

On the spectral theory of the Schrödinger operator with electromagnetic potential

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¹Partly supported by the Bulgarian Science Foundation under Grant MM 8/91

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1 Introduction

In the present survey we consider various aspects of the spectral theory of the Schrödinger operator with electromagnetic potential

$$H \equiv H_{h,\mu,g}(A, V) := (ih\nabla + \mu A)^2 + gV \tag{1.1}$$

where $V : \mathbf{R}^m \rightarrow \mathbf{R}$ is the electric (scalar) potential, $A = (A_1, \dots, A_m) : \mathbf{R}^m \rightarrow \mathbf{R}^m$ is the magnetic (vector) potential, h is the Planck constant, μ is the magnetic-field coupling constant, and g is the electric-field coupling constant. Under some natural assumptions about A and V , the operator H supplied, if necessary, with appropriate boundary conditions, is well-defined as an operator selfadjoint in $L^2(\Omega)$ where $\Omega \subseteq \mathbf{R}^m$, $m \geq 2$, is an open set. Usually, we assume for simplicity $\Omega = \mathbf{R}^m$.

The main goal of the survey is to outline some of the specific spectral properties of H which are due to the fact that the magnetic potential A generates a nonvanishing magnetic field

$$B \equiv \text{curl } A := \{B_{j,k}\}_{j,k=1}^m, B_{j,k} := \partial_j A_k - \partial_k A_j, \tag{1.2}$$

where $\partial_j := \partial/\partial X_j$, $j = 1, \dots, m$.

Amongst the magnetic fields of physical interest we would mention the constant one, and the magnetic field whose norm

$$|B(X)| := \left(\sum_{j,k=1}^m |B_{j,k}(X)|^2 \right)^{1/2} \tag{1.3}$$

grows unboundedly as $|X| \rightarrow \infty$. The most striking difference between the case $\mu = 0$ (which has been studied intensively during the last four decades) and the case $\mu \neq 0$ (which is the one considered in this paper) is the fact that the classical-mechanics approximations concerning the spectrum of H , generically valid for $\mu = 0$, are false for practically all magnetic potentials of physical interest if $\mu \neq 0$. Two of the simplest and best known statements concerning these approximations (which can be regarded as special versions of the correspondence principle in quantum mechanics) are formulated below in Conjectures 1.1 and 1.2.

For $\lambda \in \mathbf{R}$ introduce the quantity

$$\mathcal{V}(\lambda) \equiv \mathcal{V}_{h,\mu,g}(\lambda) := (2\pi)^{-m} \text{vol} \left\{ (X, \Xi) \in T^*\Omega : |h\Xi - \mu A(X)|^2 + gV(X) < \lambda \right\}$$

which coincides with the classical phase-space volume associated with the interval $(-\infty, \lambda)$, i.e. the volume of this part of $T^*\Omega$ where the values of the complete symbol of the operator $H_{h,\mu,g}$ are smaller than λ . Note that the quantity $\mathcal{V}_{h,\mu,g}(\lambda)$ is independent of μ ; in particular, $\mathcal{V}_{h,\mu,g}(\lambda) = \mathcal{V}_{h,0,g}(\lambda)$ for all real μ and λ .

Conjecture 1.1 *The essential spectrum $\sigma_{\text{ess}}(H_{h,\mu,g})$ of the operator $H_{h,\mu,g}$ does not intersect with the semiaxis $(-\infty, \lambda)$, $\lambda \in \mathbf{R}$, if and only if the quantity $\mathcal{V}_{h,\mu,g}(\lambda)$ is finite.*

Assume that the spectrum of the operator $H_{h,\mu,g}$ on the interval $(-\infty, \lambda)$ is purely discrete. Then $\mathcal{N}_{h,\mu,g}(\lambda)$ denotes the total multiplicity of the eigenvalues of $H_{h,\mu,g}(\lambda)$ smaller than λ .

Conjecture 1.2 *The quantities $\mathcal{N}_{h,\mu,g}(\lambda)$ and $\mathcal{V}_{h,\mu,g}(\lambda)$ are asymptotically equivalent, i.e. we have*

$$\mathcal{N}_{h,\mu,g}(\lambda) = \mathcal{V}_{h,\mu,g}(\lambda)(1 + o(1)),$$

as $h \downarrow 0$, or as $g \rightarrow \infty$, or as λ approaches the lower bound of $\sigma_{\text{ess}}(H)$ (or as $\lambda \rightarrow \infty$, if the spectrum of H is purely discrete.)

It has been well-known for a long time that if $\mu = 0$, then Conjectures 1.1 and 1.2 are valid for a wide class of electric potentials V including the ones which grow unboundedly as $|X| \rightarrow \infty$, and the ones which tend to zero at infinity (see [Re.Sim 4, Chapter XIII]). Later, it has been found that for some more sophisticated potentials V , Conjecture 1.2 is false even if $\mu = 0$ (see [Sim 3], [Rob], [So], [Gur], [Rai 1]). In the case $\mu \neq 0$ the situation changes in a drastic way: – if the magnetic field is not identically zero, Conjectures 1.1 and 1.2 are not valid even in the simplest cases. For example, Conjecture 1.1 predicts that the essential spectrum of the operator $H_{1,\mu,0}$ coincides with the semiaxis $[0, \infty)$ for all $\mu \in \mathbf{R}$; as a matter of fact, if $\mu \neq 0$, then $\sigma_{\text{ess}}(H_{1,\mu,0})$ may be empty, may consist of countably many points or bands lying on the positive semiaxis, or may coincide with $[\Lambda, \infty)$ for some $\Lambda \geq 0$, depending on the properties of the magnetic field B .

The development of the spectral theory of the Schrödinger operator with magnetic field has been given a strong impetus by the series of three papers [Av.Her.Sim 1] – [Av.Her.Sim 3]. After them, a great lot of interesting works on this topic has been published. Some of the results contained in these works have already been summarized in the surveys [Hun], [Cy.Fr.Ki.Sim] and [Hel]. For these reasons, it seemed to us unreasonable to try to include into the present survey all the significant results

on the spectral theory of the Schrödinger operator with magnetic field, so that we have restricted our attention to those several topics which are closest to our own research interests. These topics are connected with the circle of problems concerned in Conjectures 1.1 – 1.2, and could be put into the following three groups:

(i) Localization of the essential spectrum of the operator H and, in particular, establishment of necessary and sufficient conditions for the resolvent compactness of H ;

(ii) Investigation of the discrete-spectrum behaviour of H near the possible “accumulation points”;

(iii) Study of the spectrum of H in the semiclassical limit, in the strong and weak electric-field limits, and in the strong and weak magnetic-field limits.

In particular, one of the main purposes of this survey is to offer the correct analogues of Conjectures 1.1 and 1.2.

Let us summarize briefly the contents of the paper.

In Section 2 we have put together some basic properties of the operator H such as selfadjointness and gauge invariance. Here we also discuss briefly some of the interesting effects arising in the case where H acts in $L^2(M)$, M being a manifold with nontrivial first or, respectively, second cohomology group; namely, these effects are the Aharonov-Bohm effect, and, respectively, the Dirac quantization of the magnetic field. Further, in Subsection 2.3 we allocate special attention to the case of a constant magnetic field since such a field is important from an applied point of view. Besides, the results of Subsection 2.3 are employed essentially in the following sections.

Finally, in Subsection 2.4 we describe “the magnetic version” of the Floquet-Bloch theory for the operator $H(A, V)$ with periodic $B = \text{curl } A$ and V .

Many of the results of Section 2 have been published in detail in various works. Nevertheless, we prove the most important ones among them in order to make our exposition comparatively self-contained.

In Section 3 we establish some necessary and sufficient conditions which guarantee $\sigma_{\text{ess}}(H) = \emptyset$; in other words, these conditions follow from, or entail the resolvent compactness of H .

Moreover, in Section 3 we present some results which allow to localize the essential spectrum of H in the cases where the resolvent of this operator is not compact. Some examples of Schrödinger operators with remarkable essential spectra are given here too. Most of the results in Section 3 are due to [Hel.Moh] and [Iwa 4].

In Section 4 we discuss the asymptotics of the isolated eigenvalues of H in a vicinity of the possible “accumulation points”.

In the case where the resolvent of H is compact, we describe the results of [CdV], [Tam 1] and [Mat 2] on the asymptotics of $\mathcal{N}_{1,1,1}(\lambda)$ as $\lambda \rightarrow \infty$.

Further, we consider the case of constant magnetic fields and electric potentials which decay at infinity, and present a series of results describing the behaviour of the isolated eigenvalues of H near its essential-spectrum tips. These results are due to [Sob 1] – [Sob 4], [Tam 2] and [Rai 2] – [Rai 3].

The rest of Section 4 contains the asymptotic estimates of the quantity $\mathcal{N}_{1,1,1}(\lambda)$ as λ approaches a given possible "accumulation point". These estimates are due to [Moh.Nou] and [Moh 1].

Finally, Section 5 contains results on the approximations of the spectrum of $H_{h,\mu,g}$ depending on the characteristic behaviour of the parameters h , μ and g .

a) Strong-electric-field limit ($g \rightarrow \infty$). Here we consider the case of arbitrary A and electric potentials V rapidly decaying at infinity (i.e. $V \in L^{m/2}(\mathbf{R}^m)$, $m \geq 3$), and the case of constant B and slowly decaying V (i.e. $V(X) \asymp -|X|^{-\alpha}$ as $|X| \rightarrow \infty$, with $\alpha \in (0, 2]$). We obtain the first asymptotic term of the quantity $\mathcal{N}_{1,1,g}(\lambda)$ as $g \rightarrow \infty$ for a fixed λ lying strictly under the essential-spectrum lower bound of the operator $H_{1,1,0}$. Besides, we introduce an analogue of $\mathcal{N}_{1,1,g}(\lambda)$ defined for arbitrary real λ in the resolvent set $\rho(H_{1,1,0})$ of the operator $H_{1,1,0}$, and investigate its asymptotic behaviour as $g \rightarrow \infty$. These results on the strong-electric-field approximation are due to [Rai 4], [Bir.Rai] and [Rai 5].

b) Weak-electric-field limit ($g \downarrow 0$). For the case $m = 3$ we consider constant magnetic fields and axisymmetric electric potentials which decay at infinity. Then the operator H is unitarily equivalent to the orthogonal sum $\sum_{l \in \mathbf{Z}}^{\oplus} H^{(l)}$ where $H^{(l)}$ is the restriction of H onto the subspace corresponding to a fixed magnetic wavenumber $l \in \mathbf{Z}$. We study the asymptotics of the ground energy of $H^{(l)}$ with fixed $l \in \mathbf{Z}$ as $g \downarrow 0$. These results are due to [Sol].

c) Strong-magnetic-field limit ($\mu \rightarrow \infty$). First, in the case $m = 3$ we consider the operator $H_{1,\mu,1}(A, V)$ with constant magnetic field and Coulomb electric potential. In this case H is again unitarily equivalent to the orthogonal sum $\sum_{l \in \mathbf{Z}}^{\oplus} H^{(l)}$. We offer without proof a result concerning the asymptotics of the ground energy of $H^{(l)}$, $l \in \mathbf{Z}$, as $\mu \rightarrow \infty$. This result is due to [Av.Her.Sim 3]. Besides, we describe briefly the approach of B.Helffer and J. Sjöstrand (see [Hel.Sjö 1], [Hel.Sjö 3], [Hel.Sjö 5]) to the study of the spectrum of the operator $H_{1,\mu,1}$ in the case $m = 2$, constant magnetic field B and electric potential V which is bounded together with all its derivatives.

