Reviews in Mathematical Physics, Vol. 14, No. 10 (2002) 1051–1072

QUASI-CLASSICAL VERSUS NON-CLASSICAL SPECTRAL ASYMPTOTICS FOR MAGNETIC SCHRÖDINGER OPERATORS WITH DECREASING ELECTRIC POTENTIALS

GEORGI D. RAIKOV

Departamento de Matemáticas, Universidad de Chile,
Las Palmeras 3425, Casilla 653, Santiago, Chile
graykov@uchile.cl

SIMONE WARZEL

Institut für Theoretische Physik, Universität Erlangen-Nürnberg,
Staudtstrasse 7, D-91058 Erlangen, Germany
simone.warzel@physik.uni-erlangen.de

Received 25 December 2001
Revised 28 June 2002

We consider the Schrödinger operator $H(V)$ on $L^2(\mathbb{R}^2)$ or $L^2(\mathbb{R}^3)$ with constant magnetic field, and electric potential V which typically decays at infinity exponentially fast or has a compact support. We investigate the asymptotic behaviour of the discrete spectrum of $H(V)$ near the boundary points of its essential spectrum. If the decay of V is Gaussian or faster, this behaviour is non-classical in the sense that it is not described by the quasi-classical formulas known for the case where V admits a power-like decay.

Keywords: Magnetic Schrödinger operators; spectral asymptotics.

Mathematics Subject Classification 2000: 35P20, 47B35

1. Introduction

Let $H(0) := (-i\nabla - A)^2$ be the Schrödinger operator with constant magnetic field of strength $b > 0$, essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, $d = 2, 3$. The magnetic potential A is chosen in the form

$$A(\mathbf{x}) = \begin{cases} \left(-\dfrac{by}{2}, \dfrac{bx}{2}\right) & \text{if } d = 2\,, \\ \left(-\dfrac{by}{2}, \dfrac{bx}{2}, 0\right) & \text{if } d = 3\,. \end{cases}$$

In the two-dimensional case we identify the magnetic field with $\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = b$, while in the three-dimensional case we identify it with curl $A = (0, 0, b)$. Moreover, if $d = 2$, we write $\mathbf{x} = (x, y) \in \mathbb{R}^2$, and if $d = 3$, we write $\mathbf{x} = (X_\perp, z)$ with $X_\perp = (x, y) \in \mathbb{R}^2$ and $z \in \mathbb{R}$. Thus, in the latter case, z is the variable along the magnetic field, while $X_\perp$ are the variables on the plane perpendicular to it. Introducing the sequence of Landau levels $E_q := (2q+1)b$, $q \in \mathbb{Z}_+ := \{0, 1, \ldots\}$, we recall [7, 3] that

$$\sigma(H(0)) = \sigma_{\text{ess}}(H(0)) = \begin{cases} \cup_{q=0}^{\infty}\{E_q\} & \text{if } d = 2\,, \\ [E_0, \infty) & \text{if } d = 3\,. \end{cases} \tag{1.1}$$

Here $\sigma(H(0))$ denotes the spectrum of the operator $H(0)$, and $\sigma_{\text{ess}}(H(0))$ denotes its essential spectrum.

Let $V : \mathbb{R}^d \to \mathbb{R}$ be a measurable, non-negative function which decays at infinity in a suitable sense, so that the operator $V^{1/2}H(0)^{-1/2}$ is compact. By Weyl's theorem, $\sigma_{\text{ess}}(H(0)) = \sigma_{\text{ess}}(H(\pm V))$ where $H(\pm V) := H(0) \pm V$, and $\pm V$ is the electric potential of constant (positive or negative) sign.

The aim of the article is to investigate the behaviour of the discrete spectrum of the operator $H(\pm V)$ near the boundary points of its essential spectrum. This behaviour has been extensively studied in the literature in case where V admits power-like or slower decay at infinity (see [15, 16, 17, 18], [12, Chaps. 11 and 12]) and also in the special case where $d = 3$ and V is axially symmetric with respect to the magnetic field (see [3, 21]). The novelty in the present paper is that we consider V's which decay exponentially fast or have compact support and which at most asymptotically obey a certain symmetry. If $d = 3$, this type of decay of V is supposed to take place in the directions perpendicular to the magnetic field while the decay in the z-direction could be much more general (see Theorems 2.3 and 2.4 below). If the decay of V in the (x, y)-directions is Gaussian or super-Gaussian, we show that the discrete-spectrum behaviour of $H(\pm V)$ is not described by quasi-classical formulas known for the case of power-like decay.

The results of the present paper have been announced in [20]. After the initial submission of the paper, we became aware of the preprint [19]. It deals with the eigenvalue asymptotics for the Schrödinger and Dirac operators with full-rank magnetic fields, and compactly supported electric potentials of fixed sign. In particular, [19] extends our Theorem 2.2 to the case of *full-rank* magnetic fields in arbitrary even dimension. The methods of proof applied in [19] are variational ones similar to those used in the present paper.

This paper is organized as follows. In Sec. 2 we formulate our main results. Section 3 is devoted to the analysis of the eigenvalue asymptotics for compact operators of Toeplitz type. Section 4 contains the proofs of the results concerning the two-dimensional case. Finally, the proofs of the results for the three-dimensional case can be found in Sec. 5.

2. Formulation of Main Results

2.1. *Basic notation*

In order to formulate our main results we need the following notations. Let T be a linear self-adjoint operator. Denote by $\mathbb{P}_I(T)$ the spectral projection of T

corresponding to the open interval $I \subset \mathbb{R}$. Set

$$N(\lambda_1, \lambda_2; T) := \text{rank } \mathbb{P}_{(\lambda_1, \lambda_2)}(T)\,, \quad \lambda_1, \lambda_2 \in \mathbb{R}\,, \quad \lambda_1 < \lambda_2\,,$$
$$N(\lambda; T) := \text{rank } \mathbb{P}_{(-\infty, \lambda)}(T)\,, \quad \lambda \in \mathbb{R}\,.$$

If T is compact, we will also use the notations

$$n_{\pm}(s; T) := \text{rank } \mathbb{P}_{(s,\infty)}(\pm T)\,, \quad s > 0\,. \tag{2.1}$$

By $\|.\|$ we denote the usual operator norm, and by $\|.\|_{\mathrm{HS}}$ the Hilbert–Schmidt norm.

2.2. *Main results for two dimensions*

This subsection contains our main results related to the two-dimensional case.

Theorem 2.1. *Let V be bounded and non-negative on $\mathbb{R}^2$. Assume that there exist two constants $0 < \mu < \infty$ and $0 < \beta < \infty$ such that*

$$\lim_{|\mathbf{x}| \to \infty} \frac{\ln V(\mathbf{x})}{|\mathbf{x}|^{2\beta}} = -\mu\,. \tag{2.2}$$

Moreover, fix a Landau level E_q, $q \in \mathbb{Z}_+$, and an energy $E' \in (E_q, E_{q+1})$.

(i) *If $0 < \beta < 1$, then we have*

$$\lim_{E \downarrow 0} \frac{N(E_q + E, E'; H(V))}{|\ln E|^{1/\beta}} = \frac{b}{2\mu^{1/\beta}}\,. \tag{2.3}$$

(ii) *If $\beta = 1$, then we have*

$$\lim_{E \downarrow 0} \frac{N(E_q + E, E'; H(V))}{|\ln E|} = \frac{1}{\ln(1 + 2\mu/b)}\,. \tag{2.4}$$

(iii) *If $1 < \beta < \infty$, then we have*

$$\lim_{E \downarrow 0} \frac{N(E_q + E, E'; H(V))}{(\ln|\ln E|)^{-1}|\ln E|} = \frac{\beta}{\beta - 1}\,. \tag{2.5}$$

The proof of Theorem 2.1 can be found in Sec. 4.2. It is evident from this proof that Theorem 2.1(iii) admits the following generalization as the asymptotic coefficient in (2.5) is independent of μ.

Corollary 2.1. *Let V be bounded and non-negative on $\mathbb{R}^2$. Assume that there exist $0 < \mu_1 < \mu_2 < \infty$ and $1 < \beta < \infty$ such that*

$$-\mu_2 \le \liminf_{|\mathbf{x}| \to \infty} \frac{\ln V(\mathbf{x})}{|\mathbf{x}|^{2\beta}}\,, \quad \limsup_{|\mathbf{x}| \to \infty} \frac{\ln V(\mathbf{x})}{|\mathbf{x}|^{2\beta}} \le -\mu_1\,.$$

Then (2.5) remains valid.

The last theorem of this subsection concerns the case where V has a compact support.

Theorem 2.2. *Let V be bounded and non-negative on $\mathbb{R}^2$. Assume that the support of V is compact, and there exists a constant $C_- > 0$ such that $V \geq C_-$ on an open non-empty subset of $\mathbb{R}^2$. Moreover, let $q \in \mathbb{Z}_+$ and $E' \in (E_q, E_{q+1})$. Then we have*

$$\lim_{E \downarrow 0} \frac{N(E_q + E, E'; H(V))}{(\ln|\ln E|)^{-1}|\ln E|} = 1 \,. \tag{2.6}$$

The proof of Theorem 2.2 is contained in Sec. 4.3.

Remark 2.1. Under the hypotheses of Theorems 2.1 or 2.2 we have $V \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. It is well-known that this inclusion implies that the operator $V^{1/2}(-\Delta + 1)^{-1/2}$ is compact. Hence, it follows from the diamagnetic inequality (see e.g. [3]) that the operator $V^{1/2}H(0)^{-1/2}$ is compact as well.

