Low Energy Asymptotics of the SSF for Pauli Operators with Nonconstant Magnetic Fields

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1. Introduction

Let

$\mathbf{A} \in C^1(\mathbb{R}^3; \mathbb{R}^3)$ be a magnetic potential,

$\mathbf{B} := \operatorname{curl} \mathbf{A} \in C(\mathbb{R}^3; \mathbb{R}^3) \cap L^\infty(\mathbb{R}^3; \mathbb{R}^3)$ be the magnetic field generated by $\mathbf{A}$.

Denote $\widehat{\sigma} := (\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\sigma}_3)$ where

$\widehat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\widehat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\widehat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

are the Pauli matrices.

Then

$$H_0 := (\widehat{\sigma} \cdot (-i\nabla - \mathbf{A}))^2$$

is the unperturbed 3D Pauli operator, essentially self-adjoint on $C_0^\infty(\mathbb{R}^3; \mathbb{C}^2)$.

Introduce the Hermitian matrix

$$V(\mathbf{x}) := \begin{pmatrix} v_{11}(\mathbf{x}) & v_{12}(\mathbf{x}) \\ v_{21}(\mathbf{x}) & v_{22}(\mathbf{x}) \end{pmatrix}, \quad \mathbf{x} \in \mathbb{R}^3,$$

with $v_{jk} \in L^\infty(\mathbb{R}^3)$, $j, k = 1, 2$.

On the domain of H_0 define the perturbed 3D Pauli operator

$$H := H_0 + V.$$

2. The Spectral Shift Function (SSF)

Assume that

$$(H - i)^{-1} - (H_0 - i)^{-1} \in S_1$$

where S_1 denotes the trace class.

A sufficient condition:

$$v_{jk} \in L^1(\mathbb{R}^3), \quad j, k = 1, 2.$$

Then there exists a unique $\xi = \xi(\cdot; H, H_0) \in L^1(\mathbb{R}; (1 + E^2)^{-1} dE)$ which vanishes identically on $(-\infty, \inf \sigma(H))$, such that *the Lifshits-Krein trace formula*

$$\mathrm{Tr}\,(f(H) - f(H_0)) = \int_{\mathbb{R}} \xi(E; H, H_0) f'(E) dE$$

holds for each $f \in C_0^\infty(\mathbb{R})$.

The function $\xi(\cdot; H, H_0)$ is called *the spectral shift function* (SSF) for the operator pair (H, H_0).

For almost every $E < \inf \sigma(H_0)$ we have

$$\xi(E; H, H_0) = -\mathrm{Tr}\, \mathbf{1}_{(-\infty,E)}(H).$$

On the other hand, for almost every $E \in \sigma_{\mathrm{ac}}(H_0)$, the SSF $\xi(E; H, H_0)$ is related to the scattering determinant $\det S(E; H, H_0)$ by *the Birman-Krein formula*

$$\det S(E; H, H_0) = e^{-2\pi i \xi(E;H,H_0)}. \tag{1}$$

Our further assumptions on B will imply

$$\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = [0, \infty).$$

Moreover, it can be shown that under these assumptions there exists $\mathcal{C} > 0$, such that the SSF $\xi(\cdot; H, H_0)$ is bounded on every compact subset of $(-\infty, 0) \cup (0, \mathcal{C})$, and continuous on $(-\infty, \mathcal{C}) \setminus (\{0\} \cup \sigma_{\rm pp}(H))$.

The problem:

For perturbations V of definite sign V, investigate the asymptotic behavior of the SSF $\xi(E; H, H_0)$ as $E \downarrow 0$ and as $E \uparrow 0$.