d) Weak-magnetic-field limit ($\mu \downarrow 0$). In the case of electric potentials which decay at infinity we prove some results relative to the stability of the negative eigenvalues of the operator $H_{1,0,1}$ with respect to perturbations by small magnetic potentials. More precisely, we show that for generic nonpositive λ the quantity $\mathcal{N}_{1,\mu,1}(\lambda)$ tends to $\mathcal{N}_{1,0,1}(\lambda)$ as $\mu \downarrow 0$ in the case where the quantity $\mathcal{N}_{1,0,1}(\lambda)$ is finite. In the case where $\mathcal{N}_{1,0,1}(0)$ is infinite, we consider the asymptotics of $\mathcal{N}_{1,\mu,1}(0)$ as $\mu \downarrow 0$. These results are due to [Rai 6]. Moreover, we consider the case of constant magnetic fields and periodic electric potentials and briefly discuss the approach of B.Helffer and J.Sjöstrand (see [Hel.Sjö 2], [Hel.Sjö 3], [Hel.Sjö 5]) to the weak-magnetic-field approximation.

e) Semiclassical limit ($h \downarrow 0$). First, we consider the semiclassical asymptotics of the quantity $\mathcal{N}_{h,1,0}(\lambda)$ with fixed λ for the case where the resolvent of $H_{h,1,0}$ is compact. These results are essentially due to [Tam 1] and [CdV]. Further, following [Rai 6], we study the behaviour of $\mathcal{N}_{h,1,1}(\lambda)$, $\lambda \leq 0$, as $h \downarrow 0$, for the case of arbitrary magnetic potentials A and rapidly decaying V , and for the case of constant magnetic fields and

electric potentials V which decay slowly at infinity.

Finally, we discuss the results of [Hel, Subsect.7.2.2] on the semiclassical analysis of the Aharonov-Bohm effect for a bound state of H .

Many results similar to the ones of Sections 4–5 concerning the cases $m = 2, 3$, but containing more precise asymptotic information, have been announced by V.Ya.Ivrii in [Ivr 1] – [Ivr 5]. The proofs of Ivrii’s results however require a sophisticated technique and for this reason these results remain out of the scope of the present survey. They have been summarized in the preprints [Ivr 6] and will soon appear in a monograph by the same author.

Moreover, in the present survey we consider only the Schrödinger operator $H(A, V)$ acting in the space of scalar functions.

Physically, such an operator can be regarded as the Hamiltonian of a spinless quantum particle. The matrix analogues of the operator $H(A, V)$ are studied in [Man.Shi], [Shi] and [Cy.Fr.Ki.Sim, Chapters 6, 12]. Some of these analogues can be interpreted from physics point of view as the Hamiltonians of a quantum particle of spin $\frac{1}{2}$.

At the end of the introduction we would say a few words about the mathematical methods used in the survey. A considerable part of the results presented here has been established by means of a variational technique of Weyl-Courant type (see [Bir.Sol 1] and [Re.Sim 4]). In the particular case of the Schrödinger operator in magnetic field such a technique has been developed in [Sol], [Sob 1] – [Sob 4], [CdV], [Tam 2] – [Tam 3] and [Rai 2] – [Rai 6]. Moreover, we utilize essentially the machinery of pseudodifferential operators (Ψ DO) with Weyl and antiwick symbols (see [Hör 2] and [Shu 1]). This machinery has been already used in the spectral theory of the Schrödinger operator with magnetic field in [Ivr 1] – [Ivr 6], [Tam 1] – [Tam 3], [Hel.Sjö 1] – [Hel.Sjö 5], [Rai 2], [Rai 3], [Rai 6] and [Sob 5].

The idea to write this survey was born during the visit of the second author to the University of Nantes in the spring of 1989 in accordance with the agreement for scientific collaboration between the Institute of Mathematics of the Bulgarian Academy of Sciences and the Department of Mathematics of the University of Nantes. He would like to thank his colleagues in Nantes for their hospitality and, especially, Prof. B. Helffer for many stimulating discussions.

2 General spectral properties of the Schrödinger operator with electromagnetic potential

In the first three sections of the paper we assume $\hbar = \mu = g = 1$ and omit these indices in the notation of H .

This section contains preliminary information about the spectral properties of the operator $H(A, V)$.

In Subsection 2.1 we establish the most general conditions which guarantee either the closability and the lower-boundedness of the quadratic form of H on $C_0^\infty(\Omega)$,

$\Omega \subseteq \mathbf{R}^m$, or the essential selfadjointness of H on $C_0^\infty(\mathbf{R}^m)$. Thus we make applicable the standard approaches to the selfadjoint realizations of H . Besides, in Subsection 2.1 we describe some of the most general estimates of the spectrum of $H(A, V)$.

Subsection 2.2 contains a discussion of the unitary invariance of H under gauge transformations. We show that if the domain Ω is simply connected, then the operators $H(A^{(1)}, V)$ and $H(A^{(2)}, V)$ are unitary equivalent under gauge transformations if and only if $\text{curl} (A^{(1)} - A^{(2)}) \equiv 0$. Moreover, we show that if Ω is not simply connected, then the spectral properties of $H(A, V)$ may differ from those of $H(0, V)$ even if $\text{curl} A \equiv 0$ (the Aharonov-Bohm effect). Finally, we discuss the possibility to define correctly a Hamiltonian with prescribed magnetic field B over a closed Riemannian manifold and establish the admissible class of such fields B (the Dirac quantization of the magnetic field).

Subsection 2.3 is devoted to the special case of constant magnetic fields. In the case $\Omega = \mathbf{R}^m$ we establish the unitary equivalence between the operator H and another operator which is usually easier to be dealt with. Besides, we consider the case where Ω is a cube and derive some important estimates of the eigenvalue-distribution function for H due to Y. Colin de Verdière ([CdV]).

In Subsection 2.4 we give a brief description of the Floquet-Bloch theory concerning the operator $H(A, V)$ with periodic $B = \text{curl} A$ and V .

2.1 The selfadjointness of H

Most of the results of this section can be found in [Av.Her.Sim 1] or in [Cy.Fr.Ki.Sim, Sect. 1.3].

Throughout the subsection except §2.1.3 we assume $\Omega = \mathbf{R}^m$.

2.1.1. In many aspects of the spectral theory of the operator $H(A, V)$, it is convenient to compare this operator with $H(0, V)$, and then to employ the well-known properties of $H(0, V)$. A central role in this approach is played by the Kato-Simon inequality (called sometimes the diamagnetic inequality.) Its formulation (see Theorem 2.1 below) is preceded by two important auxiliary assertions.

Assume that the following conditions are fulfilled

$$A = (A_1, \dots, A_m) \in L_{loc}^2(\mathbf{R}^m; \mathbf{R}^m), V_+ \in L_{loc}^1(\mathbf{R}^m; \mathbf{R}), V_+ \geq 0. \quad (2.1)$$

Introduce the extended momentum operators

$$L_j \equiv L_j(A) := -i\partial_j - A_j, j = 1, \dots, m. \quad (2.2)$$

Further, introduce the quadratic form

$$h_+[u] := \int_{\mathbf{R}^m} \left(\sum_{j=1}^m |L_j u|^2 + V_+ |u|^2 \right) dX, \quad (2.3)$$

on the domain

$$\left\{ u \in L^2(\mathbf{R}^m) : L_j u \in L^2(\mathbf{R}^m), j = 1, \dots, m, V_+^{1/2} u \in L^2(\mathbf{R}^m) \right\}.$$

Obviously, h_+ is closed in $L^2(\mathbf{R}^m)$. Denote by H_+ the selfadjoint operator generated by the quadratic form h_+ in $L^2(\mathbf{R}^m)$.

Lemma 2.1 [Lei.Si, Theorem 1] *Let the conditions (2.1) hold. Then $C_0^\infty(\mathbf{R}^m)$ is a form core for the operator H_+ .*

Sketch of the proof. First we show that $D[h_+] \cap L^\infty(\mathbf{R}^m)$ is dense in $D[h_+]$ with respect to the norm $\|\cdot\|_h$ generated by the quadratic form $h_+[u] + \|u\|_{L^2(\mathbf{R}^m)}^2$.

Set $\phi_n(t) = 1$ if $0 \leq t \leq n$, $\phi_n(t) = n/t$ if $t > n$, $n \in \mathbf{N}_+ \equiv \mathbf{N} \setminus \{0\}$, and $u_n = \phi_n(|u|)u$, $u \in D[h_+]$. Note that the identity $\partial_j |u| = \operatorname{Re}(i\bar{u}|u|^{-1}L_j u)$ implies the inequality

$$|\partial_j |u|| \leq |L_j u|, j = 1, \dots, m, \quad (2.4)$$

which holds almost everywhere in \mathbf{R}^m . Hence, we get $\lim_{n \rightarrow \infty} \|u_n - u\|_h = 0$.

Further, we prove that the linear set $\mathcal{C}_0 := \{\phi u : \phi \in C_0^\infty(\mathbf{R}^m), u \in D[h_+] \cap L^\infty(\mathbf{R}^m)\}$ is dense in $D[h_+]$ (see [Lei.Si, Lemma 3].)

Let $\phi \in C_0^\infty(\mathbf{R}^m)$ and $\phi \equiv 1$ in a neighbourhood of the origin. Put $\phi_n(X) = \phi(X/n)$, $n \in \mathbf{N}_+$, and $u_n = \phi_n u$, $u \in D[h_+] \cap L^\infty(\mathbf{R}^m)$. By virtue of the relations

$$iL_j(\Phi v) = i\Phi L_j v + v\partial_j \Phi,$$

$$\|V_+^{1/2}\Phi v\|_{L^2(\mathbf{R}^m)} \leq \|V_+^{1/2}v\|_{L^2(\mathbf{R}^m)}\|\Phi\|_{L^\infty(\mathbf{R}^m)}, \forall v \in D[h_+], \forall \Phi \in C_0^\infty(\mathbf{R}^m),$$

we find that $\lim_{n \rightarrow \infty} \|u_n - u\|_h = 0$.

Finally, we prove that $C_0^\infty(\mathbf{R}^m)$, is dense in $D[h_+]$. Take $u \in \mathcal{C}_0$ and set $u_\varepsilon := \mathbf{J}_\varepsilon u$ where $\mathbf{J}_\varepsilon, \varepsilon > 0$, is the standard Friedrichs mollifier. Then we have $\lim_{\varepsilon \downarrow 0} \|u_\varepsilon - u\|_h = 0$.

