A New Family of Slash-Distributions with Elliptical Contours

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Abstract

We introduce a new family of univariate and multivariate slash-distributions. Our construction is based on elliptical distributions. We define the new family by means of a stochastic representation as the scale mixture of an elliptically distributed random variable with respect to the power of a $U(0,1)$ random variable. The same idea is extended to the multivariate case. We study general properties of the resulting families, including their moments. We illustrate special cases of interest, such as Normal, Cauchy, Student-t, Type II Pearson and Kotz-Type distributions.

KEY WORDS: Kurtosis, Scale Mixtures of Elliptical Distributions, Symmetric Distributions.

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1 Introduction

Symmetric distributions generalizing normality have been the subject of much study in the statistical literature. One such extension is the class of standard slash distributions, which is described next: we say that S has standard slash distribution with parameter $q > 0$ if S can be expressed as

$$S = \frac{Z}{U^{\frac{1}{q}}}, \tag{1}$$

where $Z \sim N(0, 1)$ and $U \sim U(0, 1)$ are independent random variables. We denote this as $S \sim SL(q)$. Setting $q = 1$ yields the canonical slash distribution, whereas the normal distribution corresponds to the limit case $q \to \infty$. While keeping symmetry, this distribution typically has heavier tails than the standard normal, i.e., it provides increased kurtosis.

The simplicity of the stochastic representation (1) turns out to be quite useful for studying properties of the corresponding family. For instance, the standard density function for the general case is readily found to be given by

$$f(x) = q \int_0^1 u^q \phi(xu)\, du,$$

where $\phi(t) = (2\pi)^{-1/2} \exp(-t^2/2)$ is the standard normal density function. When $q = 1$ the canonical slash density follows:

$$f(x) = \begin{cases} \frac{\phi(0)-\phi(x)}{x^2} & x \neq 0 \\ \frac{1}{2}\phi(0) & x = 0. \end{cases}$$

See Johnson, Kotz and Balakrishnan (1995). General properties of this distribution are studied in Rogers and Tukey (1972) and in Mostelles and Tukey (1977). Maximum likelihood estimates (MLEs) of the related location-scale family are discussed in Kadafar (1982). Recently, Wang and Genton (2006) described multivariate and skew-multivariate extensions of the slash distribution.

From (1) we obtain

$$Z = U^{\frac{1}{q}} S \sim N(0, 1). \tag{2}$$

The class of distributions we introduce are based on generalizing (2), letting the distribution of Z be more general than normal. We do this by considering elliptical distributions. The theory and properties of such distributions have been the subject of much research in the statistical literature. The early works by Kelker (1970), and Cambanis et al. (1981) contain many interesting properties and results. A general compilation of such theory is given in Fang et al. (1990). Specifically, a random variable X has elliptical distribution with location μ and scale parameter σ, denoted as $X \sim El(\mu, \sigma; g)$, if X has density function given by

$$f_X(x) = \frac{1}{\sigma}\, g\left(\left(\frac{x-\mu}{\sigma}\right)^2\right),$$

for some nonnegative function $g(u)$, $u \geq 0$ (referred to as the *density generator*), satisfying $\int_0^\infty u^{-\frac{1}{2}} g(u)\, du = 1$.

Any member of the new class of distributions can be represented as $W = ZU^{-1/q}$ for some $q > 0$, where $Z \sim El(0, 1; g)$ is independent of $U \sim U(0, 1)$. We use the notation $W \sim SEl(0, 1; g)$. Members of this class are called *slash-elliptical distributions.* A key advantage of our construction is the simplicity by which standard and well-known symmetric distributions can be modified to support increased kurtosis. We study the main properties of the corresponding induced location-scale family, discussing an appropriate multivariate extension.

The rest of this article is organized as follows. Section 2 gives the main properties of slash-elliptical distributions, studying in particular the density function and the corresponding moments. We also indicate how to compute moment estimates and give an illustration comparing these with the MLEs. Section 3 gives the multivariate extension and derives some related properties. Section 4 concludes with a few final remarks and some further extensions of our work.