For further references, we introduce some additional notation which allows us to unify (2.3)–(2.6) into a single formula. For $\kappa \in (e, \infty)$ define the increasing functions $a_\mu^{(\beta)}$ by

$$a_\mu^{(\beta)}(\kappa) := \begin{cases} \dfrac{b}{2}\left(\dfrac{\kappa}{\mu}\right)^{1/\beta} & \text{if } 0 < \beta < 1 \,, \\[2mm] \dfrac{\kappa}{\ln(1 + 2\mu/b)} & \text{if } \beta = 1 \,, \\[2mm] \dfrac{\beta}{\beta - 1}\dfrac{\kappa}{\ln \kappa} & \text{if } 1 < \beta < \infty \,, \\[2mm] \dfrac{\kappa}{\ln \kappa} & \text{if } \beta = \infty \,. \end{cases} \tag{2.7}$$

Then asymptotic relations (2.3)–(2.6) can be re-written as

$$\lim_{E \downarrow 0} \frac{N(E_q + E, E'; H(V))}{a_\mu^{(\beta)}(|\ln E|)} = 1 \,, \quad 0 < \beta \leq \infty \,. \tag{2.8}$$

Remark 2.2. Whenever we refer to functions (2.7) with $1 < \beta \leq \infty$, we will write $a^{(\beta)}(\kappa)$ instead of $a_\mu^{(\beta)}(\kappa)$ because in this case they are independent of μ.

Let us discuss the results of Theorems 2.1 and 2.2.

(1) Asymptotic relation (2.8) describes the behaviour of the infinite sequence of discrete eigenvalues of the operator $H(V)$ accumulating to the Landau level E_q, $q \in \mathbb{Z}_+$, from the right. Analogous results hold if we consider the eigenvalues of $H(-V)$ accumulating to E_q from the left. Namely, (2.8) remains valid if we replace $N(E_q + E, E'; H(V))$ by $N(E'', E_q - E; H(-V))$ with some $E'' \in (E_{q-1}, E_q)$ if $q > 0$, or by $N(E_0 - E; H(-V))$ if $q = 0$.

(2) Introduce the quasi-classical quantity

$$\mathcal{N}_{\rm cl}(E) := \frac{b}{2\pi}|\{\mathbf{x} \in \mathbb{R}^2 | V(\mathbf{x}) > E\}| \,, \quad E > 0 \,,$$

where $|\cdot|$ denotes the Lebesgue measure. If $V \geq 0$ satisfies the asymptotics $V(\mathbf{x}) = \frac{\mathbf{v}(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^\alpha}(1 + o(1))$ as $|\mathbf{x}| \to \infty$ with some $\mathbf{v} \in C(\mathbb{S}^1)$, $\mathbf{v} > 0$, and some $0 < \alpha < \infty$, then $\lim_{E\downarrow 0} E^{2/\alpha}\mathcal{N}_{\rm cl}(E) = \frac{b}{4\pi}\int_{\mathbb{S}^1} \mathbf{v}(s)^{2/\alpha}\, ds$, and it has been shown that

$$\lim_{E\downarrow 0} \frac{N(E_q + E, E'; H(V))}{\mathcal{N}_{\rm cl}(E)} = 1\,, \tag{2.9}$$

assuming some regularity of $\mathcal{N}_{\rm cl}(E)$ as $E \downarrow 0$ (see [15, Theorem 2.6], [12, Chap. 11]). On the other hand, if V satisfies the assumptions of Theorem 2.1, then

$$\lim_{E\downarrow 0} \frac{\mathcal{N}_{\rm cl}(E)}{|\ln E|^{1/\beta}} = \frac{b}{2\mu^{1/\beta}}\,, \quad 0 < \beta < \infty\,,$$

and if V satisfies the assumptions of Theorem 2.2, then

$$\mathcal{N}_{\rm cl}(E) = O(1)\,, \quad E \downarrow 0\,.$$

Comparing (2.8) and (2.9), we see that they are different if and only if $1 \leq \beta \leq \infty$. In case $\beta = 1$ the asymptotic orders of (2.8) and (2.9) coincide but their coefficients differ although they have the same main asymptotic term in the strong magnetic field regime $b \to \infty$. In brief, asymptotic relation (2.8) is quasi-classical for potentials V whose decay is slower than Gaussian ($0 < \beta < 1$), and it is non-classical for potentials whose decay is faster than Gaussian ($1 < \beta \leq \infty$), while the Gaussian decay ($\beta = 1$) of V is the border-line case.

A similar transition from quasi-classical to non-classical behaviour as a function of the decay of the single-site potential with Gaussian decay as the border-line case has been detected in [10]. There the leading low-energy fall-off of the integrated density of states of a charged quantum particle in $\mathbb{R}^2$ subject to a perpendicular constant magnetic field and repulsive impurities randomly distributed according to Poisson's law has been considered.

(3) The assumptions of Theorems 2.1 and 2.2 that V be bounded and non-negative are not quite essential. For example, both theorems remain valid if we consider potentials $|\mathbf{x}|^{-\alpha}V(\mathbf{x})$ where $0 < \alpha < 2$, and V satisfies the hypotheses of Theorems 2.1 or 2.2. Similarly, Theorem 2.1 holds also in the case where V is allowed to change sign on a compact subset of $\mathbb{R}^2$.

(4) Let $\pi(\lambda)$ be the number of primes less than $\lambda > 0$. It is well-known that

$$\lim_{\lambda\to\infty} \frac{\pi(\lambda)}{(\ln\lambda)^{-1}\lambda} = 1$$

(see e.g. [9, Sec. 1.8, Theorem 6]). Hence, (2.6) can be re-written as

$$\lim_{E\downarrow 0} \frac{N(E_q + E, E'; H(V))}{\pi(|\ln E|)} = 1\,.$$

2.3. *Main results for three dimensions*

In this subsection we formulate our main results concerning the case $d = 3$. In this case we will analyze the behaviour of $N(E_0 - E; H(-V))$ as $E \downarrow 0$. In order to define properly the operator $H(-V)$ we need the following lemma.

Lemma 2.1. *Let $U \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, and $v \in L^1(\mathbb{R})$. Assume that $0 \leq V(X_\perp, z) \leq U(X_\perp)v(z)$, $X_\perp \in \mathbb{R}^2$, $z \in \mathbb{R}$. Then the operator $V^{1/2}H(0)^{-1/2}$ is compact.*

The proof of the lemma is elementary. Nevertheless, for the reader's convenience we include it in Sec. 5.2.

Denote by $H(-V)$ the self-adjoint operator generated in $L^2(\mathbb{R}^3)$ by the quadratic form

$$\int_{\mathbb{R}^3} \{|i\nabla u + Au|^2 - V|u|^2\}\, d\mathbf{x}\,, \quad u \in D(H(0)^{1/2})\,,$$

which is closed and lower bounded in $L^2(\mathbb{R}^3)$ since the operator $V^{1/2}H(0)^{-1/2}$ is compact by Lemma 2.1.

Theorem 2.3. *Let $0 < \mu < \infty$ and $0 < \beta < \infty$. Assume that there exist a constant $C > 0$ and a function $v \in L^1(\mathbb{R}; (1 + |z|)\, dz)$, which does not vanish identically, such that*

$$0 \leq V(\mathbf{x}) \leq Cv(z)\,, \quad \mathbf{x} = (X_\perp, z) \in \mathbb{R}^3\,.$$

Moreover, suppose that for every $\delta > 0$ there exist a constant $r_\delta > 0$ and two non-negative functions $v_\delta^\pm \in L^1(\mathbb{R}; (1 + |z|)\, dz)$, which do not vanish identically, such that

$$e^{-\delta|X_\perp|^{2\beta}} v_\delta^-(z) \leq e^{\mu|X_\perp|^{2\beta}} V(X_\perp, z) \leq e^{\delta|X_\perp|^{2\beta}} v_\delta^+(z)$$

for all $|X_\perp| \geq r_\delta$ and all $z \in \mathbb{R}$. Then we have

$$\lim_{E\downarrow 0} \frac{N(E_0 - E; H(-V))}{a_\mu^{(\beta)}(|\ln\sqrt{E}|)} = 1\,. \tag{2.10}$$

The proof of Theorem 2.3 can be found in Sec. 5.4.

Our last theorem treats the case where the projection of the support of V onto the plane perpendicular to the magnetic field is compact. Denote by $\chi_{r,X'_\perp} : \mathbb{R}^2 \to \mathbb{R}$ the characteristic function of the disk $\{X_\perp \in \mathbb{R}^2 |\, |X_\perp - X'_\perp| < r\}$ of radius $r > 0$, centered at $X'_\perp \in \mathbb{R}^2$. If $X'_\perp = 0$, we will write χ_r instead of $\chi_{r,0}$.

Theorem 2.4. *Assume that there exist four constants $r_\pm > 0$, $X_\perp^\pm \in \mathbb{R}^2$, and two non-negative functions $v^\pm \in L^1(\mathbb{R}; (1 + |z|)\, dz)$, which do not vanish identically, such that V obeys the estimates*

$$\chi_{r_-,X_\perp^-}(X_\perp)v^-(z) \leq V(\mathbf{x}) \leq \chi_{r_+,X_\perp^+}(X_\perp)v^+(z)\,, \quad \mathbf{x} = (X_\perp, z) \in \mathbb{R}^3\,.$$

Then we have

$$\lim_{E\downarrow 0} \frac{N(E_0 - E; H(-V))}{a^{(\infty)}(|\ln\sqrt{E}|)} = 1\,. \tag{2.11}$$

The proof of Theorem 2.4 is contained in Sec. 5.5.

Let us discuss briefly the above results.

(1) In particular, Theorem 2.3 covers bounded negative potentials $-V$ which decay at infinity exponentially fast, i.e.

$$\lim_{|\mathbf{x}|\to\infty} \frac{\ln V(\mathbf{x})}{|\mathbf{x}|^{2\beta}} = -\mu\,, \tag{2.12}$$

with some $0 < \beta < \infty$ and $0 < \mu < \infty$.