Corollary 2.1 [Lei.Si, Lemma 5] *Let (2.1) hold. Assume $A^{(n)} \in C_0^\infty(\mathbf{R}^m; \mathbf{R}^m)$, $A^{(n)} \rightarrow A$ as $n \rightarrow \infty$ in $L_{loc}^2(\mathbf{R}^m)^m$ and $V^{(n)} \in C_0^\infty(\mathbf{R}^m; \mathbf{R}_+)$, $V^{(n)} \rightarrow V$ as $n \rightarrow \infty$ in $L_{loc}^1(\mathbf{R}^m)$. Then we have $H^{(n)} := H(A^{(n)}, V^{(n)}) \rightarrow H(A, V)$ as $n \rightarrow \infty$ in the strong resolvent sense.*

Proof. Define the quadratic form $h^{(n)}$ replacing in (2.3) A by $A^{(n)}$ and V_+ by $V^{(n)}$. Take $f \in L^2(\mathbf{R}^m)$ and set $u_n = (H^{(n)} + i)^{-1}f$. Then we have $\|u_n\|_{L^2} \leq \|f\|_{L^2}$ and $h^{(n)}[u_n] \leq \|f\|_{L^2}$. Therefore, $\{u_n\}$ contains a subsequence (denoted again by $\{u_n\}$) such that

$$\lim_{n \rightarrow \infty} u_n = u, \lim_{n \rightarrow \infty} (V^{(n)})^{1/2}u_n = v, \lim_{n \rightarrow \infty} L_j(A^{(n)})u_n = w_j, j = 1, \dots, m,$$

weakly in $L^2(\mathbf{R}^m)$. It is easy to check that $u \in D[h_+]$ and, besides, $v = V_+^{1/2}u$, $w_j = L_j u, j = 1, \dots, m$. Moreover, we have

$$h_+[u, \varphi] = (f - iu, \varphi), \forall \varphi \in C_0^\infty(\mathbf{R}^m),$$

where $h_+[\cdot, \cdot]$ is the sesquilinear form generated by the quadratic one $h_+[\cdot]$, and (\cdot, \cdot) is the inner product in $L^2(\mathbf{R}^m)$.

Hence, by virtue of Lemma 2.1, we find that $u = (H + i)^{-1}f$. Since we could have started with an arbitrary subsequence of $\{u_n\}$, we obtain the relation

$$\lim_{n \rightarrow \infty} ((H^{(n)} + i)^{-1}f, g) = ((H + i)^{-1}f, g), \forall f \in L^2(\mathbf{R}^m), \forall g \in L^2(\mathbf{R}^m),$$

implying the strong resolvent convergence (see [Re.Sim 1, Sect. VIII.7]).

Let T and Q be two bounded operators acting in the Lebesgue space $L^2(M, d\mu)$ where M is a space with measure μ . We write $T \preceq Q$ if and only if the estimate

$$|(Tu)(X)| \leq (Q|u|)(X)$$

is valid for almost every $X \in M$ and for each $u \in L^2(M, d\mu)$.

Theorem 2.1 *Assume that the conditions (2.1) are fulfilled. Then the Kato-Simon inequality*

$$\exp(-tH_+) \preceq \exp(-t(-\Delta + V_+)) \leq \exp(t\Delta) \quad (2.5)$$

holds for each $t \geq 0$.

Proof. Bearing in mind Corollary 2.1, we may assume $A \in C_0^\infty(\mathbf{R}^m; \mathbf{R}^m)$ and $V_+ \in C_0^\infty(\mathbf{R}^m; \mathbf{R}_+)$, without any loss of generality. By the generalized Trotter's product formula (see [Kat.Mas]), we have

$$\begin{aligned} \exp\{-tH(A, V_+)\} = \\ s - \lim_{n \rightarrow \infty} \left[\exp\left(-\frac{t}{n}L_1^2\right) \dots \exp\left(-\frac{t}{n}L_m^2\right) \exp\left(-\frac{t}{n}V_+\right) \right]^n. \end{aligned} \quad (2.6)$$

Set $\Phi_j(X) := \int_0^{X_j} A_j(X_j, \dots, X_{j-1}, Y, X_{j+1}, \dots, X_m) dY, j = 1, \dots, m$. Thus we obtain $L_j = \exp(i\Phi_j)(-i\partial_j)\exp(-i\Phi_j)$ and, hence,

$$\exp\left(-\frac{t}{n}L_j^2\right) = \exp(i\Phi_j)\exp\left(\frac{t}{n}\partial_j^2\right)\exp(-i\Phi_j), j = 1, \dots, m. \quad (2.7)$$

Now, (2.6) - (2.7) entail (2.5).

Remark. The validity of the inequality (2.5) follows also from the Feynman-Kac-Itô formula for the distributional kernel $K_{A,V_+}(X, Y; t)$ of the operator $\exp(-tH_+)$ in the terms of a path integral with respect to the Wiener measure. Namely, if the potential (A, V_+) satisfies (2.1) and, moreover, $\operatorname{div} A = 0$ in the distributional sense, then we have

$$K_{A,V_+}(X, Y; t) = \int dE_{0,X;t,Y}(\omega(s)) \exp \left\{ -i \int_0^t A(\omega(s)) \cdot d\omega(s) - \int_0^t V_+(\omega(s)) ds \right\} \quad (2.8)$$

where $E_{0,X;t,Y}(\omega(s))$ is the conditional Wiener measure over the set of paths $\{\omega(s), s \in [0, t]\}$ satisfying $\omega(0) = X, \omega(t) = Y$, (see [Sim 2, Theorem 15.5]). In particular, we have

$$|K_{A,V_+}(X, Y; t)| \leq K_{0,V_+}(X, Y; t) \quad (2.9)$$

for each $t \geq 0$ and almost every $(X, Y) \in \mathbf{R}^{2m}$.

2.1.2. Applying the well-known operator identity

$$(Q + E)^{-\gamma} = \int_0^\infty e^{-tQ} e^{-tE} t^{\gamma-1} dt / \Gamma(\gamma), \quad E > 0, \gamma > 0, \quad (2.10)$$

where $Q \geq 0$ is a selfadjoint operator, we find that (2.5) entails the validity of the inequality

$$(H_+ + E)^{-\gamma} \preceq (-\Delta + V_+ + E)^{-\gamma} \leq (-\Delta + E)^{-\gamma}, \quad \forall E > 0, \forall \gamma > 0. \quad (2.11)$$

Proposition 2.1 *Let $V_- \geq 0$ be a measurable function over \mathbf{R}^m . Assume that the conditions (2.1) are fulfilled. If the multiplier by V_- is Δ -bounded (resp., $-\Delta$ -form-bounded) with relative bound a , then V_- is H_+ -bounded (resp., H_+ -form-bounded) with relative bound (resp., relative form-bound) at most a .*

Proof. Using (2.11) with $\gamma = 1$ (resp., with $\gamma = 1/2$), we get $\|V_-(H_+ + E)^{-1}\| \leq \|V_-(-\Delta + E)^{-1}\|$ (resp., $\|V_-^{1/2}(H_+ + E)^{-1}\| \leq \|V_-^{1/2}(-\Delta + E)^{-1/2}\|$). In order to complete the proof, we have to recall that if T and Q are nonnegative selfadjoint operators and T is Q -bound (resp., Q -form-bound), then the Q -relative bound (resp., Q -relative form-bound) of T can be written as $\lim_{E \rightarrow \infty} \|T(Q + E)^{-1}\|$ (resp., as $\lim_{E \rightarrow \infty} \|T^{1/2}(Q + E)^{-1/2}\|^2$.)

The following theorem concerns the selfadjoint realizations of the operator $H(A, V)$ with $V = V_+ - V_-$, $V_+ \geq 0$ and $V_- \geq 0$.

Theorem 2.2 *Let the potential (A, V_+) satisfy (2.1). Assume that the multiplier by the measurable function $V_- : \mathbf{R}^m \rightarrow \mathbf{R}_+$ is Δ -bounded (resp., $-\Delta$ -form-bounded) with relative bound (resp., relative form bound) smaller than one. Set $V = V_+ - V_-$. Then the operator sum (resp., form sum) $H_+ - V_- \equiv H(A, V)$ is selfadjoint in $L^2(\mathbf{R}^m)$. Moreover we have*

$$\exp(-tH(A, V)) \stackrel{\leq}{\leq} \exp(-tH(0, V)), \forall t \geq 0. \quad (2.12)$$

Proof. The selfadjointness of $H(A, V)$ follows immediately from Proposition 2.1 and the Kato-Rellich theorem [Re.Sim 2, Theorem X.12] (resp. the KLMN theorem [Re.Sim 2, Theorem X.17]).

For the proof of (2.12) see [Av.Her.Sim 1, Theorems 2.4-2.5].)

Remark. The form version of Theorem 2.2 entails implicitly that the quadratic form

$$h[u; A, V] := (H(A, V)u, u) \equiv h_+[u] - \int_{\mathbf{R}^m} V_- |u|^2 dX, u \in C_0^\infty(\mathbf{R}^m), \quad (2.13)$$

is semibounded from below in $L^2(\mathbf{R}^m)$. Thus, by Lemma 2.1, the selfadjoint operator $H(A, V)$ defined as a form sum in (2.13), can be considered as the Friedrichs extension of the symmetric lower-bounded operator $H_+ - V_-$ defined on $C_0^\infty(\mathbf{R}^m)$.

In particular, the quadratic form (2.13) is closed on $D[h_+] \equiv D(H_+^{1/2})$.

2.1.3. Let $\Omega \subseteq \mathbf{R}^m$ be an open set, and let $-\Delta_\Omega^D$ be the selfadjoint operator generated by the closure of the quadratic form $\int_\Omega |\nabla u|^2 dX$, $u \in C_0^\infty(\Omega)$. Assume that the potential $(A, V) \equiv (A, V_+ - V_-)$ satisfies the conditions

(i) $A \in L_{loc}^2(\Omega; \mathbf{R}^m)$ and $V_+ \in L_{loc}^1(\Omega; \mathbf{R}_+)$;

(ii) the multiplier by the measurable function $V_- : \Omega \rightarrow \mathbf{R}_+$ is $-\Delta_\Omega^D$ -form-bounded with relative form-bound smaller than one.

Define the quadratic form h_Ω by analogy with (2.13) and (2.3), substituting \mathbf{R}^m for Ω . As above, we find that the quadratic form h_Ω (defined originally on $C_0^\infty(\Omega)$) is closable and lower-bounded in $L^2(\Omega)$. Denote by $H_\Omega^D(A, V)$ the selfadjoint operator generated by h_Ω .

Corollary 2.2 *Let $\Omega \subseteq \mathbf{R}^m$ be an open set. Assume that the potential (A, V) satisfies the conditions (i) - (ii). Then we have*

$$\inf \sigma(H_\Omega^D(A, V)) \geq \inf \sigma(H_\Omega^D(0, V)). \quad (2.14)$$

Sketch of the proof. The estimate (2.14) follows easily from the analogue of (2.4) for $u \in D[h_\Omega]$, and a simple variational argument.

2.1.4. Note that the form version of Theorem 2.2 provides a purely form-generated selfadjoint realization of the operator $H(A, V)$ since H_+ is defined as the operator

generated by the quadratic form h_+ , and $H(A, V)$ is defined as the form sum of H_+ and $-V_-$. The operator version of the same theorem provides a mixed form-operator realization of H since H_+ is still defined through its quadratic form, but H itself is treated as an operator sum. Now, we present a result which provides a purely operator-generated selfadjoint realization of $H(A, V)$.

Theorem 2.3 [Lei.Si, Theorem 3] *Assume $A \in L_{loc}^4(\mathbf{R}^m; \mathbf{R}^m)$, $\operatorname{div} A \in L_{loc}^2(\mathbf{R}^m)$ and $V_+ \in L_{loc}^2(\mathbf{R}^m; \mathbf{R}_+)$. Moreover, let the multiplier be the measurable function $V_- : \mathbf{R}^m \rightarrow \mathbf{R}_+$ be Δ -bounded with relative bound smaller than one. Set $V = V_+ - V_-$. Then the operator $H(A, V)$ defined in the operator version of Theorem 2.2 is essentially selfadjoint on $C_0^\infty(\mathbf{R}^m)$.*

Sketch of the proof. The assumptions about A and V imply that the function $v := (i\nabla + A)^2 w + Vw \equiv -\Delta w + 2iA \cdot \nabla w + (|A|^2 + V + i \operatorname{div} A)w$ is in $L^2(\mathbf{R}^m)$ if $w \in C_0^\infty(\mathbf{R}^m)$. Moreover, it is easy to see that for $w \in D(H_+)$ we have $v = Hw$.

