A Trace Formula for the Eigenvalue Clusters of the Perturbed Landau Hamiltonian

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1. The Landau Hamiltonian

The Landau Hamiltonian is the operator

$$H_0 = (-i\nabla - A)^2,$$

self-adjoint in $L^2(\mathbb{R}^2)$.

Here:

- $A = (A_1, A_2) := \frac{B}{2}(-x_2, x_1)$
  is the magnetic potential;

- $B := \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} > 0$
  is the constant positive magnetic field.

Spectrum of H_0 (V. Fock (1928), L. Landau (1930)):

$$\sigma(H_0) = \sigma_{\text{ess}}(H_0) = \bigcup_{q=0}^{\infty} \{\lambda_q\}$$

where

$$\lambda_q := B(2q+1), \quad q \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\},$$

are the so-called *Landau levels*. Moreover,

$$\dim \operatorname{Ker}(H_0 - \lambda_q) = \infty, \quad q \in \mathbb{Z}_+.$$

Idea of the proof: we have

$$H_0 = a^* a + B, \quad [a, a^*] = 2B, \quad \dim \operatorname{Ker} a = \infty.$$

Here

$$a = -2ie^{-B|\mathbf{x}|^2/4} \frac{\partial}{\partial \bar{z}} e^{B|\mathbf{x}|^2/4}, \; \bar{z} = x_1 - ix_2,$$

is the magnetic annihilation operator, and

$$a^* = -2ie^{B|\mathbf{x}|^2/4} \frac{\partial}{\partial z} e^{-B|\mathbf{x}|^2/4}, \; z = x_1 + ix_2,$$

is the magnetic creation operator.

Let P_q be the orthogonal projection onto $\mathrm{Ker}\,(H_0 - \lambda_q)$, $q \in \mathbb{Z}_+$. We have

$$P_q L^2(\mathbb{R}^2) = (a^*)^q \mathrm{Ker}\, a, \quad q \in \mathbb{Z}_+,$$

and

$$\mathrm{Ker}\, a =$$

$$\left\{ f \in L^2(\mathbb{R}^2) \,\middle|\, f(\mathbf{x}) = g(\mathbf{x}) e^{-B|\mathbf{x}|^2/4}, \ \frac{\partial g}{\partial \bar{z}} = 0 \right\}.$$

2. The perturbation

Let V be a short range electric potential, i.e $V \in C(\mathbb{R}^2; \mathbb{R})$ satisfies the estimate

$$|V(\mathbf{x})| \leq C(1 + |\mathbf{x}|)^{-\kappa}, \quad \mathbf{x} \in \mathbb{R}^2, \quad \kappa > 1. \quad (1)$$

Set

$$H = H_0 + V.$$

We have

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = \bigcup_{q=0}^{\infty} \{\lambda_q\}.$$

However, in the case of H the Landau levels are typically accumulation points of the discrete eigenvalues (e.g. in the case of V of a fixed sign: F. Klopp, G. Raikov, IMRN (2009)).

Evidently,

$$\sigma(H) \subset \bigcup_{q=0}^{\infty} \left[\lambda_q - \|V\|_{L^\infty(\mathbb{R}^2)}, \lambda_q + \|V\|_{L^\infty(\mathbb{R}^2)}\right].$$

In fact we have more:

Proposition 1. *Assume* (1)*; then there exist* $C_1 > 0$ *such that*

$$\sigma(H) \cap [\lambda_q - B, \lambda_q + B] \subset \left(\lambda_q - C_1\lambda_q^{-1/2}, \lambda_q + C_1\lambda_q^{-1/2}\right) \qquad (2)$$

for all $q \in \mathbb{Z}_+$*, i.e for large* q *the eigenvalue clusters shrink to the Landau levels.*

In the case $V \in C_0^\infty(\mathbb{R}^2)$, inclusion (2) was first proved by E. Korotyaev, A. Pushnitski JFA (2004).

3. Main results

3.1. Goal of the talk

The goal of talk is to discuss the asymptotic distribution of the eigenvalues of H within the qth cluster, as $q \to \infty$.