3. Admissible Magnetic Fields

3.1. Admissible magnetic fields
Suppose that the magnetic field $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ has a constant direction, i.e.

$$\mathbf{B} = (0, 0, b). \tag{2}$$

By the Maxwell equation, $\operatorname{div} \mathbf{B} = 0$, we should then have $\frac{\partial b}{\partial x_3} = 0$. Assume that

$$b \in C(\mathbb{R}^2; \mathbb{R}) \cap L^\infty(\mathbb{R}^2),$$

and $b = b_0 + \tilde{b}$ where $b_0 > 0$ is a constant, while $\tilde{b} : \mathbb{R}^2 \to \mathbb{R}$ is such that the Poisson equation

$$\Delta \tilde{\varphi} = \tilde{b} \tag{3}$$

admits a solution $\tilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}$, continuous and bounded together with its derivatives of order up to two. Then we will say that b is *an admissible* magnetic field.

Leading example:

$$\tilde{b}(x) = \int_{\mathbb{R}^2} e^{i\lambda\cdot x} d\nu(\lambda), \quad x \in \mathbb{R}^2,$$

where ν is a Borel charge on $\mathbb{R}^2$ such that:

- $|\nu|(\mathbb{R}^2) < \infty$;
- $\nu(\delta) = \overline{\nu(-\delta)}$ for each Borel set $\delta \subset \mathbb{R}^2$;
- $\nu(\{0\}) = 0$;
- $\int_{\mathbb{R}^2} |\lambda|^{-2} d|\nu|(\lambda) < \infty$.

Then the Poisson equation (3) admits the solution

$$\tilde{\varphi}(x) := -\int_{\mathbb{R}^2} |\lambda|^{-2} e^{i\lambda\cdot x} d\nu(\lambda), \quad x \in \mathbb{R}^2, \quad (4)$$

which has all the required properties.

Set

$$\varphi_0(x) := b_0(x_1^2 + x_2^2)/4, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

$$\varphi := \varphi_0 + \tilde{\varphi},$$

Then $\Delta\varphi_0 = b_0$ and $\Delta\varphi = b$.

Put $\mathbf{A} := (A_1, A_2, A_3)$ with

$$A_1 := -\frac{\partial\varphi}{\partial x_2}, \quad A_2 := \frac{\partial\varphi}{\partial x_1}, \quad A_3 = 0. \tag{5}$$

Then

$$\operatorname{curl}\mathbf{A} = \mathbf{B} = (0, 0, b).$$

Changing, if necessary, the gauge, we will assume that the magnetic potential $\mathbf{A}$ in H_0 is given by (5).

3.2. Spectral properties of the operator H_0

Let

$$a = a(b) := -2ie^{-\varphi}\frac{\partial}{\partial \overline{z}}\, e^{\varphi},$$

$$a^* = a(b)^* := -2ie^{\varphi}\frac{\partial}{\partial z}\, e^{-\varphi},$$

with

$$z := x_1 + ix_2, \quad \overline{z} := x_1 - ix_2,$$

be respectively the annihilation and the creation operators. Set

$$H_\perp^- = H_\perp^-(b) := a^*a, \quad H_\perp^+ = H_\perp^+(b) := aa^*,$$

$$H_\perp = H_\perp(b) := \begin{pmatrix} H_\perp^- & 0 \\ 0 & H_\perp^+ \end{pmatrix} = H_\perp^- \oplus H_\perp^+.$$

Then we have

$$\text{Ker } H_\perp^- = \text{Ker } a =$$

$$\left\{ u \in L^2(\mathbb{R}^2) | u = g e^{-\varphi},\ \frac{\partial g}{\partial \overline{z}} = 0 \right\},$$

$$\text{Ker } H_\perp^+ = \text{Ker } a^* =$$

$$\left\{ u \in L^2(\mathbb{R}^2) | u = g e^{\varphi},\ \frac{\partial g}{\partial z} = 0 \right\},$$

$$\text{Ker } H_\perp =$$

$$\left\{ \mathbf{u} = (u_1, u_2) | u_1 \in \text{Ker } H_\perp^-,\ u_2 \in \text{Ker } H_\perp^+ \right\}.$$