(2) Assume that $V \geq 0$ satisfies the asymptotics $V(\mathbf{x}) = \frac{\mathbf{v}(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^\alpha}(1 + o(1))$ as $|\mathbf{x}| \to \infty$ with some $\mathbf{v} \in C(\mathbb{S}^2)$, $\mathbf{v} > 0$, and some $2 < \alpha < \infty$. For $E > 0$ set

$$\tilde{\mathcal{N}}_{\mathrm{cl}}(E) := \frac{b}{2\pi} \left| \left\{ X_\perp \in \mathbb{R}^2 \,\middle|\, \int_{\mathbb{R}} V(X_\perp, z)\, dz > 2\sqrt{E} \right\} \right| .$$

Under some supplementary regularity assumptions concerning the behaviour of $\tilde{\mathcal{N}}_{\mathrm{cl}}(E)$ as $E \downarrow 0$ we have

$$\lim_{E\downarrow 0} \frac{N(E_0 - E; H(-V))}{\tilde{\mathcal{N}}_{\mathrm{cl}}(E)} = 1 \tag{2.13}$$

(see [17], [18, Theorem 1(ii)], [15, Theorem 2.4(i)], [12, Chap. 12]). Theorem 2.3 shows that (2.13) remains valid if the decay of V is slower than Gaussian in the sense that (2.12) holds with $0 < \beta < 1$. On the other hand, if this decay is Gaussian or faster in the sense that (2.12) holds with $\beta = 1$ or $1 < \beta \leq \infty$, the asymptotics of $N(E_0 - E; H(-V))$ as $E \downarrow 0$ differs from (2.13).

3. Spectra of Auxiliary Operators of Toeplitz Type

3.1. *Landau Hamiltonian and angular-momentum eigenstates*

Let $d = 2$. In this case, by (1.1) the spectrum of $H(0)$ consists of the eigenvalues E_q, $q \in \mathbb{Z}_+$, which are of infinite multiplicity. Denote by P_q, $q \in \mathbb{Z}_+$, the spectral projection of $H(0)$ corresponding to the eigenvalue E_q. Our next goal is to introduce convenient orthonormal bases of the subspaces $P_q L^2(\mathbb{R}^2)$.

For $\mathbf{x} \in \mathbb{R}^2$, $q \in \mathbb{Z}_+$, and $k \in \mathbb{Z}_+ - q := \{-q, -q+1, \ldots\}$ set

$$\varphi_{q,k}(\mathbf{x}) := \sqrt{\frac{q!}{(k+q)!}} \left[\sqrt{\frac{b}{2}}(x + iy) \right]^k L_q^{(k)}\left(\frac{b|\mathbf{x}|^2}{2} \right) \sqrt{\frac{b}{2\pi}} \exp\left(-\frac{b|\mathbf{x}|^2}{4} \right) \tag{3.1}$$

where

$$L_q^{(\alpha)}(\xi) := \sum_{m=0}^{q} \binom{q+\alpha}{q-m} \frac{(-\xi)^m}{m!}\,, \quad \xi \geq 0\,, \tag{3.2}$$

are the generalized Laguerre polynomials (see e.g. [8, Sec. 8.97]) which are defined in terms of the binomial coefficients $\binom{\alpha}{m} := \alpha(\alpha-1)\cdot\ldots\cdot(\alpha-m+1)/m!$ if $m \in \mathbb{Z}_+\backslash\{0\}$, and $\binom{\alpha}{0} := 1$, for all $\alpha \in \mathbb{R}$. It is well-known that the functions $\varphi_{q,k}$, $k \in \mathbb{Z}_+ - q$, constitute an orthonormal basis in the qth Landau-level eigenspace $P_q L^2(\mathbb{R}^2)$, $q \in \mathbb{Z}_+$ (see e.g. [7, 11]). In fact, $\varphi_{q,k}$ is also an eigenfunction of the angular-momentum operator $-i(x\partial/\partial y - y\partial/\partial x)$ with eigenvalue k.

For further references we establish some useful properties of the Laguerre polynomials $L_q^{(\alpha)}$. We first recall [1, Sec. 22.2.12] their orthogonality relation

$$\int_0^\infty \xi^\alpha e^{-\xi} L_q^{(\alpha)}(\xi) L_{q'}^{(\alpha)}(\xi)\, d\xi = \frac{\Gamma(\alpha+q+1)}{q!}\delta_{q,q'} \tag{3.3}$$

valid for all q, $q' \in \mathbb{Z}_+$ and $\alpha > -1$. Here we have introduced Kronecker's delta $\delta_{q,q'}$ and Euler's gamma function $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,dt$, $s > 0$, such that $\Gamma(k+1) = k!$ if $k \in \mathbb{Z}_+$, see e.g. [1, Chap. 6].

Lemma 3.1. *Let $q \in \mathbb{Z}_+$. Then*

$$|L_q^{(k)}(\xi)| \le (k+q)^q e^{\xi/(k+q)} \tag{3.4}$$

holds for all $\xi \ge 0$ and all $k \ge 1-q$. Moreover, one has the uniform convergence

$$\lim_{k\to\infty} k^{-q} L_q^{(k)}(k\xi) = \frac{(1-\xi)^q}{q!} \tag{3.5}$$

for all $0 \le \xi \le 1$.

Remark 3.1. An immediate consequence of (3.5) is the following lower bound on the pre-limit expression

$$k^{-q} L_q^{(k)}(k\xi) \ge \frac{(1-\xi_0)^q}{2q!} \tag{3.6}$$

which is valid for all $0 \le \xi \le \xi_0 < 1$ and sufficiently large k.

Proof of Lemma 3.1. The rough upper bound (3.4) is taken from [11, Eq. (42)]. For a proof of (3.5) we use (3.2) to obtain

$$k^{-q} L_q^{(k)}(k\xi) = \sum_{m=0}^{q} k^{m-q} \binom{q+k}{q-m} \frac{(-\xi)^m}{m!}\,. \tag{3.7}$$

Asymptotic relation [1, Eq. 6.1.46] entails

$$\lim_{k\to\infty} k^{m-q} \frac{\Gamma(k+q)}{\Gamma(k+m)} = 1\,. \tag{3.8}$$

The r.h.s. of (3.7) thus converges (uniformly on $[0,1]$) towards $\sum_{m=0}^{q}\binom{q}{m}(-\xi)^m/q! = (1-\xi)^q/q!$ by the binomial formula. □

For $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$ denote by $K_q(\mathbf{x},\mathbf{x}') := \sum_{k=-q}^{\infty} \varphi_{q,k}(\mathbf{x})\overline{\varphi_{q,k}(\mathbf{x}')}$ the integral kernel of the projection P_q, $q \in \mathbb{Z}_+$. It is well-known that

$$K_q(\mathbf{x},\mathbf{x}') = \frac{b}{2\pi} L_q^{(0)}\left(\frac{b|\mathbf{x}-\mathbf{x}'|^2}{2}\right) \exp\left(-\frac{b}{4}(|\mathbf{x}-\mathbf{x}'|^2 + 2i(x'y - xy'))\right) \tag{3.9}$$

(see e.g. [11]). Note that we have

$$K_q(\mathbf{x},\mathbf{x}) = \frac{b}{2\pi}\,, \quad \mathbf{x}\in\mathbb{R}^2\,, \quad q\in\mathbb{Z}_+\,. \tag{3.10}$$

3.2. *Compact operators of Toeplitz type*

In this subsection we investigate the eigenvalue asymptotics of auxiliary compact operators of Toeplitz type P_qFP_q where $q\in\mathbb{Z}_+$ and F is the multiplier by a real-valued function. The results obtained here will be essentially employed in the proofs of Theorems 2.1–2.4.

First of all, note that $P_qFP_q = e^{2(2q+1)bt}P_qe^{-tH(0)}Fe^{-tH(0)}P_q$, $t>0$, $q\in\mathbb{Z}_+$. Hence, the diamagnetic inequality implies that P_qFP_q is compact if the operator $|F|^{1/2}e^{\Delta t}$ (or, equivalently, $e^{\Delta t}|F|^{1/2}$) is compact for some $t>0$ (see [3, Theorems 2.2, 2.3]). In particular, the following lemma holds.

Lemma 3.2 ([15, Lemma 5.1]). *Let F be real-valued and $F\in L^p(\mathbb{R}^2)$ for some $p\geq 1$. Then the operator P_qFP_q, $q\in\mathbb{Z}_+$, is self-adjoint and compact.*

Lemma 3.3. *Let $F:\mathbb{R}^2\to\mathbb{R}$ satisfy the conditions of Lemma 3.2. Suppose in addition that F is radially symmetric with respect to the origin, and bounded. Then the eigenvalues of the operator P_qFP_q with domain $P_qL^2(\mathbb{R}^2)$, $q\in\mathbb{Z}_+$, are given by*

$$\langle F\varphi_{q,k},\varphi_{q,k}\rangle = \frac{q!}{(k+q)!}\int_0^\infty F((\sqrt{2\xi/b},0))e^{-\xi}\xi^k L_q^{(k)}(\xi)^2\,d\xi\,, \quad k\in\mathbb{Z}_+-q\,, \tag{3.11}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$.

Proof. It suffices to take into account (3.1) and the radial symmetry of F. □

Remark 3.2. Evidently, Lemma 3.3 is valid under more general assumptions. In particular, the boundedness condition is unnecessarily restrictive. However, we state the lemma in a simple form which is sufficient for our purposes.