2 Slash-Elliptical Distributions

We give now the general stochastic representation defining the new family, deriving from it important properties such as the density function and moments. We also discuss the corresponding moment estimates.

2.1 Density Function in the General Case

Let $Y \sim SEl(\mu, \sigma; g)$, i.e., Y is a random variable that can be stochastically represented as

$$Y = \sigma \frac{W}{U^{1/q}} + \mu, \tag{3}$$

where $W \sim El(0, 1; g)$ and $U \sim U(0, 1)$ are independent, $q > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$. In what follows we state the results for the standard case ($\mu = 0$ and $\sigma = 1$) from which extensions to the location-scale family are immediate and therefore omitted. We start with the general form of the density function.

Proposition 1 *Let $Y \sim SEl(0, 1, q; g)$. Then Y has density function given by*

$$f_Y(y; 0, 1, q) = \begin{cases} \frac{q}{2|y|^{q+1}} \int_0^{y^2} t^{\frac{q-1}{2}} g(t)\, dt & \text{if } y \neq 0 \\ \frac{q}{1+q} g(0) & \text{if } y = 0. \end{cases} \tag{4}$$

In the special canonical case, i.e. when $q = 1$, (4) reduces to

$$f_Y(y; 0, 1, 1) = \begin{cases} \frac{G(y^2)}{2y^2} & \text{if } y \neq 0 \\ \frac{1}{2} g(0) & \text{if } y = 0, \end{cases} \tag{5}$$

where $G(x) = \int_0^x g(t)\, dt$.

Proof. From (3), and using the independence of U and W, standard calculations (based on the Jacobian of the appropriate transformation) show that

$$f_Y(y; 0, 1, q) = q \int_0^1 u^q f_W(yu)\, du$$

where $f_W(x) = g(x^2)$ is the density function of W. Hence,

$$f_Y(y; 0, 1, q) = q \int_0^1 u^q g\left(y^2 u^2\right)\, du. \tag{6}$$

If $y = 0$, the result is immediate. On the other hand, if $y \neq 0$ (4) follows by changing the integration variable to $t = y^2 u^2$ in (6). $\square$

It follows immediately from Proposition 1 that $f_Y(y; 0, 1, q)$ is continuous at $y = 0$ provided that g is right-continuous at 0. Also,

$$\lim_{q \to \infty} f_Y\,(y; 0, 1, q) = g(y^2),$$

which shows that the class of slash-elliptical distributions contains the elliptical family as a limit case when $q \to \infty$.

Table 1 shows some typical special examples of generator function g together with the corresponding G function, from which (4), and in particular (5) can be easily derived. Figure 1 shows the density functions arising in some special cases of canonical slash-elliptical distributions.

Type	$g(t)$	$G(t)$
Normal	$(2\pi)^{-\frac{1}{2}} \exp\left(-\frac{t}{2}\right)$	$\sqrt{\frac{2}{\pi}}\left[1 - \exp\left(-\frac{t}{2}\right)\right]$
Cauchy	$\pi^{-1}(1+t)^{-1}$	$\pi^{-1}\log(1+t)$
Student-t	$\frac{\Gamma((1+\nu)/2)}{\Gamma(\nu/2)\sqrt{\pi\nu}}\left(1+\frac{t}{\nu}\right)^{-\frac{1+\nu}{2}}$	$\frac{2\Gamma((1+\nu)/2)\sqrt{\nu}}{\Gamma(\nu/2)(\nu-1)\sqrt{\pi}}\left[1-\left(1+\frac{t}{\nu}\right)^{-\frac{\nu-1}{2}}\right]$
Pearson	$\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha)\sqrt{\pi}}(1-t)^{\alpha-1}\ ,\ \alpha > 0$	$\frac{\Gamma(\alpha+1/2)}{\alpha\Gamma(\alpha)\sqrt{\pi}}\left[1-(1-t)^{\alpha}\right]$
Kotz-Type	$\frac{sr^{(2N-1)/2s}}{\Gamma((2N-1)/2s)}t^{N-1}\exp\left(-rt^s\right), r, s > 0, N > 1/2$	$\int_0^t \frac{sr^{(2N-1)/2s}}{\Gamma((2N-1)/2s)}x^{N-1}\exp\left(-rx^s\right)dx$