Since $b_0 > 0$, and $\tilde{\varphi} \in L^\infty(\mathbb{R}^2)$, we have

$$\dim \text{Ker } H_\perp^- = \infty, \quad \dim \text{Ker } H_\perp^+ = 0.$$

Proposition 1. *Let b be an admissible magnetic field. Then*

$$\dim \operatorname{Ker} H_\perp = \infty, \tag{6}$$

and

$$(0, \mathcal{C}) \subset \varrho(H_\perp)$$

with

$$\mathcal{C} := 2b_0 \exp\left(-2 \operatorname{osc} \tilde{\varphi}\right), \tag{7}$$

where $\operatorname{osc} \tilde{\varphi} := \sup_{x \in \mathbb{R}^2} \tilde{\varphi}(x) - \inf_{x \in \mathbb{R}^2} \tilde{\varphi}(x)$.

Further,

$$H_0 = \begin{pmatrix} H_0^- & 0 \\ 0 & H_0^+ \end{pmatrix} :=$$

$$\begin{pmatrix} (-i\nabla - \mathbf{A})^2 - b & 0 \\ 0 & (-i\nabla - \mathbf{A})^2 + b \end{pmatrix}.$$

We have

$$H_0^{\pm} = H_{\perp}^{\pm} \otimes I_{\|} + I_{\perp} \otimes H_{\|} \tag{8}$$

where $I_{\|}$ and $I_{\perp}$ are the identity operators in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$ respectively, and

$$H_{\|} := -\frac{d^2}{dx_3^2}$$

is the operator essentially self-adjoint on $C_0^{\infty}(\mathbb{R})$.

Corollary 2. *Assume that b is an admissible magnetic field. Then the spectrum $\sigma(H_0)$ of the operator H_0 coincides with $[0, \infty)$, and is purely absolutely continuous.*

4. Main Results

4.1. Berezin-Toeplitz operators

Let $p(b)$ be the spectral projection onto $\mathrm{Ker}\, a$.

The low-energy asymptotic behaviour of the SSF will be described in the terms of effective Hamiltonians which are compact operators of Berezin-Toeplitz type $p(b)Up(b)$ with $U \in L^\infty(\mathbb{R}^2;\mathbb{R})$, $\lim_{|x|\to\infty} U(x) = 0$.

Lemma 3. *Let $U \in L^q(\mathbb{R}^2)$, $q \in [1,\infty)$. Assume that b is an admissible magnetic field. Then $p(b)Up(b) \in S_q$ where S_q denotes the Schatten - von Neumann class, and*

$$\|p(b)Up(b)\|_q^q \leq \frac{b_0}{2\pi} e^{2\,\mathrm{osc}\,\tilde{\varphi}} \|U\|_{L^q}^q.$$

4.2. Statement of the main results

For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ we set $x := (x_1, x_2)$. For $j, k = 1, 2$ assume

$$v_{jk} \in C(\mathbb{R}^3),$$

$$|v_{jk}(\mathbf{x})| \leq C_0 \langle x \rangle^{-m_\perp} \langle x_3 \rangle^{-m_3}, \quad \mathbf{x} \in \mathbb{R}^3, \quad (9)$$

with $C_0 > 0$, $m_\perp > 2$, $m_3 > 1$, or

$$v_{jk} \in C(\mathbb{R}^3), \quad |v_{jk}(\mathbf{x})| \leq C_0 \langle \mathbf{x} \rangle^{-m}, \quad \mathbf{x} \in \mathbb{R}^3, \quad (10)$$

with $m > 3$. Note that (10) implies (9) with any $m_3 \in (0, m)$ and $m_\perp = m - m_3$.
Suppose, moreover,

$$V(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathbb{R}^3. \quad (11)$$

In the sequel we will consider the operators $H_0 + V$ or $H_0 - V$.