Both Theorems 2.2 and 2.3 deal with selfadjoint operators $H(A, V)$ which are lower-bounded. However the lower-boundedness of $H(A, V)$ is not a necessary condition for its selfadjointness. One can find examples in [Iwa 5] of operators $H(A, V)$ which are selfadjoint although they are not lower-bounded.

2.1.5. In what follows we denote by \mathbf{S}_∞ the Banach space of the linear compact operators acting in a fixed Hilbert space, and by \mathbf{S}_p , $p \in [1, \infty)$, the Banach spaces of linear compact operators supplied with the norm $\|T\|_p := (\operatorname{Tr}|T|^p)^{1/p}$, $T \in \mathbf{S}_p$, (see e.g. [Bir.Sol 3, Chapter 11]). Respectively, $\|T\|_\infty$ is the usual operator norm of a given bounded operator T .

Our next purpose is to compare the relative compactness and the relative form-compactness of the pairs (H_+, \mathcal{W}) and $(-\Delta, \mathcal{W})$ where \mathcal{W} is the multiplier by a fixed measurable real-valued function over \mathbf{R}^m . To this end, we shall make use of the following abstract lemma.

Lemma 2.2 ([Pitt], [Av.Her.Sim 1, p.850]) *Let $T \preceq Q$. Then T is compact (resp., $T \in \mathbf{S}_{2p}$, $p \in \mathbf{N}_+$) if Q is compact (resp., $Q \in \mathbf{S}_{2p}$).*

Proposition 2.2 *Let \mathcal{W} be the multiplier by a measurable real-valued function over \mathbf{R}^m . Then the following implications are valid.*

- (i) *The operator $\mathcal{W}(H_+ + 1)^{-1}$ is compact if $\mathcal{W}(-\Delta + 1)^{-1}$ is compact.*
- (ii) *The operator $|\mathcal{W}|^{1/2}(H_+ + 1)^{-n/2}$, $n \in \mathbf{N}_+$, is compact if $|\mathcal{W}|^{1/2}(-\Delta + 1)^{-n/2}$ is compact.*

Proof. The proposition is a direct consequence of Lemma 2.2 and the estimate (2.11) with $\gamma = 1$, or with $\gamma = n/2$.

Remark. In the formulation of Proposition 2.2 we write \mathcal{W} instead of V_- in order to underline that \mathcal{W} may not have a definite sign.

Example. Let $m \geq 3$. Suppose that for each $\varepsilon > 0$ the potential \mathcal{W} can be written as a sum $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ where $\mathcal{W}_1 \in L^{m/2}(\mathbf{R}^m)$ and

$$\int_{\mathbf{R}^m} |\mathcal{W}_2| |u|^2 dX \leq \varepsilon \int_{\mathbf{R}^m} \{|\nabla u|^2 + |u|^2\} dX, \forall u \in C_0^\infty(\mathbf{R}^m).$$

Then the operator $|\mathcal{W}|^{1/2}(-\Delta + 1)^{-1/2}$ is compact. If $m = 2$, this is true again if we replace $\mathcal{W}_1 \in L^{m/2}(\mathbf{R}^m)$ by $\mathcal{W}_1 \in L^p(\mathbf{R}^2)$, $p > 1$, and assume that the support of \mathcal{W}_1 is compact. The potential \mathcal{W} satisfies the above condition if, for example, $\mathcal{W} \in L_{loc}^r(\mathbf{R}^m)$ with $r = m/2$ if $m \geq 3$, $r > 1$ if $m = 2$, and $\int_{|X-Y|<1} |\mathcal{W}(Y)|^r dY \rightarrow 0$ as $|X| \rightarrow \infty$.

2.1.6. At the end of this section we formulate the analogue of the Rozenblyum-Cwikel-Lieb estimate of the negative-eigenvalues total multiplicity for the operator $-\Delta + V$, in the case where a magnetic potential $A \not\equiv 0$ is present. At first, however, we shall introduce some necessary notations which will be used throughout the paper.

Let T be a selfadjoint operator in Hilbert space, and $\mathcal{I} \subset \mathbf{R}$ be an open interval. Denote by $P_{\mathcal{I}}(T)$ the spectral projection of T corresponding to \mathcal{I} . Put

$$\mathcal{N}(\mathcal{I}|T) := \text{rank } P_{\mathcal{I}}(T);$$

$$N(\mu; T) := \mathcal{N}((-\infty, \mu)|T), \mu \in \mathbf{R};$$

$$n(\mu; T) := \mathcal{N}((\mu, +\infty)|T), \mu \geq 0.$$

Proposition 2.3 [Av.Her.Sim 1, Theorem 2.15] *Let $m \geq 3$. Assume that the conditions (2.1) are fulfilled. Moreover, suppose $V_- \in L^{m/2}(\mathbf{R}^m; \mathbf{R}_+)$. Set $V = V_+ - V_-$. Then the estimate*

$$N(0; H(A, V)) \leq c_m \int_{\mathbf{R}^m} V_-^{m/2} dX \tag{2.15}$$

holds with a constant c_m which depends only on the dimension m .

Remark. The bound $N(\lambda; H(A, V)) \leq N(\lambda; H(0, V))$, $\lambda \leq 0$, which might seem probable, is false: – see the counterexample [Av.Her.Sim 1, p.856, Ex.2].

2.2 Gauge invariance of the operator H

2.2.1. Let M be a C^∞ -smooth, m -dimensional connected Riemannian manifold. For simplicity, we assume either that $M \subseteq \mathbf{R}^m$ is a domain with C^∞ -smooth (eventually empty) boundary, or that M is a compact manifold without boundary. Let

$g = \{g^{jk}\}_{j,k=1}^m$, be the metric tensor over M . Denote by $d\mu$ the canonical measure over M . In local coordinates we have $d\mu(X) = G(X)dX^1 \dots dX^m$ where $G(X) := (\det g(X))^{-1/2}$. Let $\Lambda^p(M)$ be the set of C^∞ -smooth p -differential forms over \overline{M} , $p = 1, \dots, m$. Denote by $\|\cdot\|_p$ the canonical norm in $\Lambda_p(M)$. Fix the real form $\mathcal{A} \in \Lambda_1(M)$. Set $\mathcal{B} := d\mathcal{A}$. In local coordinates we have

$$\mathcal{A} \equiv \mathcal{A}(A) := \sum_{j=1}^m A_j dX^j, \quad (2.16)$$

$$\mathcal{B} := d\mathcal{A} \equiv \frac{1}{2} \sum_{j,k=1}^m B_{j,k} dX^j \wedge dX^k, \quad B_{j,k} := \partial_j A_k - \partial_k A_j, \quad j, k = 1, \dots, m, \quad (2.17)$$

(cf. (1.2).) Following the physics literature, we shall call the set of coefficients $A = \{A_j\}_{j=1}^m$, a magnetic potential, and the set of coefficients $B = \{B_{j,k}\}_{j,k=1}^m \equiv \text{curl } A$ a magnetic field.

Define the operator $\nabla_A : C_0^\infty(M) \rightarrow \Lambda^1(M)$ by

$$\nabla_A u = du - iu\mathcal{A}$$

Further, let $V \in C^\infty(\overline{M}; \mathbf{R})$ and $V \geq -C$ with some $C \in \mathbf{R}$. Define the quadratic form $h_M[u; A, V]$ as the closure of the quadratic form

$$h_M[u; A, V] = \|\nabla_A u\|_1^2 + \int_M V |u|^2 d\mu, \quad u \in C_0^\infty(M),$$

in $L^2(M; d\mu)$. Let $H_M(A, V)$ be the selfadjoint operator generated by $h_M(A, V)$ in $L^2(M; d\mu)$. Note that if $M \equiv \Omega$ is an open subset of \mathbf{R}^m , and $d\mu$ is the Lebesgue measure, then the operator $H_M(A, V)$ coincides with $H_\Omega^D(A, V)$ (see §2.1.3.) In local coordinates we have

$$H(A, V) = G^{-1} \sum_{j,k=1}^m L_j(A) G g^{jk} L_k(A) + V$$

where $L_j(A) = -i\partial_j - A_j$, $j = 1, \dots, m$, (cf. (2.2).)

Note the validity of the commutation relations

$$-i[L_j; L_k] = B_{j,k}, \quad j, k = 1, \dots, m. \quad (2.18)$$

Let ω be a C^∞ -smooth real p -differential form over \overline{M} , $p = 1, \dots, m$. Assume that ω is closed, i.e. $d\omega = 0$. We say that the cohomology class of ω is 2π -integral if and only if $\int_{\gamma_p} \omega \in 2\pi\mathbf{Z}$ for each p -dimensional cycle γ_p .

Let $A^{(l)}$, $l = 1, 2$, be two real vector potentials over M . Set $A = A^{(2)} - A^{(1)}$. Assume that the 1-form $\mathcal{A}(A)$ is closed, and its cohomology class is 2π -integral. Then we say that the vector potentials $A^{(1)}$ and $A^{(2)}$ are gauge-equivalent, and write $A^{(1)} \sim A^{(2)}$.

Lemma 2.3 *Let $A^{(l)}, l = 1, 2$, be two C^∞ -smooth gauge-equivalent vector potentials. Then the operators $H_M(A^{(1)}, V)$ and $H_M(A^{(2)}, V)$ are unitarily equivalent under gauge transformations, i.e. there exists a function $\mathcal{E} \in C^\infty(\overline{M})$, $|\mathcal{E}(X)| = 1, \forall X \in \overline{M}$, such that we have*

$$\mathcal{E}H_M(A^{(1)}, V) = H_M(A^{(2)}, V)\mathcal{E}. \quad (2.19)$$

Proof. Fix $X_0 \in \overline{M}$. Set $\Phi(X; \gamma) = \int_\gamma \mathcal{A}$ where $\mathcal{A} = \mathcal{A}(A)$, $A = A^{(2)} - A^{(1)}$, and γ is a path connecting the generic point $X \in \overline{M}$ with X_0 . Since the 1-form \mathcal{A} is closed, and its cohomology class is 2π -integral, we have $(\Phi(X; \gamma^{(1)}) - \Phi(X; \gamma^{(2)})) \in 2\pi\mathbf{Z}$ for any pair of paths $\gamma^{(1)}$ and $\gamma^{(2)}$ connecting X with X_0 . Hence, the function $\mathcal{E} = \exp(i\Phi)$ is well-defined over \overline{M} . Besides, $|\mathcal{E}(X)| = 1, \forall X \in \overline{M}$. Moreover, we have

$$\mathcal{E}L_j(A^{(2)}) = L_j(A^{(1)})\mathcal{E}, j = 1, \dots, m,$$

which entails (2.19).