Table 1: *Some special cases of generator function g.*

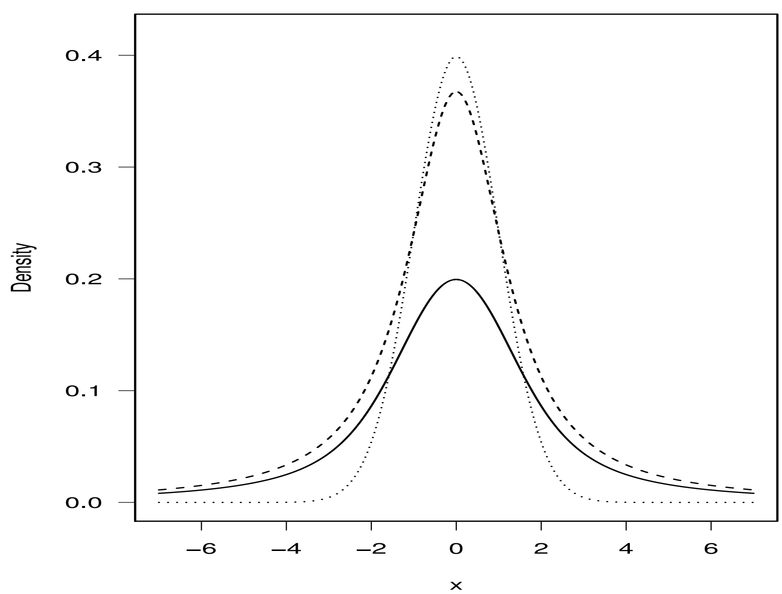

Figure 1: *Density functions for some canonical cases of slash-elliptical distributions: slash (solid line), slash-t with $\nu = 3$ (dashed line) and standard normal (dotted line)*

2.2 Some Properties of the Slash-Elliptical Distributions

Consider a random variable $X \sim SEl(0,1,q;g)$ with $q > 0$, and define $W = |X|$ and $V = X^2$. It is straightforward to see from (4) that the density functions for W and V are given by

$$f_W(w;q) = \frac{q}{w^{q+1}} \int_0^{w^2} t^{\frac{q-1}{2}} g(t)\, dt, \quad \text{and} \quad f_V(v;q) = \frac{q}{2v^{(q+2)/2}} \int_0^{v} t^{\frac{q-1}{2}} g(t)\, dt, \tag{7}$$

where $w > 0$ and $v > 0$. The first part of (7) can be seen as a generalization of the half-symmetric family, while the second part gives an extension to results involving the square of symmetric random variables.

Example 1 Consider the normal case, i.e. $g(t) = c\exp(-t/2)$ (the corresponding proportionality constant is given in Table 1). For $q > 0$, (7) then becomes

$$f_W(w;q) = \frac{cq}{w^{q+1}} \int_0^{w^2} t^{\frac{q-1}{2}} \exp(-t/2)\, dt, \quad \text{and} \quad f_V(v;q) = \frac{cq}{2v^{(q+2)/2}} \int_0^{v} t^{\frac{q-1}{2}} \exp(-t/2)\, dt,$$

for $w > 0$ and $v > 0$. In the particular case $q = 1$ we get

$$f_W(w;1) = \frac{G(w^2)}{w^2} = \frac{2(\phi(0) - \phi(w))}{w^2}, \quad \text{and} \quad f_V(v;1) = \frac{G(v)}{2v^{3/2}} = \frac{(\phi(0) - \phi(\sqrt{v}))}{v^{3/2}}.$$

We recognize $f_W(w;1)$ as the canonical half-slash density, while $f_V(v;1)$ is just the canonical slash-chi-square density with 1 degree of freedom.