Assume that (9) with $m_\perp > 2$, $m_3 > 1$, and (11) hold true. Set

$$W(x) := \int_{\mathbb{R}} v_{11}(x, x_3) dx_3, \quad x \in \mathbb{R}^2.$$

For $E > 0$ introduce the operator

$$\omega(E) := \frac{1}{2\sqrt{E}} p(b) W p(b).$$

Let $T = T^*$ be compact operator. Set

$$n_\pm(s; T) := \operatorname{Tr} \mathbf{1}_{(s,\infty)}(\pm T), \quad s > 0.$$

Theorem 4. *Let V satisfy* (10) *with $m > 3$, and* (11)*. Assume that b is an admissible magnetic field. Then for each $\varepsilon \in (0, 1)$ we have*

$$-n_+((1-\varepsilon); \omega(E)) + O(1) \leq$$

$$\xi(-E; H_0 - V, H_0) \leq$$

$$-n_+((1+\varepsilon); \omega(E)) + O(1), \quad E \downarrow 0.$$

Suppose again that V satisfies (9) with $m_\perp > 2$, $m_3 > 1$, and (11). For $E > 0$ and $x \in \mathbb{R}^2$ set

$$\mathcal{W}_E = \mathcal{W}_E(x) := \begin{pmatrix} w_{11}(x) & w_{12}(x) \\ w_{21}(x) & w_{22}(x) \end{pmatrix},$$

where

$$w_{11}(x) := \int_{\mathbb{R}} v_{11}(x, x_3) \cos^2(\sqrt{E} x_3) dx_3,$$

$$w_{22}(x) := \int_{\mathbb{R}} v_{11}(x, x_3) \sin^2(\sqrt{E} x_3) dx_3,$$

$$w_{12}(x) = w_{21}(x) :=$$

$$\int_{\mathbb{R}} v_{11}(x, x_3) \cos(\sqrt{E} x_3) \sin(\sqrt{E} x_3) dx_3.$$

Set

$$\Omega(E) := \frac{1}{2\sqrt{E}} p(b) \mathcal{W}_E\, p(b).$$

Evidently, $\Omega(E) = \Omega(E)^* \geq 0$ in $L^2(\mathbb{R}^2; \mathbb{C}^2)$, and $\Omega(E) \in S_1$.

Theorem 5. *Let V satisfy* (10) *with $m > 3$ and* (11)*. Assume that b is an admissible magnetic field. Then for each $\varepsilon \in (0,1)$ we have*

$$\pm\frac{1}{\pi}\mathrm{Tr}\,\arctan\left((1\pm\varepsilon)^{-1}\Omega(E)\right)+O(1)\leq$$

$$\xi(E;H_0\pm V,H_0)\leq$$

$$\pm\frac{1}{\pi}\mathrm{Tr}\,\arctan\left((1\mp\varepsilon)^{-1}\Omega(E)\right)+O(1),\quad E\downarrow 0.$$

4.3. Eigenvalue asymptotics for compact Berezin-Toeplitz operators

In the following three lemmas we will discuss the eigenvalue asymptotics of compact Berezin-Toeplitz operators $p(b)Up(b)$, with $0 \leq U \in L^\infty(\mathbb{R}^2; \mathbb{R})$, $U(x) \to 0$ as $|x| \to \infty$.

More precisely, we will be concerned with the asymptotics of $n_+(s; p(b)Up(b))$ as $s \downarrow 0$.

The first lemma treats the case where the decay of U at infinity is power-like.

It involves the concept of *an integrated density of states* (IDS) for the operator $H_\perp^-(b)$. Let $\chi_{T,\tau}$ be the characteristic function of the square $\tau + \left(-\frac{T}{2}, \frac{T}{2}\right)^2$ with $\tau \in \mathbb{R}^2$ and $T > 0$. Tthe non-increasing function $\varrho_b : \mathbb{R} \to [0, \infty)$ is called IDS for the operator $H_\perp^-(b)$, if for each $\tau \in \mathbb{R}^2$ it satisfies

$$\varrho_b(E) = \lim_{T\to\infty} T^{-2} \mathrm{Tr}\,(\chi_{T,\tau} \mathbf{1}_{(-\infty,E)}(H_\perp^-(b))\chi_{T,\tau})$$

at its continuity points $E \in \mathbb{R}$.