3.2. Notation

For $[\alpha, \beta] \subset \mathbb{R} \setminus \{0\}$ set

$$\mu_q([\alpha, \beta]) :=$$

$$\sum_{\lambda_q + \alpha\lambda_q^{-1/2} \leq \lambda \leq \lambda_q + \beta\lambda_q^{-1/2}} \dim \operatorname{Ker}(H - \lambda I).$$

For any fixed $[\alpha, \beta] \subset \mathbb{R} \setminus \{0\}$ we have

$$\mu_q([\alpha, \beta]) < \infty,$$

provided that $q \in \mathbb{Z}_+$ is large enough. We are interested in the asymptotics as $q \to \infty$ of the measure μ_q.

Define the Radon transform of V:

$$\tilde{V}(\omega, b) := \frac{B}{2\pi} \int_{-\infty}^{\infty} V(b\omega + t\omega^{\perp}) dt,$$

where

$$\omega = (\omega_1, \omega_2) \in \mathbb{T}, \quad \omega^{\perp} = (-\omega_2, \omega_1),$$

$$b \in \mathbb{R}.$$

Note that assumption (1) entails

$$|\tilde{V}(\omega, b)| \leq CB(1 + |b|)^{1-\kappa},\ \omega \in \mathbb{T},\ b \in \mathbb{R}.$$

For $[\alpha, \beta] \subset \mathbb{R} \setminus \{0\}$ set

$$\mu([\alpha, \beta]) := \frac{1}{2\pi} \left| \tilde{V}^{-1}([\alpha, \beta]) \right|,$$

$|\cdot|$ being the Lebesgue measure on $\mathbb{T} \times \mathbb{R}$.

3.3. Main theorem

Theorem 2. *Let $V \in C(\mathbb{R}^2;\mathbb{R})$ satisfy* (1). *Then we have*

$$\lim_{q\to\infty} \lambda_q^{-1/2} \int_{\mathbb{R}} \rho(\lambda) d\mu_q(\lambda) = \int_{\mathbb{R}} \rho(\lambda) d\mu(\lambda) \quad (3)$$

for any function $\rho \in C_0^\infty(\mathbb{R} \setminus \{0\})$.

Remark: Asymptotic relation (3) can be more explicitly written as

$$\lim_{q\to\infty} \lambda_q^{-1/2} \mathrm{Tr}\, \rho(\sqrt{\lambda_q}(H-\lambda_q)) =$$

$$\frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} \rho(\tilde{V}(\omega,b))\, db\, d\omega, \quad (4)$$

and is equivalent to

$$\lim_{q\to\infty} \lambda_q^{-1/2} \mu_q([\alpha,\beta]) = \mu([\alpha,\beta]),$$

for any α, β, such that $\alpha\beta > 0$ and

$$\mu(\{\alpha\}) = \mu(\{\beta\}) = 0.$$

4. Semiclassical interpretation

For $\boldsymbol{\xi} := (\xi, \eta) \in \mathbb{R}^2$, and $\mathbf{x} := (x, y) \in \mathbb{R}^2$, consider the classical Hamiltonian

$$\mathcal{H}(\boldsymbol{\xi}, \mathbf{x}) = (\xi + \tfrac{1}{2}By)^2 + (\eta - \tfrac{1}{2}Bx)^2.$$

The orbits of the Hamiltonian flow of $\mathcal{H}$ are circles of radius $\sqrt{E}/B$ in the configuration space; here $E > 0$ is the value of the energy corresponding to the orbit. The classical particles move around these circles with period $T_B = \pi/B$. These orbits can be parameterized by the energy $E > 0$ and the center $\mathbf{c} \in \mathbb{R}^2$ of a circle. Denote by $\gamma(\mathbf{c}, E, t)$, $t \in [0, T_B)$, the path corresponding to such an orbit, and set

$$\langle V \rangle(\mathbf{c}, E) = \frac{1}{T_B} \int_0^{T_B} V(\gamma(\mathbf{c}, E, t))dt.$$