Proposition 2 *Let $Y|U = u \sim El(0, u^{-1/q}; g)$ and $U \sim U(0,1)$. Then $Y \sim SEl(0,1,q;g)$*

Proof.

$$f_Y(y;q) = \int_0^1 f_{Y|U}(y|u) f_U(u)\, du = \int_0^1 u^{1/q} g\left(\left(\frac{y}{u^{1/q}}\right)^2\right) du.$$

The result for $y = 0$ is an immediate consequence of this. The result for the case $y \neq 0$ follows by taking the transformation $t = \left(\frac{y}{u^{1/q}}\right)^2$. $\square$

Remark 1 Proposition 2 shows that the slash-elliptical family can be represented as a particular scale-mixture of elliptical distributions and the $U(0,1)$ distribution. The result is also useful for generating $SEl(0,1,q;g)$ deviates. Figure 2 shows that the canonical half-slash and canonical slash-$\chi^2(1)$ models of Example 1 have greater kurtosis than the half-normal and $\chi^2(1)$ distributions, respectively.

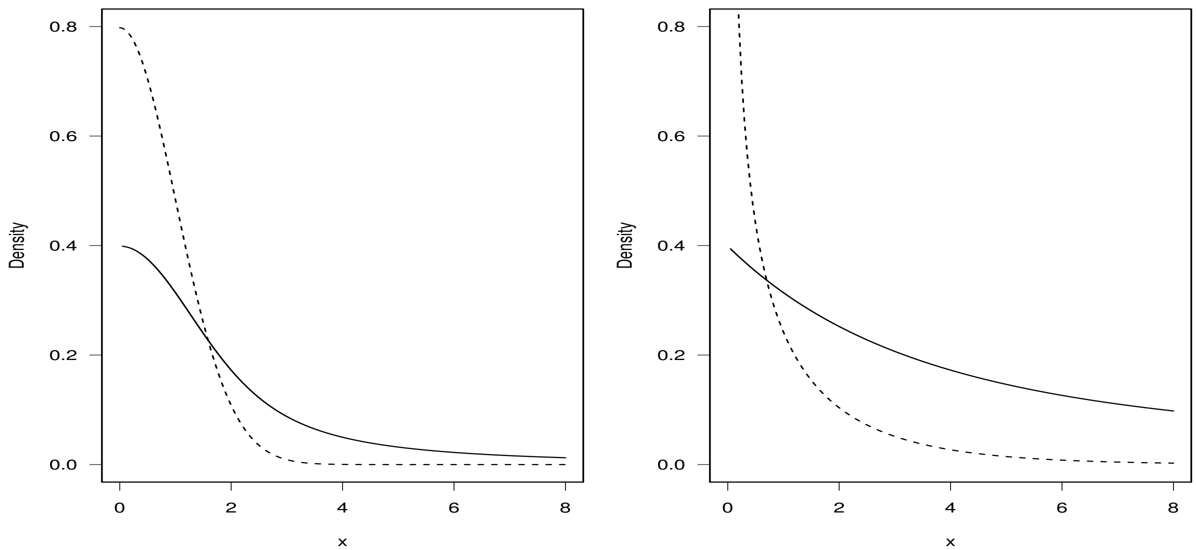

Figure 2: *The left panel compares the canonical half-slash (solid line) with the half-normal (dashed line) distributions. The right panel shows the canonical slash-*$\chi^2(1)$ *(solid line) and the* χ_1^2 *(dashed line) distributions.*

2.3 Moments and Kurtosis

We study now properties of moments and kurtosis of the slash-elliptical family. From (3), and using results by Gupta and Varga (1993), it is easy to show that for $X \sim SEl(0,1,q;g)$, $Y \sim SEl(\mu,\sigma,q;g)$, and $q > r$ we get

$$\mu_r = \begin{cases} \left(\frac{q}{i^r(q-r)}\right) \frac{r!}{(r/2)!} a_{r/2} & r \text{ even} \\ 0 & r \text{ odd} \end{cases} \qquad \text{and} \qquad \mu'_r = \sum_{k=0}^{r} \binom{r}{k} \sigma^k \mu^{r-k} \mu_k. \tag{8}$$

with $\mu_r = E(X^r)$, $\mu'_r = E(Y^r)$, $i = \sqrt{-1}$, and $a_{r/2} = d^{r/2}\, g(x)/d\, x^{r/2}\, |_{x=0}$ for $r = 1, 2, \ldots$.