Lemma 6. *Let* $U \in C^1(\mathbb{R}^2)$ *satisfy*

$$0 \le U(x) \le C_1 \langle x \rangle^{-\alpha},$$

$$|\nabla U(x)| \le C_1 \langle x \rangle^{-\alpha-1}, \quad x \in \mathbb{R}^2,$$

for $\alpha > 0$ *and* $C_1 > 0$*. Assume, moreover, that:*

- $U(x) = u_0(x/|x|)|x|^{-\alpha}(1 + o(1))$ *as* $|x| \to \infty$*, where* $0 < u_0 \in C(\mathbb{S}^1)$*;*
- b *is an admissible magnetic field;*
- *there exists an IDS* ϱ_b *for the operator* $H_\perp^-(b)$*.*

Then we have

$$n_+(s; p(b)Up(b)) = \Psi_\alpha(s)\,(1 + o(1)),\ s \downarrow 0,$$

where

$$\Psi_\alpha(s) = \Psi_\alpha(s; u_0, b_0) := s^{-2/\alpha} \frac{b_0}{4\pi} \int_{\mathbb{S}^1} u_0(\theta)^{2/\alpha} d\theta.$$

Our second lemma concerns the case where U decays exponentially at infinity.

Lemma 7. *Let* $0 \le U \in L^\infty(\mathbb{R}^2)$. *Assume that*

$$\ln U(x) = -\eta |x|^{2\beta}(1 + o(1)), \quad |x| \to \infty,$$

for some $\beta \in (0, \infty)$, $\eta \in (0, \infty)$. *Let* b *be an admissible magnetic field. Then we have*

$$n_+(s; p(b)Up(b)) = \Phi_\beta(s)(1 + o(1)), \quad s \downarrow 0,$$

where

$$\Phi_\beta(s) = \Phi_\beta(s; \eta, b_0) :=$$

$$\begin{cases} \frac{b_0}{2\eta^{1/\beta}} |\ln s|^{1/\beta} & \text{if} \quad 0 < \beta < 1, \\ \frac{1}{\ln(1 + 2\eta/b_0)} |\ln s| & \text{if} \quad \beta = 1, \\ \frac{\beta}{\beta - 1} (\ln |\ln s|)^{-1} |\ln s| & \text{if} \quad 1 < \beta < \infty. \end{cases}$$

Our third lemma concerns the case where U has a compact support.

Lemma 8. *Let* $0 \leq U \in L^\infty(\mathbb{R}^2)$. *Assume that the support of* U *is compact, and there exists a constant* $C > 0$ *such that* $U \geq C$ *on an open non-empty subset of* $\mathbb{R}^2$. *Let* b *be an admissible magnetic field. Then we have*

$$n_+(s; p(b)Up(b)) = \Phi_\infty(s)\,(1 + o(1)), \quad s \downarrow 0,$$

where

$$\Phi_\infty(s) := (\ln|\ln s|)^{-1}|\ln s|. \qquad (12)$$

4.4. Corollaries from the main results

Corollary 9. *Let V satisfy* (10) *with $m > 3$, and* (11).
(i) Assume that the hypotheses of Lemma 6 hold with $U = W$ and $\alpha = m - 1$. Then

$$\xi(-E; H_0 - V, H_0) = -\Psi_{m-1}(2\sqrt{E}; u_0, b_0)\,(1 + o(1))$$

as $E \downarrow 0$.
(ii) Assume that the hypotheses of Lemma 7 hold with $U = W$. Then we have

$$\xi(-E; H_0 - V, H_0) = -\Phi_\beta(2\sqrt{E}; \eta, b_0)\,(1 + o(1))$$

for $\beta \in (0, \infty)$ as $E \downarrow 0$.
(iii) Assume that the hypotheses of Lemma 8 hold with $U = W$. Then we have

$$\xi(-E; H_0 - V, H_0) = -\Phi_\infty(2\sqrt{E})\,(1 + o(1))$$

as $E \downarrow 0$.