3.3. *Two examples of explicit eigenvalue asymptotics*

For $\mathbf{x}\in\mathbb{R}^2$ set $G_\mu^{(\beta)}(\mathbf{x}) := \exp(-\mu|\mathbf{x}|^{2\beta})$ where $0<\mu<\infty$ and $0<\beta<\infty$. According to Lemma 3.3 the eigenvalues of $P_qG_\mu^{(\beta)}P_q$ on $P_qL^2(\mathbb{R}^2)$ are given by

$$\gamma_{q,k}^{(\beta)}(\mu) := \langle G_\mu^{(\beta)}\varphi_{q,k},\varphi_{q,k}\rangle\,, \quad k\in\mathbb{Z}_+-q\,. \tag{3.12}$$

Let $(a_\mu^{(\beta)})^{-1}$ denote the inverse function of $a_\mu^{(\beta)}$ defined in (2.7). Evidently,

$$(a_\mu^{(\beta)})^{-1}(k) = \begin{cases} \mu\left(\dfrac{2k}{b}\right)^\beta & \text{if } 0<\beta<1\,, \\ k\,\ln(1+2\mu/b) & \text{if } \beta=1\,. \end{cases} \tag{3.13}$$

Moreover, it is straightforward to verify that

$$\lim_{k\to\infty} \frac{(a^{(\beta)})^{-1}(k)}{k \ln k} = \begin{cases} \dfrac{\beta-1}{\beta} & \text{if } 1<\beta<\infty\,, \\ 1 & \text{if } \beta=\infty\,. \end{cases} \tag{3.14}$$

The next proposition treats the asymptotics of $\gamma_{q,k}^{(\beta)}(\mu)$, $q \in \mathbb{Z}_+$, as $k \to \infty$. For $q = 0$ and $0 < \beta \le 1/2$ closely related asymptotic evaluations can be found in [21, Appendix].

Proposition 3.1. *Let $q \in \mathbb{Z}_+$, $0 < \mu < \infty$, and $0 < \beta < \infty$. Then we have*

$$\lim_{k\to\infty} \frac{\ln \gamma_{q,k}^{(\beta)}(\mu)}{(a_\mu^{(\beta)})^{-1}(k)} = -1\,. \tag{3.15}$$

Proof. From (3.12) and Lemma 3.3 it follows that

$$\gamma_{q,k}^{(\beta)}(\mu) = \frac{q!\,k!}{(k+q)!} \mathcal{J}^{(\beta)}(k, \mu(2/b)^\beta)$$

where we have introduced the notation

$$\mathcal{J}^{(\beta)}(k,\lambda) := \frac{1}{k!} \int_0^\infty \xi^k e^{-\lambda\xi^\beta - \xi} L_q^{(k)}(\xi)^2 \, d\xi\,. \tag{3.16}$$

Thanks to asymptotic relation (3.8) it remains to study the asymptotic behaviour of $\mathcal{J}^{(\beta)}$ for large values of its first argument. For this purpose we distinguish three cases.

Case 0 $<\beta<$ 1. The claim follows from (3.8) and (3.13) with $0 < \beta < 1$, together with the asymptotic relation

$$\lim_{k\to\infty} \frac{\ln \mathcal{J}^{(\beta)}(k,\lambda)}{k^\beta} = -\lambda \tag{3.17}$$

valid for $\lambda > 0$ in this case. For a proof of (3.17) we construct asymptotically coinciding lower and upper bounds. To obtain a lower bound we suppose $k > -1$. The orthogonality relation (3.3) implies that $\xi^k e^{-\xi} L_q^{(k)}(\xi)^2 q!/(k+q)!\, d\xi$ induces a probability measure on $[0,\infty]$ such that Jensen's inequality [14] yields

$$\mathcal{J}^{(\beta)}(k,\lambda) \ge \frac{(k+q)!}{k!\,q!} \exp\left\{ -\lambda \frac{q!}{(k+q)!} \int_0^\infty \xi^{k+\beta} e^{-\xi} L_q^{(k)}(\xi)^2 \, d\xi \right\}. \tag{3.18}$$

We may now employ the combinatorial identity $L_q^{(k)}(\xi) = \sum_{m=0}^q \binom{m-\beta-1}{m} L_{q-m}^{(k+\beta)}(\xi)$ [8, Eq. 8.974(2)], which implies that

$$\begin{aligned} &\frac{q!}{(k+q)!} \int_0^\infty \xi^{k+\beta} e^{-\xi} L_q^{(k)}(\xi)^2 \, d\xi \\ &\quad = \sum_{m,l=0}^q \binom{m-\beta-1}{m} \binom{l-\beta-1}{l} \frac{q!}{(k+q)!} \int_0^\infty \xi^{k+\beta} e^{-\xi} L_{q-m}^{(k+\beta)}(\xi) L_{q-l}^{(k+\beta)}(\xi)\, d\xi \\ &\quad = \sum_{m=0}^q \binom{m-\beta-1}{m}^2 \frac{q!}{(q-m)!} \frac{\Gamma(k+q-m+\beta+1)}{\Gamma(k+q+1)}\,. \end{aligned} \tag{3.19}$$

Here we have again used the orthogonality relation (3.3) in the last step. Using (3.8) this entails $\liminf_{k\to\infty} k^{-\beta} \ln \mathcal{J}^{(\beta)}(k,\lambda) \geq -\lambda$.

For the upper bound we suppose $k+q>2$ and choose Ξ_k as the (unique) maximum of the integrand in the r.h.s. of the estimate

$$\mathcal{J}^{(\beta)}(k,\lambda) \leq \frac{(k+q)^{2q}}{k!} \int_0^\infty \xi^k e^{-\lambda\xi^\beta - (1-2/(k+q))\xi}\, d\xi \tag{3.20}$$

which was obtained by using (3.4). More precisely, we define Ξ_k as the (unique) solution of the equation $\lambda\beta\Xi_k^\beta + (1-2/(k+q))\Xi_k = k$. Splitting the integration in (3.20) into two parts with domain of integration restricted to $[0,\Xi_k)$ and $[\Xi_k,\infty)$, the two parts are estimated separately as follows. Using monotonicity of the integrand on $[0,\Xi_k)$ we obtain the bound

$$\begin{aligned}\frac{1}{k!}\int_0^{\Xi_k} \xi^k e^{-\lambda\xi^\beta-(1-2/(k+q))\xi}\, d\xi &\leq \frac{\Xi_k^{k+1}}{k!}\exp[-\lambda\Xi_k^\beta - (1-2/(k+q))\Xi_k] \\ &= \Xi_k \frac{k^k}{k!}\exp[k\ \ln[\Xi_k/k] - (1-2/(k+q))\Xi_k - \lambda\Xi_k^\beta] \\ &\leq \Xi_k \frac{k^k}{k!} e^{-k} \exp[-\lambda\Xi_k^\beta + 2\Xi_k/(k+q)] \end{aligned}\tag{3.21}$$

on the first part. For the last inequality we have used the fact that $\ln\xi \leq \xi - 1$ for all $\xi > 0$. The second part is bounded according to

$$\begin{aligned}\frac{1}{k!}\int_{\Xi_k}^\infty \xi^k e^{-\lambda\xi^\beta-(1-2/(k+q))\xi}\, d\xi &\leq \exp[-\lambda\Xi_k^\beta]\int_0^\infty \frac{\xi^k}{k!} e^{-(1-2/(k+q))\xi}\, d\xi \\ &= (1-2/(k+q))^{-k-1}\exp[-\lambda\Xi_k^\beta]\,. \end{aligned}\tag{3.22}$$

The sandwiching bounds $1-\lambda\beta k^{\beta-1} \leq (1-2/(k+q))\Xi_k/k \leq 1$ imply $\lim_{k\to\infty}\Xi_k/k = 1$. Using this in (3.21) and (3.22), employing Stirling's asymptotic formula [1, Eq. 6.1.37]

$$\lim_{k\to\infty} \frac{k^{k-1/2}}{\Gamma(k)} e^{-k} = (2\pi)^{-1/2}\,, \tag{3.23}$$

and the fact that $\lim_{k\to\infty}(1+2/k)^k = e^2$, we obtain $\limsup_{k\to\infty} k^{-\beta}\ln\mathcal{J}^{(\beta)}(k,\lambda) \leq -\lambda$. This concludes the proof of (3.17).

Case $\boldsymbol{\beta} = \mathbf{1}$. An explicit calculation yields

$$\begin{aligned}\mathcal{J}^{(1)}(k,\lambda) &= \frac{1}{k!}\int_0^\infty \xi^k e^{-(1+\lambda)\xi} L_q^{(k)}(\xi)^2\, d\xi \\ &= \frac{1}{k!}\sum_{m,l=0}^{q} \binom{q+k}{q-m}\binom{q+k}{q-l}\frac{(-1)^{m+l}}{m!\,l!}\int_0^\infty \xi^{k+m+l}e^{-(1+\lambda)\xi}\, d\xi \\ &= \sum_{m,l=0}^{q} \binom{q+k}{q-m}\binom{q+k}{q-l}\frac{(-1)^{m+l}}{m!\,l!}\frac{(k+l+m)!}{k!}(1+\lambda)^{-k-m-l-1}\,. \end{aligned}\tag{3.24}$$

Using (3.8) and proceeding similarly as in the second part of the proof of Lemma 3.1 one shows that the r.h.s. is asymptotically equal to

$$(1+\lambda)^{-k-1}\frac{k^{2q}}{(q!)^2}\left[\sum_{m=0}^{q}\binom{q}{m}\frac{(-1)^m}{(1+\lambda)^m}\right]^2 = (1+\lambda)^{-k-2q-1}\frac{(\lambda k)^{2q}}{(q!)^2} \tag{3.25}$$

which in turn implies that $\lim_{k\to\infty} k^{-1}\ln\mathcal{J}^{(\beta)}(k,\lambda) = -\ln(1+\lambda)$.

