{-1}([0, 1])$, so if we define $v : X \longrightarrow [-1, 1]$ by $v(x) = f([\nu(x)]_I)$, where $\nu(x)$ is the canonical projection of X into $\mathbf{F}_{L^*}(X)$, then $\overline{v}(\alpha) < 0$, so $\alpha \notin \emptyset^{\models}$.

2. This follows from *1.* and the corollary to the Deduction Theorem, 3.15.

3. The proof of $\Theta^{\vdash} \subseteq \Theta^{\models}$ is straightforward.

In order to prove that in general we do not have equality, we adapt Wójcicki's example, see [**?**]. Let $\gamma_n(x) = -\mathbf{1} \to \neg^n x$, where $\neg^0 x = x$ and $\neg^{n+1} x = \neg^n x \to x$ and let $\Gamma = \{\gamma_n(x) : n \geq 1\}$. We observe that a valuation v satisfies $\gamma_n(x)$ if and only if $v(x) \geq \frac{-1}{n}$ so we have that $\Gamma \models x$. Nevertheless, if $\Gamma \vdash x$ there is a finite set $\Gamma_0 = \{\gamma_{n1}, \ldots, \gamma_{nk}\} \subseteq \Gamma$ such that $\Gamma_0 \vdash x$. Letting m be the largest of the subindexes, define the valuation $v(x) = \frac{-1}{m} < 0$. But then, since $\overline{v}(\gamma_{ni}(x)) \geq 0$ for all $\gamma \in \Gamma_0$, we have $\Gamma_0 \not\models x$, contradicting *2.* above.

4. We observe that $\Theta^{\models} = \bigcap_{\alpha \notin \Theta^{\models}} \overline{v}_\alpha^{-1}(\{0\})\uparrow$, where v_α is a valuation such that $\overline{v}_\alpha(\Theta) \subseteq [0, 1]$ and $\overline{v}_\alpha(\alpha) < 0$. Recall that by Lemma 3.11 $\overline{v}_\alpha^{-1}(\{0\})\uparrow$ is a maximal Ł*–filter. This implies that if we assume that $\Theta^{\vdash} = \Theta^{\models}$, then $[\Theta^{\vdash}]$ is an intersection of maximal Ł* filters.

Suppose now that $[\Theta^{\vdash}] = \bigcap_{j \in J} M_j$, where each M_j is a maximal Ł* filters. By the correspondence between Ł*–filters and ideals, we can define the quotient of the algebra $\mathbf{A}$ modulo the filter M in the obvious way, that is, $\mathbf{A}/M = \mathbf{A}/(M \cap -M)$. Now recall that an MV^*–algebra $\mathbf{A}/M$ is simple if and only if M is a maximal filter ($M \cap -M$ is a maximal ideal). By [8], Theorem 30, since the MV^*–algebra $\mathbf{A}/M$ is simple it is isomorphic to a subalgebra of $[-1, 1]$. For each $j \in J$ let $h_j : \mathbf{A} \longrightarrow [-1, 1]$ be the homomorphism such that $h_j^{-1}(\{0\})\uparrow = M_j$. Observe that if $[\vartheta] \in [\Theta]$, then $h_j(\vartheta) \geq 0$.

So if $[\alpha] \notin [\Theta^{\vdash}]$, for some $j \in J$, $\alpha \notin h_j^{-1}(\{0\})\uparrow$, or what is the same, $h_j(\alpha) < 0$. But h_j may be regarded as a valuation, so $\alpha \notin \Theta^{\models}$. This finishes the proof of $\Theta^{\models} \subseteq \Theta^{\vdash}$ and together with *3.* above, this part of the theorem.

■

4 The tautology problem for Ł*

In this section we consider the complexity of the tautology problem for the logic Ł* . The fact that the problem is at least co-NP-complete is proved by means of a comparison with the tautology problem for Łukasiewicz's logic Ł, that is well known to be co-NP-complete.

The following three (or four) results are taken *mutatis mutandis* from chapter 9 section 3 of [4]. We shall denote by $|\varphi|$ the number of occurrences of symbols in φ. We shall also use the notation $|x|$ for the absolute value of a real number x. For each Ł*–formula $\alpha(x_1, \ldots, x_n)$ denote by $f_\alpha = \alpha^{[-1,1]} : [-1, 1]^n \longrightarrow [-1, 1]$ its associated McNaughton* function. See [8] Theorem 39.

Theorem 4.1
Let $\alpha(x_1, \ldots, x_n)$ and $\mathbf{x}$, $\mathbf{y} \in [-1, 1]^n$. Then the (one sided) directional derivative $f'_\alpha(\mathbf{x}; \mathbf{d})$ at $\mathbf{x}$ along the direction $\mathbf{d} = \mathbf{y} - \mathbf{x}$ is well defined and the following inequality holds.

$$|f'_\alpha(\mathbf{x}; \mathbf{d})| \leq \|\mathbf{d}\| \cdot |\alpha| \,,$$

where $\|\mathbf{d}\|$ denotes the Euclidean norm in $\mathbf{R}^n$.