Remark 2 An immediate consequence of (8) is that the mean and variance of a random variable $Y \sim SEl(\mu,\sigma,q;g)$ are given by

$$E(Y) = \mu \qquad \text{and} \qquad Var(Y) = \frac{-2q}{q-2} a_1 \sigma^2, \qquad q > 2. \tag{9}$$

Similarly, the kurtosis coefficient of a random variable $Y \sim SEl(\mu,\sigma,q;g)$ is given by

$$\beta_2 = \frac{3(q-2)^2 a_2}{q(q-4)a_1^2}, \qquad q > 4. \tag{10}$$

Table 2 shows the values of the a_1 and a_2 functions and the supported range (over q) of kurtosis (10) for some usual cases.

Slash-Distribution	a_1	a_2	Kurtosis Range
Normal	$-1/2$	$1/4$	$3 < \beta_2 < \infty$
Cauchy	-1	2	$6 < \beta_2 < \infty$
t-Student	$-\frac{1+\nu}{2\nu}$	$\frac{(1+\nu)(3+\nu)}{4\nu^2}$	$\frac{3(3+\nu)}{(1+\nu)} < \beta_2 < \infty$
Pearson	$1-\alpha$	$(\alpha-1)(\alpha-2)$	$\frac{3(\alpha-2)}{(\alpha-1)} < \beta_2 < \infty$

Table 2: *Kurtosis range for some special cases.*

Proposition 3 *Let $Y_1, \ldots, Y_n$ be a random sample from the $SEl(\mu, \sigma, q; g)$ distribution. Then, the moment estimates of $\boldsymbol{\theta} = (\mu, \sigma^2, q)$ are given by*

$$\widehat{\mu}_n = \overline{Y}, \qquad \widehat{\sigma}_n^2 = \frac{(2-\widehat{q}_n)S^2}{2\widehat{q}_n a_1}, \qquad \textit{and} \qquad \widehat{q}_n = 2\left(1 + |a_1|\sqrt{\frac{b_2}{H}}\right),$$

where b_2 is the sample kurtosis coefficient, $\overline{Y}$ and S^2 are the sample mean and variance, respectively, and $H = b_2 a_1^2 - 3a_2 > 0$.

Proof. From (9) it follows that

$$\mu = E(Y) \qquad \text{and} \qquad \sigma^2 = \frac{(2-q)Var(Y)}{2qa_1}, \tag{11}$$

and plugging the sample kurtosis coefficient b_2 in (10) we get

$$b_2 = \frac{3(\widehat{q}-2)^2 a_2}{\widehat{q}(\widehat{q}-4)a_1^2}. \tag{12}$$

Then, solving (12) for $\widehat{q}$ we find

$$\widehat{q}_n = 2\left(1 + |a_1|\sqrt{\frac{b_2}{b_2 a_1^2 - 3a_2}}\right).$$

Finally, the result comes from plugging the corresponding sampling quantities in (11). □