Case $1<\beta<\infty$. The claim follows from (3.8) and (3.14) together with the asymptotic relation

$$\lim_{k\to\infty}\frac{\ln\mathcal{J}^{(\beta)}(k,\lambda)}{k\,\ln k} = -\frac{\beta-1}{\beta} \tag{3.26}$$

valid for $\lambda>0$ in this case. For a proof of (3.26) we construct asymptotically coinciding lower and upper bounds. The lower bound reads

$$\begin{aligned}\mathcal{J}^{(\beta)}(k,\lambda) &\geq e^{-\lambda k - k^{1/\beta}}\frac{1}{k!}\int_0^{k^{1/\beta}}\xi^k L_q^{(k)}(\xi)^2\,d\xi\\ &\geq e^{-\lambda k - k^{1/\beta}}\frac{k^{k+1}}{k!}\int_0^{k^{\frac{1-\beta}{\beta}}}\xi^k L_q^{(k)}(k\xi)^2\,d\xi\\ &\geq e^{-\lambda k - k^{1/\beta}}\frac{k^{k+1/\beta}}{(k+1)!}k^{k\frac{1-\beta}{\beta}}\frac{k^{2q}}{4^{q+1}(q!)^2}\,.\end{aligned} \tag{3.27}$$

Here the last inequality follows from (3.6) with $\xi_0 = 1/2$, and is valid for sufficiently large k only. Using Stirling's asymptotic formula (3.23) in (3.27), we obtain $\liminf_{k\to\infty}(k\,\ln k)^{-1}\ln\mathcal{J}^{(\beta)}(k,\lambda) \geq \frac{1-\beta}{\beta}$.

For the upper bound we suppose $k+q>2$ and use (3.4) in order to estimate the integrand in (3.20) from above. Thus we obtain

$$\mathcal{J}^{(\beta)}(k,\lambda) \leq \frac{(k+q)^{2q}}{k!}\int_0^\infty \xi^k e^{-\lambda\xi^\beta}\,d\xi = \frac{(k+q)^{2q}}{\beta\lambda^{(k+1)/\beta}k!}\Gamma\left(\frac{k+1}{\beta}\right)\,. \tag{3.28}$$

Stirling's formula (3.23) finally yields $\limsup_{k\to\infty}(k\,\ln k)^{-1}\ln\mathcal{J}^{(\beta)}(k,\lambda) \leq \frac{1-\beta}{\beta}$. □

The last topic in this section is the derivation of an asymptotic property of the eigenvalues

$$\nu_{q,k}(r) := \langle\chi_r\varphi_{q,k},\varphi_{q,k}\rangle\,,\quad k\in\mathbb{Z}_+-q\,,\quad q\in\mathbb{Z}_+\,,\quad r>0\,, \tag{3.29}$$

of the operator $P_q\chi_r P_q$ (see Lemma 3.3).

Proposition 3.2. *Let $q\in\mathbb{Z}_+$ and $r>0$. Then we have*

$$\lim_{k\to\infty}\frac{\ln\nu_{q,k}(r)}{k\,\ln k} = -1\,. \tag{3.30}$$

Remark 3.3. It follows from (3.30), (3.15), (3.13), and (3.14) with $\beta < \infty$, that

$$\nu_{q,k}(r) = o(\gamma_{q,k}^{(\beta)}(\mu)) , \quad k \to \infty , \tag{3.31}$$

for all $0 < \mu < \infty$ and $0 < \beta < \infty$.

Proof of Proposition 3.2. From Lemma 3.3 it follows that

$$\nu_{q,k}(r) = \frac{q!}{(k+q)!} \int_0^{br^2/2} \xi^k e^{-\xi} L_q^{(k)}(\xi)^2 \, d\xi . \tag{3.32}$$

In its turn, the integral in (3.32) is estimated as follows

$$\begin{aligned} \int_0^{br^2/2} \xi^k e^{-\xi} L_q^{(k)}(\xi)^2 \, d\xi &\geq e^{-br^2/2} k^{k+1} \int_0^{br^2/(2k)} \xi^k L_q^{(k)}(k\xi)^2 \, d\xi \\ &\geq e^{-br^2/2} \frac{k^{k+1}}{k+1} \left(\frac{br^2}{2k} \right)^{k+1} \frac{k^{2q}}{4^{q+1}(q!)^2} . \end{aligned} \tag{3.33}$$

Here the last inequality again is implied by (3.6), and is valid for sufficiently large k. Moreover, we may use (3.4) to estimate

$$\begin{aligned} \int_0^{br^2/2} \xi^k e^{-\xi} L_q^{(k)}(\xi)^2 \, d\xi &\leq (k+q)^{2q} \int_0^{br^2/2} \xi^k e^{-(1-2/(k+q))\xi} \\ &\leq \frac{(k+q)^{2q}}{k+1} \left(\frac{br^2}{2} \right)^{k+1} \end{aligned} \tag{3.34}$$

for all $k + q \geq 2$. The claim again follows with the help of Stirling's formula (3.23). □

4. Proof of the Main Results for Two Dimensions

4.1. *Reduction to a single Landau-level eigenspace*

In this subsection we establish asymptotic estimates of $N(E_q + E, E'; H(V))$ as $E \downarrow 0$, which play a crucial role in the proof of Theorems 2.1 and 2.2. For this purpose, we recall in the following lemma a suitable version of the well-known Weyl inequalities for the eigenvalues of self-adjoint compact operators.

Lemma 4.1 ([5, Sec. 9.2, Theorem 9]). *Let T_1 and T_2 be linear self-adjoint compact operators on a Hilbert space. Then for each $s > 0$ and $\varepsilon \in (0,1)$ we have*

$$\begin{aligned} n_\pm(s(1+\varepsilon); T_1) - n_\mp(s\varepsilon; T_2) &\leq n_\pm(s; T_1 + T_2) \\ &\leq n_\pm(s(1-\varepsilon); T_1) + n_\pm(s\varepsilon; T_2) , \end{aligned} \tag{4.1}$$

the counting functions $n_\pm$ being defined in (2.1).

Proposition 4.1. *Let $E' \in (E_q, E_{q+1})$, $q \in \mathbb{Z}_+$. Assume that V satisfies the hypotheses of Theorem* 2.1 *or Theorem* 2.2. *Then for every $\varepsilon \in (0,1)$ we have*

$$n_+(E;(1-\varepsilon)P_qVP_q) + O(1) \le N(E_q+E, E'; H(V)) \le n_+(E;(1+\varepsilon)P_qVP_q) + O(1)\,, \quad E \downarrow 0\,. \tag{4.2}$$

Proof. First of all, note that under the hypotheses of Theorems 2.1 and 2.2, V satisfies the assumptions of Lemma 3.2, so that the operator P_qVP_q is compact.

Next, the generalized Birman–Schwinger principle (see e.g. [2, Theorem 1.3]) entails

$$\begin{aligned} N(E_q+E, E'; H(V)) &= n_+(1; V^{1/2}(E_q+E-H(0))^{-1}V^{1/2}) \\ &\quad - n_+(1; V^{1/2}(E'-H(0))^{-1}V^{1/2}) - \dim \operatorname{Ker}\,(H(V)-E')\,. \end{aligned} \tag{4.3}$$

Since the operator $V^{1/2}H(0)^{-1/2}$ is compact, the last two terms at the r.h.s. of (4.3), which are independent of E, are finite.

Fix $\varepsilon \in (0,1)$ and set $Q_q := \mathrm{Id} - P_q$. Applying (4.1) with $T_1 := V^{1/2}(E_q + E - H(0))^{-1}P_qV^{1/2}$ and $T_2 := V^{1/2}(E_q+E-H(0))^{-1}Q_qV^{1/2}$, we obtain

$$\begin{aligned} n_+(1; V^{1/2}(E_q+E-H(0))^{-1}V^{1/2}) &\ge n_+(1/(1-\varepsilon); V^{1/2}(E_q+E-H(0))^{-1}P_qV^{1/2}) \\ &\quad - n_-(\varepsilon/(1-\varepsilon); V^{1/2}(E_q+E-H(0))^{-1}Q_qV^{1/2})\,, \end{aligned} \tag{4.4}$$

$$\begin{aligned} n_+(1; V^{1/2}(E_q+E-H(0))^{-1}V^{1/2}) &\le n_+(1/(1+\varepsilon); V^{1/2}(E_q+E-H(0))^{-1}P_qV^{1/2}) \\ &\quad + n_+(\varepsilon/(1+\varepsilon); V^{1/2}(E_q+E-H(0))^{-1}Q_qV^{1/2})\,. \end{aligned} \tag{4.5}$$

Next, we deal with the first terms on the r.h.s. of (4.4) and (4.5). Since the non-zero singular numbers of the compact operators $P_qV^{1/2}$ and $V^{1/2}P_q$ coincide, we get

$$\begin{aligned} n_+(1/(1\pm\varepsilon); V^{1/2}(E_q+E-H(0))^{-1}P_qV^{1/2}) &= n_+(E;(1\pm\varepsilon)V^{1/2}P_qV^{1/2}) \\ &= n_+(E;(1\pm\varepsilon)P_qVP_q)\,. \end{aligned} \tag{4.6}$$

Further, we estimate the second terms on the r.h.s. of (4.4) and (4.5). The operator inequality

$$\begin{aligned} |E_q+E-H(0)|^{-1}Q_q &= \sum_{\substack{l\in\mathbb{Z}_+ \\ l\neq q}} |E_q+E-E_l|^{-1}P_l \\ &\le C_q \sum_{l\in\mathbb{Z}_+} E_l^{-1}P_l = C_qH(0)^{-1}\,, \end{aligned} \tag{4.7}$$

valid for $E \in (0, E' - E_q)$, $E' \in (E_q, E_{q+1})$, and $C_q := E_{q+1}/(E_{q+1} - E')$, implies

$$\begin{aligned} &n_\pm(\varepsilon/(1 \pm \varepsilon); V^{1/2}(E_q + E - H(0))^{-1} Q_q V^{1/2}) \\ &\quad \le n_+(\varepsilon/(1 \pm \varepsilon); C_q V^{1/2} H(0)^{-1} V^{1/2}) \,. \end{aligned} \tag{4.8}$$