Remark. Assume that M is simply connected. Let \mathcal{A} be a closed 1-form over \overline{M} . Then \mathcal{A} is exact, i.e. there exists a well-defined function Φ over \overline{M} such that $\mathcal{A} = d\Phi$. In particular, in this case the cohomology class of \mathcal{A} is zero. Thus, if M is simply connected, then two vector potentials $A^{(1)}$ and $A^{(2)}$ are gauge-equivalent if and only if there exists a well-defined function Φ over \overline{M} such that we have

$$A^{(1)} = A^{(2)} + \nabla\Phi. \quad (2.20)$$

The function Φ can be defined in the same way as in the proof of Lemma 2.3. However, under the hypotheses of that lemma only $\mathcal{E} = \exp(i\Phi)$ (but not necessarily Φ itself) is well-defined over \overline{M} , while the simple-connectedness of M implies that Φ itself is well-defined over \overline{M} .

2.2.2. In this subsection we assume that $\Omega \subseteq \mathbf{R}^m$ is a connected open set and denote by $\mathcal{D}' = \mathcal{D}'(\Omega)$ the class of distributions dual to $C_0^\infty(\Omega)$.

Proposition 2.4 [Lei, Lemma 1.1 (i)] *Assume $A^{(l)} \in L_{loc}^p(\Omega)^m, l = 1, 2, p \in [1, \infty)$. If there exists a distribution $\Phi \in \mathcal{D}'(\Omega)$ satisfying (2.20), then $\Phi \in W_{1,loc}^p(\Omega)$.*

Theorem 2.4 *Let $A^{(l)} \in L_{loc}^2(\Omega)^m, l = 1, 2, V \in L_{loc}^1(\Omega)$. Assume that (2.20) holds for some Φ . Then the quadratic forms of the operators $H_\Omega^D(A^{(l)}, V), l = 1, 2$, are closable on $C_0^\infty(\Omega)$ simultaneously. Moreover, the operators $H_\Omega^D(A^{(1)}, V)$, and $H_\Omega^D(A^{(2)}, V)$, are unitarily equivalent under gauge transformations, i.e. we have*

$$\exp(i\Phi)H_\Omega^D(A^{(1)}, V)\exp(-i\Phi) = H_\Omega^D(A^{(2)}, V).$$

Theorem 2.5 *Let $A^{(l)} \in L_{loc}^4(\mathbf{R}^m)^m$, $\operatorname{div} A^{(l)} \in L_{loc}^2(\mathbf{R}^m)$, $l = 1, 2$, $V \in L_{loc}^1(\mathbf{R}^m)$. Assume that (2.20) holds for some Φ . Then the operators $H(A^{(l)}, V)$, $l = 1, 2$, are essentially selfadjoint on $C_0^\infty(\mathbf{R}^m)$ simultaneously. Moreover, the operators $H(A^{(1)}, V)$ and $H(A^{(2)}, V)$ are unitarily equivalent under gauge transformations.*

In the case where $A^{(l)}$, $l = 1, 2$, are C^∞ -smooth, Theorems 2.4 - 2.5 are trivial. For the general case see the proofs of [Lei, Theorems 1.2 - 1.3]. Here we only note that by Proposition 2.4, the hypotheses of Theorems 2.4 - 2.5 imply $\Phi \in W_{1,loc}^p$; hence, Φ is a measurable function, and the multiplier by $\exp(i\Phi)$ is a unitary operator.

Proposition 2.5 *Let $\Omega \subseteq \mathbf{R}^m$ be a simply connected open set, and $A^{(l)} \in L_{loc}^p(\Omega)^m$, $l = 1, 2$, $p \in [1, \infty)$. Then the existence of a function $\Phi \in W_{1,loc}^p$ for which (2.20) holds, is equivalent to $\operatorname{curl} A^{(1)} = \operatorname{curl} A^{(2)}$.*

Sketch of the proof. In the case $A^{(l)} \in C^\infty(\bar{\Omega})$, $l = 1, 2$, the proposition follows trivially from the remark at the end of §2.2.1. For the case of general nonsmooth potentials $A^{(l)}$, see the proof of [Lei, Lemma 1.1 (i)].

The physical meaning of Theorems 2.4 - 2.5 and Proposition 2.5 is that the influence of the magnetic potential A upon the spectral properties of the Hamiltonian $H_\Omega^D(A, V)$ for simply connected Ω , is determined thoroughly by the magnetic field $B = \operatorname{curl} A$. At the end of this paragraph we discuss the possibility to obtain a divergence-free magnetic potential A by means of a gauge transform. As a particular consequence, it turns that the requirements about the regularity of $\operatorname{div} A$ occurring in Theorem 2.3 and Theorem 2.5 are not quite essential.

Proposition 2.6 [Lei, Lemma 1.1 (ii)] *Let $\Omega \subseteq \mathbf{R}^m$ be an open set. For any $A \in L_{loc}^p(\Omega)^m$, $p \in [2, \infty)$, there exists $\tilde{A} \in L_{loc}^p(\Omega)^m$ such that $\tilde{A} = A + \nabla\Phi$ for some Φ and*

$$\operatorname{div} \tilde{A} = 0. \tag{2.21}$$

Proof. It suffices to define Φ as a distributional solution of the equation $-\Delta\Phi = \operatorname{div} A$; it is well-known that such solutions exist and belong to $W_{1,loc}^p(\Omega)$ (see e.g. [Hör 1, Chapters 3–4].)

If the potential \tilde{A} satisfies (2.21), then it is said in the physics literature that \tilde{A} obeys the Coulomb gauge condition.

Proposition 2.6 combined with Proposition 2.5 implies that the estimate (2.9) holds for general magnetic potentials $A \in L_{loc}^2(\mathbf{R}^m)^m$ including those which do not satisfy $\operatorname{div} A = 0$ in distributional sense.

Corollary 2.3 [Lei, Corollary 1.4] *Let $A \in L^4_{loc}(\mathbf{R}^m)^m$, $V_+ \in L^2_{loc}(\mathbf{R}^m)$. and V_- be Δ -bounded with relative bound less than 1. Let $\Phi \in \mathcal{D}'(\mathbf{R}^m)$ be any solution of $-\Delta\Phi = \operatorname{div} A$. Then the operator $H(A, V)$ is essentially selfadjoint on $\exp(-i\Phi)C_0^\infty(\mathbf{R}^m)$.*

2.2.3. Our main goal in this paragraph is to show that the closedness of the 1-form $\mathcal{A}(A)$ is essentially insufficient for the unitary equivalence of the operators $H_\Omega^D(A, V)$ and $H_\Omega^D(0, V)$ if the domain Ω is not simply connected. In this case the spectral properties of $H_\Omega^D(A, V)$ and $H_\Omega^D(0, V)$ differ for certain when the cohomology class of \mathcal{A} is not 2π -integral. Physically, this means that if Ω is not simply connected, then there exist vector potentials A generating zero magnetic field such that the Hamiltonians $H_\Omega^D(A, V)$ and $H_\Omega^D(0, V)$ have different spectral properties (the Aharonov-Bohm effect).

Theorem 2.6 [Hel, Theorem 7.2.1.1] *Let $\Omega \subseteq \mathbf{R}^m$ be a connected open set with C^∞ -smooth (eventually empty) boundary. Suppose that $(A, V) \in C^\infty(\bar{\Omega}; \mathbf{R}^{m+1})$ and $V \geq C > -\infty$.*

Assume that the lower bounds of the spectra of the operators $H_\Omega^D(A, V)$ and $H_\Omega^D(0, V)$ are isolated eigenvalues of finite multiplicity denoted respectively by λ_A and λ_0 .

Then the following conditions are equivalent:

- (i) $\lambda_A = \lambda_0$;
- (ii) the operators $H_\Omega^D(A, V)$ and $H_\Omega^D(0, V)$ are unitary equivalent;
- (iii) the potential A is gauge-equivalent to the zero potential.

Proof. The implication (ii) \Rightarrow (i) is trivial. The implication (iii) \Rightarrow (ii) follows directly from Lemma 2.3.

We prove the implication (i) \Rightarrow (iii). Assume $\lambda_A = \lambda_0$. It is well-known that λ_0 is a simple eigenvalue of $H_\Omega^D(0, V)$, and the eigenfunction u_0 corresponding to λ_0 can be chosen to be strictly positive in Ω ; we assume also that u_0 is normalized in $L^2(\Omega)$.

Integrating by parts, we obtain the Lavine – O’Carroll identity

$$\|(\nabla - iA - \nabla u_0/u_0)u\|^2 = h_\Omega[u; A, V] - \lambda_0\|u\|^2, \forall u \in C_0^\infty(\Omega), \quad (2.22)$$

where $\|\cdot\|$ denotes the norm in $L^2(\Omega)$ (see [Lav.O’Ca]).

Further, let u_A denote some normalized in $L^2(\Omega)$ eigenfunction of $H_\Omega^D(A, V)$ corresponding to λ_A . Then the identity (2.22) and the condition (i) easily yield

$$(\nabla - iA - \nabla u_0/u_0)u_A = 0 \text{ in } C^\infty(\bar{\Omega}). \quad (2.23)$$

Set $v_A := u_A/u_0$. Then (2.23) implies

$$dv_A = iv_A \mathcal{A}. \quad (2.24)$$

Since A is real, (2.24) entails $\nabla|v_A| = 0$. Hence, we have $v_A = c_0 e^{i\varphi}$ where c_0 is a positive constant and φ is a real function defined modulo 2π . Thus we get $\nabla\varphi = A$, and, therefore, the cohomology class of \mathcal{A} is 2π -integral.

Remark. The Lavine - O'Carroll identity (2.22) entails, in particular, the inequality (2.14) for the case where $\inf \sigma(H_\Omega^D(0, V))$ and $\inf \sigma(H_\Omega^D(A, V))$ are eigenvalues of $H_\Omega^D(0, V)$ and $H_\Omega^D(A, V)$.

Note that we cannot generalize Theorem 2.6 in a symmetric form as Lemma 2.3. In other words, if we have two nonzero vector potentials $A^{(2)}$ and $A^{(1)}$, we cannot deduce their gauge equivalence (and, hence, the unitary equivalence of $H_\Omega^D(A^{(1)}, V)$ and $H_\Omega^D(A^{(2)}, V)$) just from the coincidence of the first eigenvalues of the operators $H_\Omega^D(A^{(1)}, V)$ and $H_\Omega^D(A^{(2)}, V)$. The intrinsic reason is the fact that the first eigenvalue of the operator $H_\Omega^D(A, V)$ with $A \neq 0$ may be degenerate (see the numerous examples and comments in [Lav.O'Ca], [Av.Her.Sim 1, Sect. 5], [Av.Her.Sim 3, Sects. 3-4], [Hel, Sect. 7.3]).

2.2.4. In this paragraph we assume for simplicity that M is a compact Riemannian manifold without boundary, and discuss the possibility to postulate the axioms of quantum dynamics with prescribed magnetic-field 2-differential form over M . In other words, if $\mathcal{B} = \frac{1}{2} \sum_{j,k=1}^m B_{j,k} dX^j \wedge dX^k$ is a given closed 2-form with real C^∞ -smooth coefficients over M , then the problem is how to define the momentum operators $L_j, j = 1, \dots, m$, which possess the following properties (see [Grü]):

- a) the commutation relations (2.18) are satisfied;
- b) some version of the canonical Heisenberg commutation relations involving the momentum and the coordinate operators holds;
- c) the coordinate operators are pairwise commuting;
- d) the free Hamiltonian H_M is defined as a selfadjoint realization of $\sum_{j=1}^m L_j^2$.