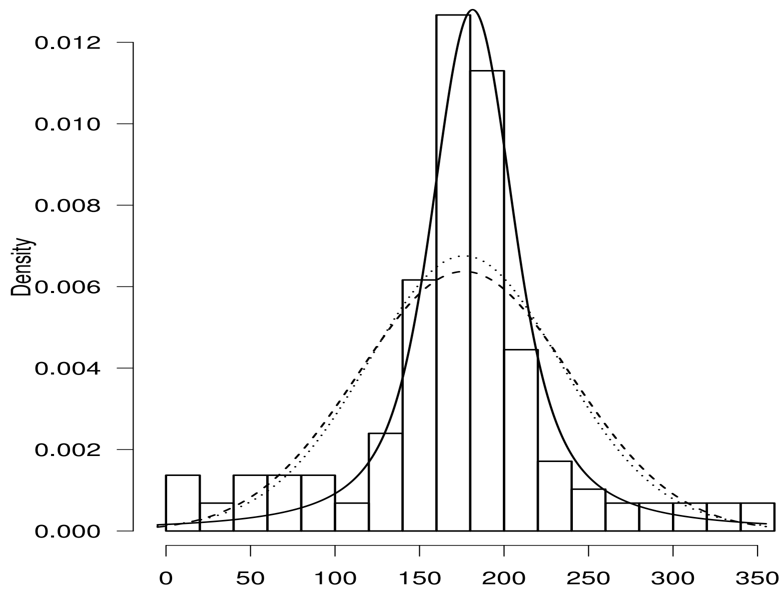

Figure 3: *Histogram of initial directions for $n = 730$ ants. Overlaid on top is the slash-normal density with parameters estimated via maximum likelihood (solid line) and the method of moments (dotted line). For comparison, we include a standard normal (dashed line).*

Remark 3 Unlike the method of moments, it is not generally possible, for a given generator g, to derive explicit expressions for the MLEs corresponding to a sample $Y_1, \ldots, Y_n \sim SEl(\mu, \sigma, q; g)$. The corresponding likelihood function $L(\mu, \sigma, q)$ is too complex, even for the simplest choices of g, and therefore numerical methods must be used to compute the MLEs.

Example 2 We consider here data coming from an entomology experiment concerning ants. A total of $n = 730$ ants were placed singly in the center of an arena. The measurements correspond to the initial direction in which they moved in relation to a visual stimulus at an angle of 180 degrees from the zero direction, rounded to the nearest 10 degrees. The data were first presented in Jander (1957), and later analyzed in Batschelet (1981), SenGupta and Pal (2001) and in Jones and Pewsey (2004). Figure 3 shows the histogram of these data, including estimated densities under a regular normal model and a slash-normal model, using maximum likelihood and the method of moments. Moments estimates for μ, σ and q are, respectively, 176.44, 49.68, and 5.3. We present these here only for reference. See Remark 4 below. The other point estimates are summarized in Table 3. The lines are defined as the corresponding density function, substituting in the parameters by their estimated values. It can be clearly seen that the solid line (MLE) fits the data much better than the other two.

Parameter	MLE (normal)	MLE (Slash-Normal)
μ	176.44 (2.317)	181.43 (1.269)
σ	62.60 (3.277)	16.80 (1.246)
q	-	1.17 (0.086)

Table 3: *Point estimates of parameters in Example 2, with standard errors within parentheses.*

Remark 4 An interesting aspect of the inferential procedure is that the existence of moment estimates assumes $q > 4$, which is reflected on the estimated values. The MLEs do not require this restriction and can therefore adjust better to the data being analyzed.

3 Multivariate Slash-Elliptical Distributions

We consider now the multivariate extension of the slash-elliptical family introduced earlier. Recall that $\mathbf{Y} = (Y_1, \ldots, Y_k)^T$ has elliptical distribution with location vector $\boldsymbol{\mu}$ and positive definite scale matrix $\boldsymbol{\Sigma}$ if its joint density is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = |\boldsymbol{\Sigma}|^{-1/2} g\left((\mathbf{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right), \qquad \mathbf{y} \in \mathbb{R}^k,$$

where now g represents the k-variate density generator function, assumed to satisfy the condition $\int_0^\infty u^{k-1} g(u^2)\, du < \infty$. We use the notation $\mathbf{Y} \sim El_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$. If the appropriate moments of $\mathbf{Y}$ are finite, then $E(\mathbf{Y}) = \boldsymbol{\mu}$ and $Var(\mathbf{Y}) = \alpha\boldsymbol{\Sigma}$, where α is a positive constant that depends on g (see, e.g., Fang et al. 1990).

Motivated by the univariate representation (3), we say that a random vector $\mathbf{Y} \in \mathbb{R}^k$ has slash-elliptical multivariate distribution with vector location parameter $\boldsymbol{\mu}$, positive definite matrix scale parameter $\mathbf{\Sigma}$, and tail parameter $q > 0$, if it can be represented as

$$\mathbf{Y} = \mathbf{\Sigma}^{1/2} \frac{\mathbf{X}}{U^{1/q}} + \boldsymbol{\mu}, \tag{13}$$

where $\mathbf{X} \sim El_k(\mathbf{0}, \mathbf{I}_k; g)$ is assumed independent of $U \sim U(0,1)$. We denote this as $\mathbf{Y} \sim SEl_k(\boldsymbol{\mu}, \mathbf{\Sigma}, q; g)$. In (13) and in what follows $\mathbf{\Sigma}^{1/2}$ refers to the *square root* of $\mathbf{\Sigma}$, that is, the symmetric matrix verifying $\mathbf{\Sigma}^{1/2}\mathbf{\Sigma}^{1/2} = \mathbf{\Sigma}$.