Then we have

$$\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}}\rho(\tilde{V}(\omega,b))db\,d\omega =$$

$$\frac{B}{2\pi}\lim_{E\to\infty}\frac{1}{\sqrt{E}}\int_{\mathbb{R}^2}\rho(\sqrt{E}\,\langle V\rangle(\mathbf{c},E))\,dc_1\,dc_2, \quad (5)$$

and the combination of (4) and (5) implies

$$\lim_{q\to\infty}\frac{1}{\sqrt{\lambda_q}}\mathrm{Tr}\,\rho(\sqrt{\lambda_q}(H-\lambda_q)) =$$

$$\frac{B}{2\pi}\lim_{E\to\infty}\frac{1}{\sqrt{E}}\int_{\mathbb{R}^2}\rho(\sqrt{E}\,\langle V\rangle(\mathbf{c},E))\,dc_1\,dc_2$$

which corresponds to the *averaging principle* for systems close to integrable ones.

5. Related work

5.1. Manifolds with periodic geodesics

Let H_0 be the Laplace-Beltrami operator, self-adjoint in $L^2(\mathbb{S}^2)$. We have

$$\sigma(H_0) = \bigcup_{q=0}^{\infty} \{q(q+1)\},$$

$$\dim \operatorname{Ker}(H_0 - q(q+1)) = 2q+1.$$

Let $V \in C(\mathbb{S}^2; \mathbb{R})$. Set

$$H = H_0 + V.$$

Fix $\omega \in \mathbb{S}^2$, and denote by $\gamma_\omega \subset \mathbb{S}^2$ the great circle lying in the plane orthogonal to ω; we parametrize γ_ω by the angle $\theta \in [0, 2\pi)$. Set

$$\tilde{V}(\omega) := \frac{1}{2\pi} \int_0^{2\pi} V(\gamma_\omega(\theta))\, d\theta, \qquad \omega \in \mathbb{S}^2,$$

$$\mu([\alpha, \beta]) := \frac{1}{4\pi} \left| \tilde{V}^{-1}([\alpha, \beta]) \right|, \qquad [\alpha, \beta] \subset \mathbb{R}.$$

Theorem 3. [A. Weinstein, Duke Math. J. (1977)] *Let* $V \in C(\mathbb{S}^2)$, $\rho \in C_0^\infty(\mathbb{R})$. *Then we have*

$$\lim_{q\to\infty} \frac{\mathrm{Tr}\,\rho(H - q(q+1))}{2q+1} =$$

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} \rho(\tilde{V}(\omega))dS(\omega) = \int_{\mathbb{R}} \rho(\lambda)d\mu(\lambda).$$

Remarks: (i) Theorem 3 extends to the case $\mathbb{S}^n$, $n \geq 2$, and, more generally, to Riemannian compact symmetric spaces of rank 1.
(ii) Other related names which should be mentioned in this context are Y. Colin de Verdière, H. Widom, L. Thomas, C. Villegas Blas.

5.2. Strong magnetic field asymptotics

Let us return to the perturbed Landau Hamiltonian.

Theorem 4. [G.Raikov, CPDE (1998)] *Let* $V \in L^p(\mathbb{R}^2)$, $p > 1$, $\rho \in C_0^\infty(\mathbb{R} \setminus \{0\})$, $q \in \mathbb{Z}_+$. *Then we have :*

$$\lim_{B \to \infty} B^{-1} \mathrm{Tr}\, \rho(H - \lambda_q) =$$

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \rho(V(\mathbf{x})) d\mathbf{x} = \int_{\mathbb{R}} \rho(\lambda) dm(\lambda)$$

where

$$m([\alpha, \beta]) := \frac{1}{2\pi} \left| V^{-1}([\alpha, \beta]) \right|, \quad [\alpha, \beta] \subset \mathbb{R} \setminus \{0\}.$$