Since the operator $V^{1/2} H(0)^{-1/2}$ is compact, the quantity on the r.h.s. of (4.8), which is independent of E, is finite for each $\varepsilon \in (0, 1)$. Putting together (4.3)–(4.8), we obtain (4.2). □

4.2. *Proof of Theorem* 2.1

Pick $\delta \in (0, \mu)$. From (2.2) we conclude that there exist $r_\delta > 0$ such that $G^{(\beta)}_{\mu+\delta}(\mathbf{x}) \le V(\mathbf{x}) \le G^{(\beta)}_{\mu-\delta}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$ which satisfy $|\mathbf{x}| > r_\delta$. Hence, we have

$$G^{(\beta)}_{\mu+\delta}(\mathbf{x}) - M\chi_{r_\delta}(\mathbf{x}) \le V(\mathbf{x}) \le G^{(\beta)}_{\mu-\delta}(\mathbf{x}) + M\chi_{r_\delta}(\mathbf{x}) \,, \quad \mathbf{x} \in \mathbb{R}^2 \,, \tag{4.9}$$

with $M := \max\{1, \sup_{\mathbf{x}\in\mathbb{R}^2} V(\mathbf{x})\}$ as $\sup_{\mathbf{x}\in\mathbb{R}^2} G^{(\beta)}_\lambda(\mathbf{x}) = 1$ for each $\lambda \in (0, \infty)$, $\beta \in (0, \infty)$. Let us pick $\varepsilon > 0$. According to Proposition 4.1 and (4.9) we have

$$\begin{aligned} &N(E_q + E, E'; H(V)) \\ &\quad \ge n_+(E; (1 - \varepsilon) P_q V P_q) + O(1) \,, \\ &\quad \ge n_+(E; (1 - \varepsilon) P_q [G^{(\beta)}_{\mu+\delta} - M\chi_{r_\delta}] P_q) + O(1) \,, \quad E \downarrow 0 \,, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &N(E_q + E, E'; H(V)) \\ &\quad \le n_+(E; (1 + \varepsilon) P_q V P_q) + O(1) \\ &\quad \le n_+(E; (1 + \varepsilon) P_q [G^{(\beta)}_{\mu-\delta} + M\chi_{r_\delta}] P_q) + O(1) \,, \quad E \downarrow 0 \,. \end{aligned} \tag{4.11}$$

Since $G^{(\beta)}_{\mu\pm\delta} \mp M\chi_{r_\delta}$ is bounded and radially symmetric, Lemma 3.3 implies that the eigenvalues of $P_q[G^{(\beta)}_{\mu\pm\delta} \mp M\chi_{r_\delta}]P_q$ are given by $\gamma^{(\beta)}_{q,k}(\mu \pm \delta) \mp M\nu_{q,k}(r_\delta)$, $k \in \mathbb{Z}_+ - q$, (see (3.12) and (3.29)). Therefore,

$$\begin{aligned} &n_+(E; (1 \mp \varepsilon) P_q [G^{(\beta)}_{\mu\pm\delta} \mp M\chi_{r_\delta}] P_q) \\ &\quad = \#\{k \in \mathbb{Z}_+ - q | (1 \mp \varepsilon)[\gamma^{(\beta)}_{q,k}(\mu \pm \delta) \mp M\nu_{q,k}(r_\delta)] > E\} \,. \end{aligned} \tag{4.12}$$

Thanks to Proposition 3.1 and (3.31), there exists some $K_\varepsilon \in \mathbb{Z}_+ - q$ such that

$$\begin{aligned} \gamma^{(\beta)}_{q,k}(\mu + \delta) - M\nu_{q,k}(r_\delta) &\ge (1 - \varepsilon)\gamma^{(\beta)}_{q,k}(\mu + \delta) \\ &\ge (1 - \varepsilon) \exp[-(1 + \varepsilon)(a^{(\beta)}_{\mu+\delta})^{-1}(k)] \,, \end{aligned} \tag{4.13}$$

$$\begin{aligned} \gamma^{(\beta)}_{q,k}(\mu - \delta) + M\nu_{q,k}(r_\delta) &\le (1 + \varepsilon)\gamma^{(\beta)}_{q,k}(\mu - \delta) \\ &\le (1 + \varepsilon) \exp[-(1 - \varepsilon)(a^{(\beta)}_{\mu-\delta})^{-1}(k)] \end{aligned} \tag{4.14}$$

for all $k \geq K_\varepsilon$. Using (4.10)–(4.14), we thus conclude that

$$\liminf_{E\downarrow 0} \frac{N(E_q+E, E'; H(V))}{a^{(\beta)}_{\mu+\delta}(|\ln(E/(1-\varepsilon)^2)|/(1+\varepsilon))} \geq 1\,, \tag{4.15}$$

$$\limsup_{E\downarrow 0} \frac{N(E_q+E, E'; H(V))}{a^{(\beta)}_{\mu-\delta}(|\ln(E/(1+\varepsilon)^2)|/(1-\varepsilon))} \leq 1\,. \tag{4.16}$$

Letting $\varepsilon \downarrow 0$ and afterwards $\delta \downarrow 0$ in (4.15) and (4.16), and taking into account that

$$\lim_{\varepsilon\downarrow 0}\lim_{\kappa\to\infty} \frac{a^{(\beta)}_{\mu\pm\delta}(\kappa/(1\pm\varepsilon))}{a^{(\beta)}_{\mu\pm\delta}(\kappa)} = 1\,, \quad \lim_{\delta\downarrow 0}\lim_{\kappa\to\infty} \frac{a^{(\beta)}_{\mu\pm\delta}(\kappa)}{a^{(\beta)}_{\mu}(\kappa)} = 1\,, \tag{4.17}$$

we obtain (2.8) with $\beta < \infty$ which is equivalent to (2.3)–(2.5). □

4.3. *Proof of Theorem* 2.2

Its hypotheses imply that there exist $C_\pm > 0$, $r_\pm > 0$, and $\mathbf{x}^\pm \in \mathbb{R}^2$, such that

$$C_-\chi_{r_-,\mathbf{x}^-}(\mathbf{x}) \leq V(\mathbf{x}) \leq C_+\chi_{r_+,\mathbf{x}^+}(\mathbf{x})\,, \quad \mathbf{x} \in \mathbb{R}^2\,. \tag{4.18}$$

Pick $\varepsilon \in (0,1)$. Combining (4.2), (4.18), and the minimax principle, we get

$$N(E_q+E, E'; H(V)) \geq n_+(E; (1-\varepsilon)C_-P_q\chi_{r_-,\mathbf{x}^-}P_q) + O(1)\,, \quad E\downarrow 0\,, \tag{4.19}$$

$$N(E_q+E, E'; H(V)) \leq n_+(E; (1+\varepsilon)C_+P_q\chi_{r_+,\mathbf{x}^+}P_q) + O(1)\,, \quad E\downarrow 0\,. \tag{4.20}$$

For $\mathbf{x}' = (x', y') \in \mathbb{R}^2$ define the magnetic translation $\mathcal{T}_{\mathbf{x}'}$ by

$$(\mathcal{T}_{\mathbf{x}'}u)(\mathbf{x}) := \exp\left\{i\frac{b}{2}(x'y - xy')\right\} u(\mathbf{x}-\mathbf{x}')\,, \quad \mathbf{x} = (x,y) \in \mathbb{R}^2\,.$$

The unitary operator $\mathcal{T}_{\mathbf{x}'}$ commutes with $H(0)$, and hence with the projections P_q, $q \in \mathbb{Z}_+$ (see e.g. [11, Eq. 11]). Therefore,

$$P_q\chi_{r_\pm,\mathbf{x}^\pm}P_q = P_q\mathcal{T}_{\mathbf{x}^\pm}\chi_{r_\pm}\mathcal{T}^*_{\mathbf{x}^\pm}P_q = \mathcal{T}_{\mathbf{x}^\pm}P_q\chi_{r_\pm}P_q\mathcal{T}^*_{\mathbf{x}^\pm}\,. \tag{4.21}$$

Hence, the operators $P_q\chi_{r_\pm,\mathbf{x}^\pm}P_q$ and $P_q\chi_{r_\pm}P_q$ are unitarily equivalent, and we have

$$\begin{aligned} &n_+(E; (1\pm\varepsilon)C_\pm P_q\chi_{r_\pm,\mathbf{x}^\pm}P_q) \\ &\quad = n_+(E; (1\pm\varepsilon)C_\pm P_q\chi_{r_\pm}P_q) \\ &\quad = \#\{k \in \mathbb{Z}_+ - q \,|\, (1\pm\varepsilon)C_\pm\nu_{q,k}(r_\pm) > E\} \\ &\quad = \#\{k \in \mathbb{Z}_+ - q \,|\, \ln\nu_{q,k}(r_\pm) + \ln((1\pm\varepsilon)C_\pm) > \ln E\}\,. \end{aligned} \tag{4.22}$$

Taking into account (3.30), we find that (4.22) entails

$$\lim_{E\downarrow 0} \frac{n_+(E; (1\pm\varepsilon)C_\pm P_q\chi_{r_\pm,\mathbf{x}^\pm}P_q)}{(\ln|\ln E|)^{-1}|\ln E|} = 1\,. \tag{4.23}$$

Putting together (4.19), (4.20), and (4.23), we obtain (2.5). □

5. Proof of Main Results for Three Dimensions

5.1. *Auxiliary facts about Schrödinger operators in one dimension*

This subsection contains some well-known facts from the spectral theory of one-dimensional Schrödinger operators.

Let $v \in L^1(\mathbb{R})$ be real-valued and let $h(v)$ be the self-adjoint operator generated in $L^2(\mathbb{R})$ by the quadratic form $\int_{\mathbb{R}}\{|u'|^2 - v|u|^2\}\,dz$, $u \in W_2^1(\mathbb{R})$. It is closed and lower bounded since the operator $|v|^{1/2}(h(0)+1)^{-1/2}$ is Hilbert–Schmidt, and hence compact.