We give next the general expression for the corresponding density function.

Proposition 4 *Let $\mathbf{Y} \sim SEl_k(\boldsymbol{\mu}, \mathbf{\Sigma}, q; g)$. Then, the density function of $\mathbf{Y}$ is given by*

$$f(\mathbf{y}; \boldsymbol{\mu}, \mathbf{\Sigma}, q) = \begin{cases} \frac{q|\Sigma|^{-1/2}}{2\gamma^{(q+k)/2}} \int_0^\gamma t^{\frac{q+k-2}{2}} g(t)\, dt & \mathbf{y} \neq \boldsymbol{\mu} \\ \frac{q}{k+q} |\Sigma|^{-1/2} g(0) & \mathbf{y} = \boldsymbol{\mu}, \end{cases} \tag{14}$$

where $\gamma = \|\mathbf{\Sigma}^{-1/2}(\mathbf{y} - \boldsymbol{\mu})\|^2 = (\mathbf{y} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu})$.

Proof. From (13) we can easily find the density function for $\mathbf{Y}$ to be given by

$$\begin{aligned} f(\mathbf{y}; \boldsymbol{\mu}, \mathbf{\Sigma}, q) &= q \int_0^1 u^{q+k-1} f_k(u\mathbf{y}; u\boldsymbol{\mu}, \mathbf{\Sigma})\, du \\ &= q \int_0^1 u^{q+k-1} |\mathbf{\Sigma}|^{-1/2} g\left((\mathbf{y} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu}) u^2 \right) du, \end{aligned}$$

from which the result for the case $\mathbf{y} = \boldsymbol{\mu}$ is immediate. The other case follows by considering the transformation $\mathbf{t} = (\mathbf{y} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu}) u^2$. $\square$

Example 3 Taking $g(t) = \frac{1}{(2\pi)^{k/2}} e^{-t/2}$, i.e., the generating function corresponding to the multivariate normal model, (14) becomes the multivariate slash model introduced by Wang and Genton (2006).

We give next the mean and covariance matrix of the multivariate slash-elliptical family. The proof is straightforward and therefore omitted.

Proposition 5 *Let $\mathbf{Y} \sim SEl_k(\boldsymbol{\mu}, \mathbf{\Sigma}, q; g)$. Then*

$$E(\mathbf{Y}) = \boldsymbol{\mu}, \qquad \textit{and} \qquad Var(\mathbf{Y}) = \left(\frac{q\alpha}{q-2} \right) \mathbf{\Sigma}, \qquad \textit{if } q > 2.$$

4 Discussion

We have introduced a new family of slash distributions aimed at supporting extended kurtosis for univariate and multivariate distributions. The basic idea is to replace the normal assumption underlying the definition of standard slash distributions by elliptically contoured alternatives. The resulting family is quite flexible, including all the elliptical distributions as special cases.

We presented a simple illustration showing the flexibility of one special case of slash-elliptical distribution, when compared to its non-slash version. We are currently developing further applications involving linear and non-linear regression models.

Finally, the class of slash-elliptical distributions can be further extended by considering distributions of the form $Z/h(U)$, where $Z \sim El(0,1;g)$ is independent of $U \sim U(0,1)$, and h is a positive and monotonic measurable function which may depend on additional parameters q (we discussed the case $h(u) = u^{1/q}$). Such extension would provide even more modeling flexibility. By carefully choosing h one may, for instance, obtain heavier tails. It is not clear though that in doing so we effectively get extended ranges of kurtosis, our basic motivation when proposing the slash-elliptical family. Such considerations are currently subject to additional study.

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