Proof. The existence of $f'_\alpha(\mathbf{x};\mathbf{d})$ follows from the definition of McNaughton function. The inequality is proved by induction on the number m of connectives occurring in α. The basis $m = 0$ is trivial. For the induction step, if $\alpha = \neg\beta$ for some Ł*–formula $\beta \in$ Ł*, then the conclusion follows from the identity $f_\alpha = -f_\beta$. Finally, if $\alpha = \beta \oplus \gamma$ then, assuming without loss of generality that both β, γ have the same variables, the conclusion follows by definition of truncated addition and the identity $f_\alpha = f_\beta \oplus f_\gamma$. ∎

Corollary 4.2
Let α and $p(\mathbf{x}) = c + m_1x_1 + ... + m_nx_n$ be a linear polynomial with integer coefficients c, $m_1, \ldots, m_n$ such that f_α and p coincide over an open set of $[-1,1]^n$. Then we have

$$max(|m_1|, \ldots, |m_n|) \leq |\alpha|\,.$$

Theorem 4.3
Let α be an Ł*–formula and f_α its associated McNaughton* function. Assume f_α does not coincide with the zero function over $[-1,1]^n$. Then there exists a point

$$\mathbf{a} = (a_1/b, \ldots, a_n/b) \in [-1,1]^n$$

with a_i, $b \in \mathbb{Z}$ and $0 \leq |a_i| \leq b$ (i=1,…, n) such that $f_\alpha(\mathbf{a}) \neq 0$ and $0 < b < 2^{(4|\alpha|^2)}$.

Proof. Note that since f_α is a McNaughton* function there is a finite set of n-dimensional simplexes $\mathcal{S}_1, \ldots, \mathcal{S}_m$, whose union is $[-1,1]^n$, such that f_α is a linear polynomial with integer coefficients over each simplex $\mathcal{S}_i$. Since $f_\alpha \neq 0$ then there exists $\mathbf{a} \in [-1,1]^n$ such that $f_\alpha(\mathbf{a}) \neq 0$. Without loss of generality we can assume that $\mathbf{a}$ is a vertex of $\mathcal{S}_1$. Therefore the coordinates of $\mathbf{a}$ are all rational, say $\mathbf{a} = (a_1/b, \ldots, a_n/b)$, where $0 \leq |a_i| \leq b$ and such that $\gcd(a_1, \ldots, a_n, b) = 1$ (gcd denotes the greatest common divisor). Moreover, by the above corollary, $\mathbf{a}$ is the solution of an $n \times n$ linear system and each row has its coefficients bounded by $2|\alpha|$. Since $n \leq |\alpha|$ and by Hadamard's inequality we conclude that the determinant Δ of the system satisfies the inequality

$$|\Delta| \leq (4n|\alpha|^2)^{n/2} < 2^{(4|\alpha|^2)}$$

Since $b \leq |\Delta|$, the conclusion follows. ∎

Remark: Again we have used the fact that for every McNaughton* function f there exists a decomposition $\{\mathcal{S}_i\}_{i=1}^m$ of $[-1,1]^n$ in n-dimensional simplexes (for $i \neq j$ then S_i and S_j are disjoint or they intersect in a common face, and $\bigcup_{i=1}^m \mathcal{S}_i = [-1,1]^n$) such that f is a linear polynomial with integer coefficients in each simplex.

Theorem 4.4
The tautology problem for the logic Ł* is in the class co-NP.

Proof. A non deterministic procedure quickly deciding if a formula $\alpha(x_1, \ldots, x_n)$ is not a tautology is as follows. Since $f_\alpha \not> 0$, applying the previous theorem, guess a rational point $\mathbf{x} = (a_1/b, \ldots, a_n/b) \in [-1,1]^n$ such that $f_\alpha(\mathbf{x}) < 0$ and $0 < b < 2^{(4|\alpha|^2)}$. Then check that $f_\alpha(\mathbf{x}) < 0$ can be computed by a deterministic Turing machine within a number of steps depending polynomially on the complexity of α. For a proof of this fact see [4]. ∎

In order to prove that the tautology problem is co-NP-hard we are going to use the fact that the tautology problem for the logic Ł is co-NP-complete. For a detailed proof of this see [4], Theorem 9.3.8.

For any Ł*–formula $\alpha(x_1,\ldots,x_n)$ we define $abs(\alpha)=\alpha\vee(\neg\alpha)$. Note that the identity

$$f_{abs(\alpha)}(\mathbf{x})=|f_\alpha(\mathbf{x})|$$

holds for all $\mathbf{x}\in[-1,1]^n$.

THEOREM 4.5
The tautology problem for the logic Ł* is co-NP-complete.

PROOF. We have already seen that the tautology problem for Ł* is in the class co-NP. We show that the problem is co-NP hard by comparing it with the tautology problem for Ł.

Let $\alpha(x_1,\ldots,x_n)$ be an Ł–tautology i.e, the McNaughton function $f_\alpha=\alpha^{[0,1]}$ over the MV-algebra $[0,1]$, equals the constant function $\mathbf{1}$. Then applying the translation $(\cdot)^*:\text{Ł}\to\text{Ł}^*$ defined recursively by

(i) $x_i^*=x_i$,
(ii) $(\neg\alpha)^*=\mathbf{1}\ominus\alpha^*$,
(iii) $(\alpha\oplus\beta)^*=\alpha^*\oplus\beta^*$,

see [8] Lemma 40, we obtain the Ł*–formula $\alpha^*(x_1,\ldots,x_n)$ such that the McNaughton function over the MV^*–algebra $[-1,1]$, $f^*_{\alpha^*}=(\alpha^*)^{[-1,1]}$, satisfies

$$f^*_{\alpha^*}\restriction_{[0,1]^n}=f_\alpha$$

Let $\beta\in\text{Ł}^*$ be the formula $\beta(x_1,\ldots,x_n)=\alpha^*(abs(x_1),\ldots,abs(x_n))$. Note that β is a tautology in the logic Ł*. Thus the problem of deciding if a formula is *not* a tautology in the logic Ł* is as hard as its counterpart for the logic Ł. Then by the previous theorem the conclusion follows.

■

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