Corollary 10. *(i) Let V satisfy* (10) *with $m > 3$, and* (11)*. Assume that the hypotheses of Lemma 6 are fulfilled for $U = W$ and $\alpha = m-1$. Then, $E \downarrow 0$, we have*

$$\xi(E; H_0 \pm V, H_0) =$$

$$\pm \frac{1}{2\cos(\pi/(m-1))} \Psi_{m-1}(2\sqrt{E}; u_0, b_0)\,(1+o(1)).$$

(ii) Let V satisfy (10) *with $m > 3$,* (11)*, and* (9) *for some $m_\perp > 2$ and $m_3 > 2$. Assume that the hypotheses of Lemma 7 are fulfilled for $U = W$. Then, for $E \downarrow 0$, and $\beta \in (0, \infty)$, we have*

$$\xi(E; H_0 \pm V, H_0) = \pm \frac{1}{2} \Phi_\beta(2\sqrt{E}; \eta, b_0)\,(1+o(1)).$$

(iii) Let the assumptions of the previous part be fulfilled, except that the hypotheses of Lemma 7 are replaced by those of Lemma 8. Then, for $E \downarrow 0$ we have

$$\xi(E; H_0 \pm V, H_0) = \pm \frac{1}{2} \Phi_\infty(2\sqrt{E})\,(1 + o(1)).$$

Corollary 11. *Under the assumptions of Corollary 10 (i) we have*

$$\lim_{E \downarrow 0} \frac{\xi(E; H_0 - V, H_0)}{\xi(-E; H_0 - V, H_0)} = \frac{1}{2\cos(\pi/(m-1))},$$

with $m > 3$*, while under the assumptions of Corollary 10 (ii)–(iii) we have*

$$\lim_{E \downarrow 0} \frac{\xi(E; H_0 - V, H_0)}{\xi(-E; H_0 - V, H_0)} = \frac{1}{2}.$$

The above formulae could be interpreted as *generalized Levinson formulae*. The classical Levinson formula relates the (finite) limiting values as $E \uparrow 0$ and $E \downarrow 0$ of the SSF $\xi(E; -\Delta + V; -\Delta)$ where Δ is the Laplacian in $\mathbb{R}^d$, $d \geq 1$, and $V : \mathbb{R}^d \to \mathbb{R}$ is a scalar potential which decays fast enough at infinity.

5. Idea of the Proofs of the Main Results

5.1. A representation of the SSF

Assume that V satisfies (11) and (9). Put

$$L(\mathbf{x}) = \left\{\ell_{jk}(\mathbf{x})\right\}_{j,k=1}^{2} := V(\mathbf{x})^{1/2}, \quad \mathbf{x} \in \mathbb{R}^3.$$

For $z \in \mathbb{C}_+ := \{\zeta \in \mathbb{C} \,|\, \mathrm{Im}\,\zeta > 0\}$, set

$$T(z) := L(H_0 - z)^{-1} L.$$

Then for $E \in (-\infty, 0) \cup (0, \mathcal{C})$ the operator-norm limit

$$T(E + i0) := \mathrm{n} - \lim_{\delta \downarrow 0} T(E + i\delta)$$

exists, and

$$\mathrm{Im}\, T(E + i0) \in S_1.$$

Due to a general abstract result of A. Pushnitski, we have

$$\xi(E; H_0 \pm V, H_0) =$$

$$\pm \int_{\mathbb{R}} n_{\mp}(1; \mathrm{Re}\, T(E+i0) + t\, \mathrm{Im}\, T(E+i0))\, d\mu(t),$$

for $E \in (-\infty, \mathcal{C}) \setminus \{0\}$ with $d\mu(t) := \frac{dt}{\pi(1+t^2)}$.