Lemma 5.1 ([4, Secs. 2.4, 4.6], [13]). *Let* $0 \le v \in L^1(\mathbb{R};(1+|z|)\,dz)$, $g > 0$. *Assume that* v *does not vanish identically. Then we have*

$$1 \le N(0; h(gv)) \le g\int_{\mathbb{R}} |z| v(z)\,dz + 1\,. \tag{5.1}$$

Note that if $0 < g\int_{\mathbb{R}}|z|v(z)\,dz < 1$, then by (5.1) the operator $h(gv)$ has a unique, strictly negative eigenvalue denoted in the sequel by $-\mathcal{E}(gv)$.

Lemma 5.2 ([6, Theorem 3.1], [13], [21]). *Let the hypotheses of Lemma* 5.1 *hold. Then* $\mathcal{E}(gv)$ *obeys the asymptotics*

$$\sqrt{\mathcal{E}(gv)} = \frac{g}{2}\int_{\mathbb{R}} v(z)\,dz(1+o(1))\,, \quad g \downarrow 0\,. \tag{5.2}$$

5.2. *Proof of Lemma* 2.1

Denote by $\mathcal{P}_q : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, $q \in \mathbb{Z}_+$, the orthogonal projections corresponding to the qth Landau level. In other words,

$$(\mathcal{P}_q u)(X_\perp, z) := \int_{\mathbb{R}^2} K_q(X_\perp, X'_\perp) u(X'_\perp, z)\,dX'_\perp\,, \quad (X_\perp, z) \in \mathbb{R}^3\,,$$

where $K_q(X_\perp, X'_\perp)$, $X_\perp$, $X'_\perp \in \mathbb{R}^2$, is the integral kernel of the orthogonal projection $P_q : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, introduced in (3.9).

Let $N \ge 1$ and set $T := V^{1/2}H(0)^{-1/2}$ and $T_N := T\sum_{q=0}^N \mathcal{P}_q$.

First, we show that T_N is a Hilbert–Schmidt operator. To this end we estimate

$$\|T_N\|_{\mathrm{HS}} \le \sum_{q=0}^N \|T\mathcal{P}_q\|_{\mathrm{HS}}\,.$$

Further, taking into account (3.9) and (3.10), we find that

$$\|T\mathcal{P}_q\|_{\mathrm{HS}}^2 = \frac{b}{(2\pi)^2}\int_{\mathbb{R}^3} V(\mathbf{x})\,d\mathbf{x}\int_{\mathbb{R}} \frac{d\zeta}{\zeta^2 + E_q} \le \frac{b}{4\pi}E_q^{-1/2}\|U\|_{L^1(\mathbb{R}^2)}\|v\|_{L^1(\mathbb{R})}\,. \tag{5.3}$$

Therefore, T_N is Hilbert–Schmidt, and hence compact.

Next we show that $\lim_{N\to\infty}\|T - T_N\| = 0$. Evidently,

$$\|T - T_N\| \le \|U\|_{L^\infty(\mathbb{R}^2)}^{1/2}\|\,|v|^{1/2}(h(0) + E_{N+1})^{-1/2}\|\,. \tag{5.4}$$

Since the operator $|v|^{1/2}(h(0) + 1)^{-1/2}$ is compact in $L^2(\mathbb{R})$, we have $\lim_{N\to\infty} \| |v|^{1/2}(h(0) + E_{N+1})^{-1/2}\| = 0$. Consequently, the operator T can be approximated in norm by the sequence of compact operators T_N. Hence, T is a compact operator itself. □

5.3. *Reduction to one dimension*

In this subsection we prove a proposition which can be regarded as the three-dimensional analogue of Proposition 4.1.

Proposition 5.1. *Let $V \geq 0$. Suppose that there exist four non-negative functions $v^\pm \in L^1(\mathbb{R})$ and $U^\pm \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ such that*

$$U^-(X_\perp)v^-(z) \leq V(\mathbf{x}) \leq U^+(X_\perp)v^+(z)\,, \quad \mathbf{x} = (X_\perp, z) \in \mathbb{R}^3\,. \tag{5.5}$$

Then for every $\varepsilon > 0$ we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}_+} & N(-E; h(\varkappa_k^- v^-)) \\ & \leq N(E_0 - E; H(-V)) \\ & \leq \sum_{k\in\mathbb{Z}_+} N(-E; h((1+\varepsilon)\varkappa_k^+ v^+)) + O(1)\,, \quad E \downarrow 0\,.\end{aligned} \tag{5.6}$$

Here $h(v)$ is the operator defined at the beginning of Sec. 5.1, and $\varkappa_k^\pm$, $k \in \mathbb{Z}_+$, stand for the respective eigenvalues of the compact operators $P_0U^\pm P_0$ on $P_0L^2(\mathbb{R}^2)$.

Proof. Set $\mathcal{Q}_0 := \mathrm{Id} - \mathcal{P}_0$ and denote by $\mathcal{Z}_1(V)$ (respectively, by $\mathcal{Z}_2(V)$) the self-adjoint operator generated in $\mathcal{P}_0L^2(\mathbb{R}^3)$ (respectively, in $\mathcal{Q}_0L^2(\mathbb{R}^3)$) by the closed, lower bounded quadratic form $\int_{\mathbb{R}^3}\{|i\nabla u + Au|^2 - V|u|^2\}\,d\mathbf{x}$ defined for $u \in \mathcal{P}_0D(H(0)^{1/2})$ (respectively, for $u \in \mathcal{Q}_0D(H(0)^{1/2})$). Let $\varepsilon > 0$. Since $V \geq 0$, the minimax principle yields

$$\begin{aligned}N(E_0 - E; \mathcal{Z}_1(V)) &\leq N(E_0 - E; H(-V)) \\ &\leq N(E_0 - E; \mathcal{Z}_1((1+\varepsilon)V)) \\ &\quad + N(E_0 - E; \mathcal{Z}_2((1+\varepsilon^{-1})V))\,.\end{aligned} \tag{5.7}$$

It is easy to check that $\sigma_{\mathrm{ess}}(\mathcal{Z}_2((1+\varepsilon^{-1})V)) = [E_1, \infty)$ for each $\varepsilon > 0$. Therefore,

$$N(E_0 - E; \mathcal{Z}_2((1+\varepsilon^{-1})V)) = O(1)\,, \quad E \downarrow 0\,. \tag{5.8}$$

Set $V^\pm(\mathbf{x}) := U^\pm(X_\perp)v^\pm(z)$, $\mathbf{x} = (X_\perp, z)$. Then (5.5) implies

$$N(E_0 - E; \mathcal{Z}_1(V)) \geq N(E_0 - E; \mathcal{Z}_1(V^-))\,, \tag{5.9}$$

$$N(E_0 - E; \mathcal{Z}_1((1+\varepsilon)V)) \leq N(E_0 - E; \mathcal{Z}_1((1+\varepsilon)V^+))\,. \tag{5.10}$$

Obviously, $\mathcal{Z}_1(V^-)$ is unitarily equivalent to the orthogonal sum $\sum_{k\in\mathbb{Z}_+} \oplus (h(\varkappa_k^- v^-) + E_0)$, while $\mathcal{Z}_1((1+\varepsilon)V^+)$ is unitarily equivalent to $\sum_{k\in\mathbb{Z}_+} \oplus (h((1+\varepsilon)\varkappa_k^+ v^+) + E_0)$. Thus the combination of (5.7)–(5.10) yields (5.6). □

5.4. *Proof of Theorem* 2.3

By the hypotheses of Theorem 2.3 we may pick $\delta \in (0, \mu)$ and choose $r_\delta > 0$ such that the assumptions of Proposition 5.1 are satisfied with

$$\begin{aligned} U^\pm(X_\perp) &= G^{(\beta)}_{\mu\mp\delta}(X_\perp) \pm \mathcal{M}\chi_{r_\delta}(X_\perp)\,, \\ v^+(z) = v_\delta^+(z) + v(z)\,, &\quad v^-(z) = v_\delta^-(z)\,, \end{aligned} \tag{5.11}$$

where, similarly to (4.9), $\mathcal{M} := \max\{1, C\}$, and C is the constant occurring in the formulation of Theorem 2.3. Accordingly, Lemma 3.3 implies that $\varkappa_k^\pm = \gamma^{(\beta)}_{0,k}(\mu \mp \delta) \pm \mathcal{M}\nu_{0,k}(r_\delta)$, $k \in \mathbb{Z}_+$. Now pick $\varepsilon \in (0, 1)$ and choose K_ε such that $k \geq K_\varepsilon$ entails the following inequalities

$$\begin{aligned} \gamma^{(\beta)}_{0,k}(\mu+\delta) - \mathcal{M}\nu_{0,k}(r_\delta) &\geq (1-\varepsilon)\gamma^{(\beta)}_{0,k}(\mu+\delta)\,, \\ \gamma^{(\beta)}_{0,k}(\mu-\delta) + \mathcal{M}\nu_{0,k}(r_\delta) &\leq (1+\varepsilon)\gamma^{(\beta)}_{0,k}(\mu-\delta)\,, \\ (1+\varepsilon)^2\gamma^{(\beta)}_{0,k}(\mu\mp\delta) &\textstyle\int_{\mathbb{R}} |z|v^\pm(z)dz < 1\,. \end{aligned} \tag{5.12}$$

Taking into account (5.1) and Proposition 5.1, we get

$$\begin{aligned} &N(E_0 - E; H(-V)) \\ &\quad\geq \sum_{k\in\mathbb{Z}_+} N(-E; h((\gamma^{(\beta)}_{0,k}(\mu+\delta) - \mathcal{M}\nu_{0,k}(r_\delta))v^-)) \\ &\quad\geq \#\{k \in \mathbb{Z}_+, k \geq K_\varepsilon | \mathcal{E}((1-\varepsilon)\gamma^{(\beta)}_{0,k}(\mu+\delta)v^-) > E\}\,. \end{aligned} \tag{5.13}$$