If the 2-form \mathcal{B} is exact, then there exists a C^∞ -smooth 1-form \mathcal{A} such that the equality $\mathcal{B} = d\mathcal{A}$ is valid globally over M . Hence, we can define H_M just as in §2.2.1.

In the case of nonzero cohomology class of \mathcal{B} , however, some problems with the definition of the Hamiltonian H_M arise. If the cohomology class of \mathcal{B} is 2π -integral, these problems can be overcome in the following way. Let E be a complex-line bundle (or, equivalently, a $U(1)$ -vector bundle) over M supplied with a Hermitian structure, and Hermitian linear connection ∇_B whose curvature form coincides with $i\mathcal{B}$; it is well-known that such a bundle exists (see [Kos, Proposition 2.1.1]). In this case the class $C^\infty(M; E) \equiv C_0^\infty(M; E)$ consisting of C^∞ -smooth cross-sections over E (see [CdV], [Dem]) should be used as a form core of the operator H_M instead of the class of functions $C_0^\infty(M)$ which has been used as a form-core for this operator in §2.2.1. Let $\{M_j\}_{j \geq 1}$ be its trivialization coordinate patches of M . Then E admits a trivialization $E_j = \mathbf{C} \times M_j$ over each M_j , and the canonical projection of E acts in E_j as the

projection onto the second factor. Moreover, if $z \in \mathbf{C}$ and $X \in M_j$, then the norm on E_j is defined by $|(z, X)| = |z|$. The cross-sections $u \in C^\infty(M, E)$ can be identified with the collections $\{u_j\}_{j \geq 1}$ of the trivializations $u_j \in C^\infty(\overline{M}_j)$ over the coordinate patches M_j . The connection ∇_B is defined on M_j by $\nabla_B u_j(X) = du_j(X) - iu_j(X)\mathcal{A}_j$ where \mathcal{A}_j is a 1-form over M_j which satisfies $d\mathcal{A}_j = \mathcal{B}$, and u_j is the trivialization of a cross-section $u \in C^\infty(M, E)$ over M_j .

Further, for each pair j, k there exists a transition function $\mathcal{E}_{jk} \in C^\infty(\overline{M}_j \cap \overline{M}_k)$ such that $|\mathcal{E}_{jk}| \equiv 1$ (hence, $\mathcal{E}_{jk}(X)$ for any fixed X can be identified with an element of the structural group $U(1)$), and we have

$$u_j(X) = \mathcal{E}_{jk}(X)u_k(X), \forall X \in M_j \cap M_k, \forall j, k,$$

for each $u \in C^\infty(M; E)$ possessing a collection of trivializations $\{u_j\}_{j \geq 1}$. Then the identity

$$\nabla_B u_j(X) = \mathcal{E}_{jk}(X)\nabla_B u_k(X), \forall X \in M_j \cap M_k, \forall j, k,$$

is the characteristic property of the connection ∇_B .

On $C^\infty(M; E)$ introduce the quadratic form

$$h_M^E[u; B] = \int_M \|\nabla_B u\|^2 d\mu$$

where $\|\cdot\|$ is the norm of the linear mapping $\nabla_B u(X) : T_X M \rightarrow E(X)$, and $E(X)$ is the trivialization of E over any coordinate patch containing $X \in M$. Let $\{e_j(X)\}_{j=1}^m$ be local orthonormal vector fields.

Set $L_j = -i\mathbf{L}_{e_j}$, $j = 1, \dots, m$, where \mathbf{L}_{e_j} is the corresponding Lie derivative. Then we have

$$h_M^E[u; B] = \sum_{j=1}^m \int_M |L_j u|^2 d\mu. \quad (2.25)$$

The quadratic form $h_M^E(B)$ is closable in the Hilbert space $L^2(M; E) \cong L^2(M; d\mu)$. Denote by $H_M^E(B) \equiv H_M$ the selfadjoint operator generated by the closed quadratic form $h_M^E(B)$ in $L^2(M, E)$.

Remark. The 2π -integrability of the cohomology class of \mathcal{B} guarantees just the *existence* of a complex-line bundle with connection ∇_B possessing the described properties; this construction is *unique* up to a bundle isomorphism if and only if the manifold M is simply-connected (see [Kos, Theorem 2.2.1]). Moreover, the operators $H_M^E(B)$ and $H_M^E(-B)$ are antiunitarily equivalent.

We define the Sobolev space $W_1^2(M; E)$ as the closure of $C^\infty(M; E)$ in the norm generated by the quadratic form $h_M^E[u; B] + \|u\|_{L^2(M; E)}^2$.

Example. The following example (see [Grü, Ex.2, p.117,120]) enlightens the requirement that the cohomology class of the magnetic-field 2-form \mathcal{B} is 2π -integrable; this requirement is known in the physics literature as the Dirac quantization of the magnetic field. Let M be the unit sphere $\mathbf{S}^2 = \{X \in \mathbf{R}^3 : |X| = 1\}$. Introduce two coordinate patches $M_{\pm} = \{X \in \mathbf{S}^2 : \pm X_3 \geq -1/2\}$. Let \mathcal{A}_{\pm} be solutions of $d\mathcal{A} = \mathcal{B}$ on M_{\pm} . Then on $M_+ \cap M_-$ we have locally $\mathcal{A}_+ = \mathcal{A}_- + d\Phi$ with some locally smooth function Φ . Hence, the transition functions $\mathcal{E}_{\pm}(X)$ must be in the form $\exp\{\pm i\Phi(X)\}$. Parametrize the sphere \mathcal{S}^2 by the angles $\theta \in [-\pi/2, \pi/2]$ and $\varphi \in (0, 2\pi]$ so that the equator \mathcal{O} is determined by $\{\theta = 0, \varphi \in (0, 2\pi]\}$. Since $\mathcal{E}_{\pm}(X)$ must be well-defined on $M_+ \cap M_-$, the equality

$$\Phi(\theta, \varphi)|_{\varphi=0} = \Phi(\theta, \varphi)|_{\varphi=2\pi} + 2\pi K, \forall \theta \in (-\pi/6, \pi/6),$$

must be valid for some $K \in \mathbf{Z}$. Set $S_{\pm} = \{X \in \mathbf{S}^2 : \pm X_3 > 0\}$. Applying the Stokes theorem, we get $\int_{S_{\pm}} \mathcal{B} = \pm \int_{\mathcal{O}} \mathcal{A}_{\pm}$. Hence, we have

$$\int_{\mathbf{S}^2} \mathcal{B} = \int_{\mathcal{O}} (\mathcal{A}_+ - \mathcal{A}_-) = \int_{\mathcal{O}} d\Phi = 2\pi K.$$

Thus we find that the cohomology class of \mathcal{B} must be 2π -integral.

2.3 The case of a constant magnetic field

2.3.1. Throughout the subsection we suppose that V satisfies the hypotheses of the form version of Theorem 2.2. Moreover, we assume

$$B_{j,k} \equiv \partial_j A_k - \partial_k A_j = \text{const.}, \forall j, k = 1, \dots, m, B \neq 0. \quad (2.26)$$

Note that $iB : \mathbf{C}^m \rightarrow \mathbf{C}^m$ is a Hermitian operator. Since the entries of B are real, the eigenvalues of the matrix B form a subset of the imaginary axis which is symmetric with respect to the origin. Let $b_1 \geq b_2 \geq \dots \geq b_d > 0$ be such numbers that the nonzero eigenvalues of B coincide with the imaginary numbers $-ib_j$ and ib_j , $j = 1, \dots, d$. Thus we have $2d = \text{rank } B$, $0 < 2d \leq m$. Set $k := m - 2d \equiv \dim \text{Ker } B$.

It is well-known that if $k > 0$ (resp., if $k = 0$) we can introduce on \mathbf{R}^m Cartesian coordinates (x, y, z) , $x \in \mathbf{R}^d$, $y \in \mathbf{R}^d$, $z \in \mathbf{R}^k$ (or, respectively, (x, y) with $x \in \mathbf{R}^d$, $y \in \mathbf{R}^d$), such that the 2-form \mathcal{B} can be written as $\mathcal{B} = \sum_{j=1}^d b_j dy^j \wedge dx^j$. Hence, $H(A, V)$ is unitarily equivalent to

$$\mathcal{H}_0 \equiv \mathcal{H}_0(B; V) = \sum_{j=1}^d \left\{ \left(-i \frac{\partial}{\partial x_j} - b_j y_j / 2 \right)^2 + \left(-i \frac{\partial}{\partial y_j} + b_j x_j / 2 \right)^2 \right\} - \sum_{l=1}^k \frac{\partial^2}{\partial z_l^2} + V(x, y, z);$$

if $k = 0$, the second sum at the right-hand side must be omitted. The operator \mathcal{H}_0 is defined originally on $C_0^\infty(\mathbf{R}^m)$, and then is extended by Friedrichs to a selfadjoint operator. Note that if V is polynomially bounded, then the operator \mathcal{H}_0 is essentially selfadjoint on the Schwartz class $\mathcal{S}(\mathbf{R}^m)$.

2.3.2. Our next goal is to construct an operator \mathcal{H}_1 which is unitarily equivalent to \mathcal{H}_0 but in many aspects is easier to be dealt with.

On $\mathcal{S}(\mathbf{R}_x^d)$ define the operator

$$h_1 \equiv h_1(\mathbf{b}) = \sum_{j=1}^d b_j \left(-\partial_j^2 + x_j^2 \right), \mathbf{b} := (b_1, \dots, b_d), \quad (2.27)$$

of the type of the harmonic oscillator, and then close it in $L^2(\mathbf{R}^d)$, thus obtaining a selfadjoint operator $h_1 \equiv h_1(\mathbf{b})$. For further references we state here some of the well-known properties of the operator h_1 . Let $\{\Lambda_q\}$, $q \geq 1$, be the strictly increasing sequence consisting of positive numbers in the form

$$\Lambda_q \equiv \Lambda_q(\mathbf{b}) := \sum_{j=1}^d (2n_j - 1)b_j \quad (2.28)$$

where $n_j, j = 1, \dots, d$, are positive integers. The numbers Λ_q are known in the physics literature as the Landau levels. The number κ_q of different sets $\{n_1, \dots, n_d\}$ which determine one and the same level Λ_q according to (2.28), is called the multiplicity of Λ_q . Note that we have

$$\Lambda_1 = \sum_{j=1}^d b_j, \kappa_1 = 1. \quad (2.29)$$

The spectrum of h_1 is purely discrete and its eigenvalues together with the multiplicities coincide with the Landau levels Λ_q , $q \geq 1$. Let $f_{q,j}$, $j = 1, \dots, \kappa_q$, be the real-valued eigenfunctions of h_1 which are orthonormal in $L^2(\mathbf{R}^d)$, and correspond to the eigenvalue Λ_q , $q \geq 1$. They can be written in the form

$$f_{q,j}(x) = \pi^{-d/4} \exp(-|x|^2/2) \mathcal{P}_{q,j}(x) \quad (2.30)$$

where $\mathcal{P}_{q,j}$ are some polynomials; in particular, $\mathcal{P}_{1,1} \equiv 1$. Note that the eigenfunctions of $h_1(\mathbf{b})$ are independent of \mathbf{b} .