5.2. Estimates of sandwiched resolvents

Introduce the orthogonal projections

$$P = P(b) := p \otimes I_{||}, \quad Q = Q(b) := I - P,$$

acting in $L^2(\mathbb{R}^3)$, and the orthogonal projections

$$\mathbf{P} = \mathbf{P}(b) := \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{Q} = \mathbf{Q}(b) := \mathbf{I} - \mathbf{P} = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix},$$

acting in $L^2(\mathbb{R}^3; \mathbb{C}^2)$.
For $z \in \mathbb{C}_+$ introduce the operators

$$T_<(z) := L\mathbf{P}(H_0 - z)^{-1}L,$$

$$T_>(z) := L\mathbf{Q}(H_0 - z)^{-1}L,$$

bounded on $L^2(\mathbb{R}^3; \mathbb{C}^2)$.

Proposition 12. *Let V satisfy* (9) *with $m_\perp > 2$, $m_3 > 1$, and* (11). *Then $T_<$ extends to a continuous operator-valued function $\overline{\mathbb{C}_+} \setminus \{0\} \ni z \mapsto T_<(z) \in S_1$. Moreover,*

$$\|T_<(E)\|_1 \leq C_1(1+E_+^{1/4})|E|^{-1/2}, \quad E \in \mathbb{R}\setminus\{0\},$$

with C_1 independent of E.

Proposition 13. *Let V satisfy the assumptions of Proposition 12. Then $T_>$ extends to an analytic operator-valued function $\mathbb{C} \setminus [\mathcal{C}, \infty) \ni z \mapsto T_>(z) \in S_2$. Moreover,*

$$\|T_>(E)\|_2 \leq C_2\left(1 + \frac{(E+1)_+}{\mathcal{C} - E}\right), \quad E \in (-\infty, \mathcal{C}),$$

with C_2 independent of E.

5.3. Low-energy expansion of the sandwiched resolvent of H_0

(i) In the proof of Theorem 4 we obtain the estimates

$$\xi(-E; H_0 - V, H_0) = -n_+(1; T(-E)) \sim$$

$$-n_+(1; T_<(-E)) \sim$$

$$-n_+(1; \mathcal{O}(E)) = -n_+(1; \omega(E)), \quad E \downarrow 0,$$

where $\mathcal{O}(E) : L^2(\mathbb{R}^3; \mathbb{C}^2) \to L^2(\mathbb{R}^3; \mathbb{C}^2)$ is the operator with matrix-valued integral kernel

$$\frac{1}{2\sqrt{E}} \ell_{j1}(x, x_3)\, \mathcal{P}_b(x, x') \ell_{1k}(x', x_3'),$$

$$j, k = 1, 2, \quad (x, x_3), (x', x_3') \in \mathbb{R}^3.$$

(ii) In the proof of Theorem 5 we obtain the estimates

$$\xi(E; H_0 \pm V, H_0) \sim$$

$$\pm \int_{\mathbb{R}} n_{\mp}(1; \operatorname{Re} T_<(E) + t \operatorname{Im} T_<(E))\, d\mu(t) \sim$$

$$\pm \int_{\mathbb{R}} n_{\mp}(1; t \operatorname{Im} T_<(E))\, d\mu(t) =$$

$$\pm \frac{1}{\pi} \operatorname{Tr} \arctan\, (\operatorname{Im} T_<(E)) =$$

$$\pm \frac{1}{\pi} \operatorname{Tr} \arctan\, (\Omega(E)), \quad E \downarrow 0.$$

In both cases we bear in mind the identity

$$n_+(s; K^*K) = n_+(s; KK^*)$$

where $s > 0$ and $K : X_1 \to X_2$ is a linear compact operator, acting between the Hilbert spaces X_1 and X_2.