6. Sketch of the proof of Theorem 2

6.1. General outline

We denote by S_ℓ, $\ell \in [1,\infty)$, the usual Schatten–von Neumann classes with norm

$$\|T\|_\ell := \left(\mathrm{Tr}\,(T^*T)^{\ell/2}\right)^{1/\ell}.$$

Our main technical result concerns the Berezin-Toeplitz operators $P_q V P_q$, $q \in \mathbb{Z}_+$, $V \in L^1(\mathbb{R}^2)+L^\infty(\mathbb{R}^2)$. Note that if

$$V \in L^p(\mathbb{R}^2), \quad p \in [1,\infty),$$

then

$$P_q V P_q \in S_p(\mathbb{R}^2), \quad q \in \mathbb{Z}_+.$$

Theorem 5. *Let* $V \in C(\mathbb{R}^2)$ *satisfy* (1).
(i) There exists a constant $C_2 = C_2(V, B) < \infty$ *such that*

$$\sup_{q \geq 0} \lambda_q^{1/2} \|P_q V P_q\| \leq C_2. \tag{6}$$

(ii) For any integer $\ell > 1/(\kappa - 1)$, *we have* $P_q V P_q \in S_\ell$, *and the estimate*

$$\sup_{q \geq 0} \lambda_q^{(\ell-1)/(2\ell)} \|P_q V P_q\|_\ell \leq C_3 \tag{7}$$

holds with $C_3 = C_3(V, B, \ell) < \infty$.
(iii) For any integer $\ell > 1/(\kappa - 1)$, *we have*

$$\lim_{q \to \infty} \lambda_q^{(\ell-1)/2} \mathrm{Tr}\,(P_q V P_q)^\ell =$$

$$\frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} \tilde{V}(\omega, b)^\ell \, db \, d\omega. \tag{8}$$

In the case $V \in C_0^\infty(\mathbb{R}^2)$, estimates (6) – (7), as well as the existence of the limit in (8), were first proved by E. Korotyaev, A. Pushnitski, JFA (2004).

Theorem 2 is derived from Theorem 5 by using the following

Lemma 6. *For any integer* $\ell > 1/(\varkappa - 1)$*, the operator* $(H - \lambda_q)^\ell E_H(\lambda_q - B, \lambda_q + B)$ *belongs to the trace class and*

$$\lambda_q^{(\ell-1)/2} \mathrm{Tr}\{(H - \lambda_q)^\ell E_H(\lambda_q - B, \lambda_q + B)\} =$$

$$\lambda_q^{(\ell-1)/2} \mathrm{Tr}(P_q V P_q)^\ell + o(1), \quad q \to \infty, \quad (9)$$

E_H *being the spectral projection of* H*.*

In the case $V \in C_0^\infty(\mathbb{R}^2)$, Lemma 6 was first proved by E. Korotyaev, A. Pushnitski JFA (2004); the proof in the general case is quite similar.

A combination of (9) and (8) yields (3) for a function $\rho \in C_0^\infty(\mathbb{R})$ such that $\rho(\lambda) = \lambda^\ell$ for small λ. After this, the general result of Theorem 2 follows by application of the Stone-Weierstrass theorem.

6.2. Unitary equivalence of the Berezin-Toeplitz operator $P_q V P_q$

Let $d \geq 1$, $s : \mathbb{R}^{2d} \to \mathbb{C}$ be a symbol from an appropriate class. Then $\mathrm{Op}^w(s) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denotes the Weyl pseudo-differential operator (ΨDO) with symbol s, defined by

$$(\mathrm{Op}^w(s)u)(x) =$$

$$(2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} s\left(\frac{x+x'}{2}, \xi\right) e^{i(x-x')\cdot\xi} u(x') dx' d\xi.$$

If, for instance, $s \in L^2(\mathbb{R}^{2d})$, then $\mathrm{Op}^w(s)$ is Hilbert–Schmidt, and

$$\|\mathrm{Op}^w(s)\|_2^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} |s(x,\xi)|^2 \, dx \, d\xi.$$