Moreover, if $\mu \geq \mu_0 > \Lambda_1$ and $\gamma > 0$, then the operator $(h_1 + \mu)^{-\gamma}$ can be written as a Ψ DO with antiwick symbol

$$\mathbf{r}_\gamma(x, \xi; \mu) =$$

$$\Gamma(\gamma)^{-1} \int_0^\infty t^{\gamma-1} \exp\{-t(\mu - \Lambda_1)\} \prod_{j=1}^d \exp\left\{-(x_j^2 + \xi_j^2)/(\coth b_j t - 1)\right\} dt \quad (2.31)$$

(see [Shu 1, Sect. 24] for definition of the Ψ DOs with antiwick symbols.) In order to verify (2.31), one has to recall that the Weyl symbol of the operator $\exp(-th_1)$, $t > 0$, can be written as

$$\prod_{j=1}^d (\cosh b_j t)^{-1} \exp\left\{-(x_j^2 + \xi_j^2) \tanh b_j t\right\} \quad (2.32)$$

(cf. [Ber.Shu, Chapter V, Subsect. 4.4].) Recall the connection between the Weyl symbol $a(x, \xi)$ and the antiwick symbol $\tilde{a}(x, \xi)$ of a given Ψ DO

$$a(x, \xi) = \pi^{-d} \int_{\mathbf{R}^{2d}} \exp\{-|x - x'|^2 - |\xi - \xi'|^2\} \tilde{a}(x', \xi') dx' d\xi' \quad (2.33)$$

(see [Shu 1, Sect. 24.2, Eq. (24.12)].) Then the antiwick symbol of the operator $\exp(-th_1)$, $t > 0$, coincides with

$$\exp(t\Lambda_1) \prod_{j=1}^d \exp\left\{-(x_j^2 + \xi_j^2)/(\coth b_j t - 1)\right\}.$$

Employing the operator identity (2.10), we come to (2.31).

Define the operator

$$\mathcal{H}_{1,1} := \int_{\mathbf{R}_y^d} \int_{\mathbf{R}_z^k} h_1 dy dz \quad (2.34)$$

(see e.g. [Re.Sim 4, Sect. XIII.16] for the definition and the elementary properties of the direct integral of selfadjoint operators); if $k = 0$, the integration with respect to z must be omitted. Thus, $\mathcal{H}_{1,1}$ is selfadjoint in $L^2(\mathbf{R}^m)$.

Let $k \geq 1$. On $W_2^2(\mathbf{R}^k)$ define the operator

$$h_2 := - \sum_{l=1}^k \frac{\partial^2}{\partial z_l^2} \quad (2.35)$$

which is selfadjoint in $L^2(\mathbf{R}^k)$. Set

$$\mathcal{H}_{1,2} := \int_{\mathbf{R}_{x,y}^{2d}} h_2 dx dy. \quad (2.36)$$

If $k = 0$, then $\mathcal{H}_{1,2} = 0$.

Assume

$$V \in L^\infty(\mathbf{R}^m; \mathbf{R}). \quad (2.37)$$

For $(x, y) \in \mathbf{R}^{2d}$ and $z \in \mathbf{R}^k$ set

$$V_{\mathbf{b}}(x, y, z) = V(b_1^{-1/2}x_1, \dots, b_d^{-1/2}x_d, b_1^{-1/2}y_1, \dots, b_d^{-1/2}y_d, z);$$

if $k = 0$, $V_{\mathbf{b}}$ is independent of z . Define the operator $\mathfrak{h}_3 = \mathfrak{h}_3(z)$ as a Ψ DO with Weyl symbol $V_{\mathbf{b}}(x - \eta, y - \xi, z)$, $(x, y; \xi, \eta) \in T^*\mathbf{R}^{2d}$, $z \in \mathbf{R}^k$; if $k = 0$, the symbol is independent of z . The operator $\mathfrak{h}_3(z)$ is selfadjoint and uniformly bounded on $L^2(\mathbf{R}^{2d})$ for almost every $z \in \mathbf{R}^k$ (see the remark after Lemma 2.4 below.) If $k \geq 1$, set

$$\mathcal{H}_{1,3} := \int_{\mathbf{R}^k}^{\oplus} \mathfrak{h}_3(z) dz;$$

if $k = 0$, then $\mathcal{H}_{1,3} = \mathfrak{h}_3$. Thus, for $u \in L^2(\mathbf{R}^m)$ we have

$$\begin{aligned} (\mathcal{H}_{1,3}u)(x, y, z) &= (2\pi)^{-2d} \int_{\mathbf{R}^{4d}} \exp \{i[\xi(x - x') + \eta(y - y')]\} \\ &V_{\mathbf{b}}\left(\frac{1}{2}(x + x') - \eta, \frac{1}{2}(y + y') - \xi, z\right) u(x', y', z) dx' dy' d\xi d\eta \end{aligned}$$

if $k \geq 1$; if $k = 0$, then $V_{\mathbf{b}}$ and u are independent of z .
For $k \geq 0$ set

$$\mathcal{H}_1 \equiv \mathcal{H}_1(B; V) := \mathcal{H}_{1,1} + \mathcal{H}_{1,2} + \mathcal{H}_{1,3}. \quad (2.38)$$

The selfadjoint operator \mathcal{H}_1 is defined originally on $\mathcal{S}(\mathbf{R}^m)$ and then is closed in $L^2(\mathbf{R}^m)$.

Lemma 2.4 (cf. [Tam 2, Subsect. 3.1]) *Let (2.26) and (2.37) hold. Then the operator $H(A, V)$ is unitarily equivalent to $\mathcal{H}_1(B; V)$.*

Proof. In $\mathbf{R}_{x,y,z}^m$ (or $\mathbf{R}_{x,y}^m$, if $k = 0$) change the variables

$$x_j \rightarrow b_j^{-1/2}x_j, y_j \rightarrow b_j^{-1/2}y_j, j = 1, \dots, d. \quad (2.39)$$

Let (ξ, η) be the variables dual to (x, y) . In $T^*\mathbf{R}^{2d} = \mathbf{R}^{4d}$ perform the linear symplectic transformation

$$x \rightarrow x - \eta, y \rightarrow y - \xi, \xi \rightarrow (y + \xi)/2, \eta \rightarrow (x + \eta)/2. \quad (2.40)$$

By [Hör 2, Theorem 4.3], the operator \mathcal{H}_0 is unitarily equivalent to \mathcal{H}_1 . Since $H(A, V)$ is unitarily equivalent to \mathcal{H}_0 , the proof is complete.

Remark. The operator \mathfrak{h}_3 is not a classical Ψ DO with Weyl symbol. However, the unitary operator corresponding to the symplectic transform (2.40) maps $\mathcal{S}(\mathbf{R}^{2d})$ onto $\mathcal{S}(\mathbf{R}^{2d})$ isometrically with respect to the $L^2(\mathbf{R}^{2d})$ -norm (see [Hör 2, Sect. 4].) Hence, we have

$$|(\mathfrak{h}_3(z)u, u)_{L^2(\mathbf{R}_{x,y}^{2d})}| \leq \sup_{x,y} |V_{\mathbf{b}}(x, y, z)| \|u\|_{L^2(\mathbf{R}_{x,y}^{2d})}^2, \forall u \in \mathcal{S}(\mathbf{R}_{x,y}^{2d}),$$

for almost every $z \in \mathbf{R}^k$ if $k \geq 1$. Thus we obtain the estimates

$$\|\mathfrak{h}_3(z)\|_{\infty} \leq \|V(\cdot, \cdot, z)\|_{L^{\infty}(\mathbf{R}_{x,y}^{2d})}, \|\mathcal{H}_{1,3}\|_{\infty} \leq \|V\|_{L^{\infty}(\mathbf{R}^m)}.$$

Corollary 2.4 (cf. [Av.Her.Sim 1, p.860, Eq.(3.5)]) *The operator $(\mathcal{H}_0(B; 0) + \lambda)^{-1}$, $\lambda > 0$, is an integral operator with kernel*

$$\begin{aligned} \mathcal{R}(x, y, z, x', y', z'; \lambda, B) &:= \int_0^{\infty} e^{-t\lambda} (4\pi t)^{-k/2} \exp\left(-|z - z'|^2/4t\right) \prod_{j=1}^d b_j (4\pi \sinh b_j t)^{-1} \\ &\exp\left\{ib_j(x_j y'_j - x'_j y_j)/2 - b_j\left(|x_j - x'_j|^2 + |y_j - y'_j|^2\right)/4 \tanh b_j t\right\} dt, \end{aligned} \quad (2.41)$$

if $k > 0$; if $k = 0$, the dependence on z and z' must be omitted and the factor $(4\pi t)^{-k/2} \exp(-|z - z'|^2/4t)$ must be replaced by 1.

Proof. For definiteness assume $k > 0$. Performing a linear transformation inverse to (2.40), and bearing in mind (2.32), we find that the operator $\exp(-t\mathcal{H}_0(B; 0))$ can be written as a Ψ DO with Weyl symbol

$$\exp(-\zeta^2 t) \prod_{j=1}^d (\cosh b_j t)^{-1} \exp\left\{-\left[(\eta_j + x_j/2)^2 + (\xi_j - y_j/2)^2\right] \tanh b_j t\right\}$$

where ζ is the variable dual to z ; note that the symbol is independent of z itself. Recall the relation between the Weyl symbol and the kernel of a given Ψ DO (see [Shu 1, Sect. 23.3, Eq. (23.38)].)

Perform a change of the variables inverse to (2.39). Taking into account (2.10) with $\gamma = 1$, we come to (2.41).

Remark. The formula (2.41) can be deduced also from the explicit evaluation Feynman-Kac-Itô (2.8) for the particular magnetic potential A generating a constant field.

Applying the explicit description of the spectrum of $\mathcal{H}_1(B, 0)$, and bearing in mind the unitary equivalence of the operators $H(A, 0)$ and $\mathcal{H}_1(B, 0)$ with $B = \text{curl } A$, we immediately obtain the following corollary.

Corollary 2.5 *Let $B = \text{curl } A$ satisfy (2.26). Then the spectrum of the operator $H(A, 0)$ is purely essential and we have:*

- (i) *If $k > 0$, $\sigma(H(A, 0)) = \sigma_{\text{ess}}(H(A, 0)) = [\Lambda_1, \infty)$ if $k \geq 1$;*
- (ii) *If $k = 0$, $\sigma(H(A, 0)) = \sigma_{\text{ess}}(H(A, 0)) = \bigcup_{q=1}^{\infty} \{\Lambda_q\}$.*

2.3.3. Denote by $\mathbf{H}_{B,R}$ the operator $H_{\Omega}^D(A, 0)$ where $B = \text{curl } A$ satisfies (2.26), and Ω coincides with a cube in \mathbf{R}^m , $m \geq 2$, of side-length R . Our main goal in this paragraph is to establish the two-sided estimates of the quantity $N(\lambda; \mathbf{H}_{B,R})$ due to Y. Colin de Verdière (see [CdV].) For their formulation we need the following notations. For $\lambda \in \mathbf{R}$ and $k \in \mathbf{Z}$, $k > 0$, set

$$\theta_k(\lambda) := \omega_k \lambda_+^{k/2} / (2\pi)^k$$

where $\omega_k \equiv \pi^{k/2} / \Gamma(1 + k/2)$ is the volume of the unit ball in \mathbf{R}^k ; respectively, $\theta_0(\lambda)$ is the Heaviside function continuous from the left.