Similarly, we have

$$\begin{aligned} &N(E_0 - E; H(-V)) \\ &\quad\leq \sum_{k\in\mathbb{Z}_+} N(-E; h((1+\varepsilon)(\gamma^{(\beta)}_{0,k}(\mu-\delta) + \mathcal{M}\nu_{0,k}(r_\delta))v^+) + O(1) \\ &\quad\leq \#\{k \in \mathbb{Z}_+, k \geq K_\varepsilon | \mathcal{E}((1+\varepsilon)^2\gamma^{(\beta)}_{0,k}(\mu-\delta)v^+) > E\} \\ &\qquad + O(1)\,, \quad E \downarrow 0\,. \end{aligned} \tag{5.14}$$

The last inequality in (5.14) results from splitting the series into two parts and using (5.1) to verify that the sum over $k \in \{0, 1, \ldots, K_\varepsilon - 1\}$ remains bounded as $E \downarrow 0$. Utilizing (5.2), choose $K'_\varepsilon \geq K_\varepsilon$ such that $k \geq K'_\varepsilon$ entails

$$\sqrt{\mathcal{E}((1-\varepsilon)\gamma^{(\beta)}_{0,k}(\mu+\delta)v^-)} \geq \frac{(1-\varepsilon)^2}{2}\gamma^{(\beta)}_{0,k}(\mu+\delta)\int_{\mathbb{R}} v^-(z)\,dz\,, \tag{5.15}$$

$$\sqrt{\mathcal{E}((1+\varepsilon)^2\gamma^{(\beta)}_{0,k}(\mu-\delta)v^+)} \leq \frac{(1+\varepsilon)^3}{2}\gamma^{(\beta)}_{0,k}(\mu-\delta)\int_{\mathbb{R}} v^+(z)\,dz\,. \tag{5.16}$$

Consequently,

$$\begin{aligned}
&\#\{k \in \mathbb{Z}_+, k \geq K_\varepsilon | \mathcal{E}((1-\varepsilon)\gamma_{0,k}^{(\beta)}(\mu+\delta)v^-) > E\} \\
&\qquad \geq \#\left\{k \in \mathbb{Z}_+, k \geq K'_\varepsilon \middle| \frac{(1-\varepsilon)^2}{2}\gamma_{0,k}^{(\beta)}(\mu+\delta)) \int_{\mathbb{R}} v^-(z)\,dz > \sqrt{E}\right\}, \qquad (5.17)\\
&\#\{k \in \mathbb{Z}_+, k \geq K_\varepsilon | \mathcal{E}((1+\varepsilon)^2\gamma_{0,k}^{(\beta)}(\mu-\delta)v^+) > E\} \\
&\qquad \leq \#\left\{k \in \mathbb{Z}_+, k \geq K'_\varepsilon \middle| \frac{(1+\varepsilon)^3}{2}\gamma_{0,k}^{(\beta)}(\mu-\delta)) \int_{\mathbb{R}} v^+(z)\,dz > \sqrt{E}\right\} \\
&\qquad\quad + O(1), \quad E \downarrow 0. \qquad (5.18)
\end{aligned}$$

Putting together (5.13)–(5.14) and (5.17)–(5.18), we obtain the asymptotic estimates

$$\begin{aligned}
&N(E_0 - E; H(-V)) \\
&\qquad \geq \#\{k \in \mathbb{Z}_+ |\ln \gamma_{0,k}^{(\beta)}(\mu+\delta) > \ln\sqrt{E} + O(1)\} + O(1), \qquad (5.19)\\
&N(E_0 - E; H(-V)) \\
&\qquad \leq \#\{k \in \mathbb{Z}_+ |\ln \gamma_{0,k}^{(\beta)}(\mu-\delta) > \ln\sqrt{E} + O(1)\} + O(1), \qquad (5.20)
\end{aligned}$$

valid as $E \downarrow 0$. Using Proposition 3.1 and proceeding as in the proof of Theorem 2.1, we find that (5.19) and (5.20) imply (2.10). □

5.5. *Proof of Theorem* 2.4

Finally, in this subsection we give a sketch of the proof of Theorem 2.4 which is quite similar and only easier than the proof of Theorem 2.3. First of all, note that the assumptions of Proposition 5.1 are satisfied with $U^\pm(X_\perp) = \chi_{r_\pm, X_\perp^\pm}(X_\perp)$, so that $\varkappa_k^\pm = \nu_{0,k}(r_\pm)$ thanks to the unitary equivalence of the operators $P_0\chi_{r_\pm,X_\perp^\pm}P_0$ and $P_0\chi_{r_\pm}P_0$ established in Sec. 4.3. Proposition 5.1 and Lemma 5.1 then imply the asymptotic estimates

$$\begin{aligned}
&\#\{k \in \mathbb{Z}_+ |\ln \nu_{0,k}(r_-) > \ln\sqrt{E} + O(1)\} + O(1) \\
&\qquad \leq N(E_0 - E; H(-V)) \\
&\qquad \leq \#\{k \in \mathbb{Z}_+ |\ln \nu_{0,k}(r_+) > \ln\sqrt{E} + O(1)\} + O(1), \qquad (5.21)
\end{aligned}$$

which hold for $E \downarrow 0$, and are analogous to (5.19) and (5.20). Applying (3.30) and (3.14) with $\beta = \infty$, we conclude that (5.21) implies (2.11). □

Acknowledgments

The authors are very grateful to Professor Grigori Rozenblum for indicating a gap in the proof of Propositions 3.1 and 3.2 for $q \geq 1$ in the first version of the paper.

Acknowledgements are also due to both referees whose remarks contributed to the improvement of the article.

A part of this work was done while G. Raikov was visiting the Friedrich-Alexander Universität Erlangen–Nürnberg in the summer of 2001 as a DAAD Research Fellow. The financial support of DAAD and of the Chilean Science Foundation *Fondecyt* under Grants 1020737 and 7020737, is gratefully acknowledged.

It is a pleasure for G. Raikov to express his gratitude to Professor Hajo Leschke for his warm hospitality. Both authors thank him for encouragement and several stimulating discussions.

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series 55, 1964.

[2] S. Alama, P. A. Deift and R. Hempel, Eigenvalue branches of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$, *Commun. Math. Phys.* **121** (1989), 291–321.

[3] J. Avron, I. Herbst and B. Simon, Schrödinger operators with magnetic fields. I. General interactions, *Duke Math. J.* **45** (1978), 847–883.

[4] M. S. Birman, On the spectrum of singular boundary value problems, *Mat. Sbornik* **55** (1961) 125–174 (in Russian). English translation in *Amer. Math. Soc. Transl.*, (2) **53** (1966), 23–80.

[5] M. S. Birman and M. Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Reidel, Dordrecht, 1987.

[6] R. Blankenbecler, M. L. Goldberger and B. Simon, The bound states of weakly coupled long-range one-dimensional quantum Hamiltonians, *Ann. Phys.* (NY) **108** (1977), 69–78.

[7] V. Fock, Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld, *Z. Physik* **47** (1928), 446–448 (in German).

[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, corrected and enlarged edition, Academic, San Diego, 1980.

[9] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1954.

[10] T. Hupfer, H. Leschke and S. Warzel, Poissonian obstacles with Gaussian walls discriminate between classical and quantum Lifshits tailing in magnetic fields, *J. Stat. Phys.* **97** (1999), 725–750.

[11] T. Hupfer, H. Leschke and S. Warzel, Upper bounds on the density of states of single Landau levels broadened by Gaussian random potentials, *J. Math. Phys.* **42** (2001), 5626–5641.

[12] V. Ya. Ivrii, *Microlocal Analysis and Precise Spectral Asymptotics*, Springer, Berlin, 1998.

[13] M. Klaus, On the bound state of Schrödinger operators in one dimension, *Ann. Phys.* (NY) **108** (1977), 288–300.

[14] E. H. Lieb and M. Loss, *Analysis*, 2nd edition, Amer. Math. Soc., Providence, RI, 2001.

[15] G. D. Raikov, Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential. I. Behaviour near the essential

spectrum tips, *Commun. Partial Differential Equations* **15** (1990), 407–434; Errata: *Commun. Partial Differential Equations* **18** (1993), 1977–1979.

[16] G. D. Raikov, Border-line eigenvalue asymptotics for the Schrödinger operator with electromagnetic potential, *Integral Equations Operator Theory* **14** (1991), 875–888.

[17] A. V. Sobolev, Asymptotic behavior of the energy levels of a quantum particle in a homogeneous magnetic field, perturbed by a decreasing electric field. I, *Probl. Mat. Anal.* **9** (1984), 67–84 (in Russian); English translation in *J. Sov. Math.* **35** (1986), 2201–2212.

[18] H. Tamura, Asymptotic distribution of eigenvalues for Schrödinger operators with homogeneous magnetic fields, *Osaka J. Math.* **25** (1988), 633–647.

[19] M. Melgaard and G. Rozenblum, Eigenvalue asymptotics for even-dimensional perturbed Dirac and Schrödinger operators with constant magnetic fields, preprint, 2002.

[20] G. D. Raikov and S. Warzel, Spectral asymptotics for magnetic Schrödinger operators with rapidly decreasing electric potentials, *C. R. Acad. Sci. Paris*, *Ser. I* **335** (2002), 683–688.

[21] S. N. Solnyshkin, Asymptotics of the energy of bound states of the Schrödinger operator in the presence of electric and homogeneous magnetic fields, *Probl. Mat. Fiz.* **10** (1982), 266-278 [in Russian]; English translation in: *Sel. Math. Sov.* **5** (1986), 297–306.