Introduce the harmonic oscillator

$$h := -\frac{d^2}{dx^2} + x^2,$$

self-adjoint in $L^2(\mathbb{R})$. We have

$$\sigma(h) = \bigcup_{q=0}^{\infty} \{2q+1\},$$

$$\dim \mathrm{Ker}\,(h - (2q+1)) = 1, \quad q \in \mathbb{Z}_+.$$

Let p_q be the orthogonal projection onto $\mathrm{Ker}(h-(2q+1))$, $q \in \mathbb{Z}_+$. We have

$$p_q = |\varphi_q\rangle\langle\varphi_q|$$

where

$$\varphi_q(x) := \frac{\mathrm{H}_q(x)e^{-x^2/2}}{(\sqrt{\pi}2^q q!)^{1/2}}, \quad x \in \mathbb{R}, \quad q \in \mathbb{Z}_+,$$

H_q being the Hermite polynomials. Let Ψ_q be the Wigner function associated with φ_q, i.e. the Weyl symbol of p_q. For $(x,\xi) \in \mathbb{R}^2$ and $q \in \mathbb{Z}_+$, we have

$$\Psi_q(x,\xi) = \frac{2(-1)^q}{2\pi}\mathrm{L}_q(2(x^2+\xi^2))e^{-(x^2+\xi^2)},$$

L_q being the Laguerre polynomials.

For $(x, y) \in \mathbb{R}^2$ set

$$V_B(x, y) = V(-B^{-1/2}y, -B^{-1/2}x).$$

Theorem 7. *There exists a unitary operator* $U_B : L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ *such that for any* $V \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ *and* $q \in \mathbb{Z}_+$*, we have*

$$\mathcal{U}_B^* P_q V P_q \mathcal{U}_B = p_q \otimes \mathrm{Op}^w(V_B * \Psi_q).$$

Remarks: (i) The operator $\mathcal{U}_B$ has a metaplectic nature.
(ii) The operator $\mathrm{Op}^w(V_B * \Psi_q)$ could be interpreted as a ΨDO with generalized anti-Wick symbol equal to V_B.

The proof of Theorem 7 is based on techniques from the theory of Weyl ΨDOs and ΨDOs with contravariant symbols.

6.3. Reduction of $\mathrm{Op}^w(V_B * \Psi_q)$ *to* $\mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})$

For $r > 0$ define $\delta_r \in \mathcal{S}'(\mathbb{R}^2)$ by

$$\delta_r(\varphi) := \frac{1}{2\pi}\int_0^{2\pi} \varphi(r\cos\theta, r\sin\theta)d\theta, \quad \varphi \in \mathcal{S}(\mathbb{R}^2).$$

Lemma 8. *(i) Let* $V(\mathbf{x}) = \langle \mathbf{x}\rangle^{-\kappa}$, $\mathbf{x} \in \mathbb{R}^2$, $\kappa > 1$, *or* $V \in C_0^\infty(\mathbb{R}^2)$. *Then*

$$\lim_{q\to\infty} \lambda_q^{1/2} \|\mathrm{Op}^w(V_B * \Psi_q) - \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})\| = 0.$$

(ii) Let $V \in C_0^\infty(\mathbb{R}^2)$. *Then*

$$\lim_{q\to\infty} \lambda_q^{1/4} \|\mathrm{Op}^w(V_B * \Psi_q) - \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})\|_2 = 0.$$

The proof is based on a careful analysis of the asymptotic properties of Ψ_q as $q \to \infty$.

6.4. Norm estimate of $\mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})$

Lemma 9. *Let* $V(\mathbf{x}) = \langle \mathbf{x} \rangle^{-\kappa}$, $\mathbf{x} \in \mathbb{R}^2$, $\kappa > 1$. *Then*

$$\|\mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})\| = O(\lambda_q^{-1/2}), \quad q \to \infty.$$

The proof is based on techniques from the theory of Weyl ΨDOs.

6.5. Asymptotics of traces

Theorem 10. *Let* $V \in C_0^\infty(\mathbb{R}^2)$. *Then for each* $\ell \in \mathbb{N}$, $\ell \geq 2$, *we have*

$$\lim_{q\to\infty} \lambda_q^{(\ell-1)/2} \mathrm{Tr}\ \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})^\ell =$$

$$\frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} \tilde{V}(\omega, b)^\ell \, db \, d\omega. \qquad (10)$$

The proof is based on techniques from the theory of Weyl ΨDOs, and the stationary phase method.