For $\lambda \in \mathbf{R}$ introduce the quantity

$$\Theta(\lambda) \equiv \Theta(\lambda; B) := b_1 \dots b_d \sum_{n_1=1}^{\infty} \dots \sum_{n_d=1}^{\infty} \theta_k(\lambda - \sum_{j=1}^d b_j(2n_j - 1)) / (2\pi)^d.$$

Note that the measure $d\Theta(\lambda) \equiv d\Theta(\lambda; B)$ depends continuously on the matrix B even if its rank $2d$ varies. This can be seen easily if we consider the Laplace – Stieltjes transform

$$Z_B(t) := \int_0^{\infty} e^{-\lambda t} d\Theta(\lambda; B) = (4\pi t)^{-m/2} \prod_{j=1}^d (b_j t / \sinh b_j t).$$

Theorem 2.7 [CdV, Theorem 3.1] *Fix $\lambda \in \mathbf{R}$. For each $R > 0$ and any $R_0 \in (0, R/2)$ we have*

$$N(\lambda; \mathbf{H}_{B,R}) \leq R^m \Theta(\lambda; B), \quad (2.42)$$

$$N(\lambda; \mathbf{H}_{B,R}) \geq (R - R_0)^m \Theta(\lambda - C/R_0^2; B), \quad (2.43)$$

where the constant C depends only on the dimension $m \geq 2$.

Since the estimates (2.42) - (2.43) will play a crucial role in the demonstration of many further results, we give here a detailed proof of these estimates following the original work [CdV]. The proof consists of several steps.

(i) *Calculation of the spectrum of the operator $H_M^E(B)$ in the case where the manifold M is a torus.*

Let M be the torus

$$\mathbf{R}^{2d} / \Gamma \times \mathbf{T} \quad (2.44)$$

where $\Gamma = \sum_{j=1}^d \oplus (\rho_j \mathbf{Z})^2$, $\mathbf{T} = \mathbf{R}^k / \Gamma_0$ and Γ_0 is some lattice in \mathbf{R}^k (if $k = 0$, the second factor in (2.44) must be omitted). Assume $\mathcal{B} = \sum_{j=1}^d b_j dy^j \wedge dx^j$, $b_j > 0$, and $b_j \rho_j^2 \in 2\pi \mathbf{Z}$, $j = 1, \dots, d$. Then the cohomology class of the 2-form \mathcal{B} is 2π -integer.

Lemma 2.5 *Under the preceding hypotheses the spectrum of the operator $H_M^E(B)$ consists of eigenvalues of the form*

$$\lambda_{n,q} = \sum_{j=1}^d (2n_j - 1)b_j + \mu_q \quad (2.45)$$

where $n_j \in \mathbf{N}_+$, $j = 1, \dots, m$, and μ_q are the eigenvalues of the operator $H_{\mathbf{T}}(0, 0)$; if $k = 0$, then $\mu_q = 0$. Moreover, the multiplicity of $\lambda_{n,q}$ in (2.45) equals the multiplicity of μ_q multiplied by $\prod_{j=1}^d K_j$ where $K_j = b_j \rho_j^2 / 2\pi$.

Proof. Without any loss of generality we assume $m = 2$ (i.e. $d = 1$ and $k = 0$) and $\mathcal{B} = b \, dy \wedge dx$, $b > 0$.

Note that if T is a closed operator with dense domain in a Hilbert space satisfying $[T^*; T] = 2b \, \text{Id}$, then the spectrum of the operator $Q := T^*T$ consists of eigenvalues $2nb$, $n \in \mathbf{N}_+$; moreover, all the eigenspaces $\mathcal{L}_n := \text{Ker}(Q - 2nb)$ are isomorphic since $\mathcal{L}_1 = \text{Ker } T^*$, $\mathcal{L}_n = T\mathcal{L}_{n-1}$ and $\mathcal{L}_{n-1} = T^*\mathcal{L}_n$ so that the eigenvalue multiplicities of Q are constant (see [CdV, Lemma 2.1].)

Since that the operator $H_M^E(B)$ can be written as $T^*T - b \, \text{Id}$ where $T = -iL_x + L_y$, $L_x \equiv L_1$, $L_y \equiv L_2$ (see (2.2)), and $[T^*; T] = 2b$ (see (2.18)), we find that the eigenvalues of $H_M^E(B)$ must be of the form $\lambda_n = (2n - 1)b$, $n \in \mathbf{N}_+$. The constant multiplicity of λ can be calculated if we take into account the well-known Weyl asymptotics

$$\lim_{\lambda \rightarrow \infty} \lambda^{-1} N(\lambda; H_M^E(B)) = (4\pi)^{-1} \text{vol}\{\mathbf{R}^2 / (\rho \mathbf{Z}^2)\} \equiv \rho^2 / 4\pi.$$

(ii) *Demonstration of (2.42).* Until the end of the subsection we assume for definiteness that $k \equiv \dim \text{Ker } B > 0$.

Let $\{M_s\}_{s \geq 1}$ be a sequence of tori of the type (2.44) such that all the side lengths of $\mathbf{R}^{2d} / \Gamma_s$ and $\mathbf{T}_s := \mathbf{R}^k / \Gamma_{0,s}$ tend to infinity as $s \rightarrow \infty$. Denote by Q_s the fundamental domain of the torus M_s . We assume that the sides of Q_s are parallel to the axes of the coordinates (x, y, z) . Consider the pavement of \mathbf{R}^m by the cubes $\mathbf{Q}_l = \{X \in \mathbf{R}^m : Rl_i < X_i < R(l_i + 1), i = 1, \dots, m\}$, $l = (l_1, \dots, l_m) \in \mathbf{Z}^m$. Note that all the operators $H_{\mathbf{Q}_l}^D(A, 0)$, $l \in \mathbf{Z}^m$, with $B = \text{curl } A$ are unitarily equivalent to $\mathbf{H}_{B,R}$. Set $\mathcal{I}_s = \{l \in \mathbf{Z}^m : \overline{\mathbf{Q}_l} \subset Q_s\}$. Assume that the sidelengths of Q_s are sufficiently large so that the set \mathcal{I}_s is not empty, and find a trivialization patch for each \mathbf{Q}_l , $l \in \mathcal{I}_s$. Then we have a natural isometric injection $j : \sum_{l \in \mathcal{I}_s} \oplus W_1^{2,0}(\mathbf{Q}_l) \rightarrow W_1^2(M_s; E)$ defined by $j\{\tilde{f}_l\} = \sum_{l \in \mathcal{I}_s} \tilde{f}_l$ where \tilde{f}_l is the extension of f_l by zero. Obviously, j maps isometrically $\sum_{l \in \mathcal{I}_s} \oplus L^2(\mathbf{Q}_l)$ onto $L^2(M_s; E)$. Using the minimax principle, and bearing in mind Lemma 2.5, we obtain

$$\begin{aligned} (\#\mathcal{I}_s) N(\lambda; \mathbf{H}_{B,R}) &\leq N(\lambda; H_{M_s}^E(B)) = \\ &\prod_{j=1}^d [(b_j \rho_{j,s}^2) / 2\pi] \left(\#\{(n, \mu_q) : \sum_{j=1}^d (2n_j - 1)b_j + \mu_q < \lambda\} \right), \end{aligned}$$

where $\rho_{j,s}$, $j = 1, \dots, d$, are the side-lengths of the fundamental domain of \mathbf{R}^{2d}/Γ_s , $n \in \mathbf{N}_+^d$, and $\mu_q \equiv \mu_{q,s}$ are the eigenvalues of the operator $H_{\mathbf{T}_s}(0, 0)$. Then we have

$$N(\lambda; \mathbf{H}_{B,R}) \leq R^m \left(\prod_{j=1}^d \frac{b_j}{2\pi} \right) \frac{\text{vol } Q_s}{\text{vol } \{\cup_{l \in \mathcal{I}_s} \mathbf{Q}_l\}}$$

$$\frac{\sum_{n_1=1}^{\infty} \dots \sum_{n_d=1}^{\infty} N(\lambda - \sum_{j=1}^d b_j(2n_j - 1); H_{\mathbf{T}_s}(0, 0))}{\text{vol } \{\mathbf{T}_s\}}.$$

Taking into account the obvious limiting relations

$$\lim_{s \rightarrow \infty} \frac{\text{vol } Q_s}{\text{vol } \{\cup_{l \in \mathcal{I}_s} \mathbf{Q}_l\}} = 1,$$

and

$$\lim_{s \rightarrow \infty} \frac{N(\mu; H_{\mathbf{T}_s}(0, 0))}{\text{vol } \{\mathbf{T}_s\}} = \theta_k(\mu), \forall \mu \in \mathbf{R},$$

we come to (2.42).

(iii). *Demonstration of (2.43).* Put $\hat{\mathbf{Q}}_l = \{X \in \mathbf{R}^m : (R - R_0)l_i < X_i < (R - R_0)(l_i + 1), i = 1, \dots, m\}$, $l = (l_1, \dots, l_m) \in \mathbf{Z}^m$. Denote by $\tilde{\mathbf{Q}}_l$ the cubes which have the same centres as $\hat{\mathbf{Q}}_l$ but whose side-lengths equal R . Set $\tilde{\mathcal{I}}_s = \{l \in \mathbf{Z}^m : \tilde{\mathbf{Q}}_l \cap Q_s \neq \emptyset\}$.

Further, we need the following simple lemma.

Lemma 2.6 [CdV, p.331] *There exists a family of functions $\{\varphi_l\}$, $\varphi_l \in C_0^\infty(\tilde{\mathbf{Q}}_l)$, $l \in \tilde{\mathcal{I}}_s$, such that:*

- (i) *We have $\sum_l \varphi_l^2 \equiv 1$ on M_s ;*
- (ii) *The estimate*

$$|\nabla \varphi_l| \leq C_0/R_0, \forall l \in \tilde{\mathcal{I}}_s, \quad (2.46)$$

holds with constant C_0 which depends only on m .

The family of functions $\{\varphi_l\}$ generates the injections $\mathbf{j}_l : W_1^2(M_s; E) \rightarrow W_1^{2,0}(\tilde{\mathbf{Q}}_l)$ defined by $\mathbf{j}_l(f) = \varphi_l f$ (we again assume that each $\tilde{\mathbf{Q}}_l$ is contained in a trivialization coordinate patch.) Integrating by parts, we obtain the estimate

$$\sum_{l \in \tilde{\mathcal{I}}_s} \int_{\tilde{\mathbf{Q}}_l} |\nabla_B(\varphi_l f)|^2 dX \leq \int_{M_s} \{|\nabla_B f|^2 + C_1 |f|^2\} d\mu, \forall f \in C^\infty(M_s),$$

where $C_1 = \sup_X \sum_l |\nabla \varphi_l(X)|^2$. In view of (2.46), we find that $C_1 \leq CR_0^{-2}$ where $C = C_0^2$ depends only on m . Taking into account the identity

$$\sum_{l \in \tilde{\mathcal{I}}_s} \int_{\tilde{\mathbf{Q}}_l} |\varphi_l f|^2 dX \leq \int_{M_s} |f|^2 d\mu, \forall f \in C^\infty(M_s),$$

and using the minimax principle, we get

$$N(\lambda - CR_0^{-2}; H_{M_s}^E(B)) \leq (\#\tilde{\mathcal{I}}_s)N(\lambda; \mathbf{H}_{B,R}).$$

Further the proof of (2.43) is the same as the one of (2.42).

2.4 The spectrum of the operator $H(A, V)$ with periodic $B = \text{curl } A$ and V

2.4.1. In this subsection we give a brief description of “the magnetic version” of the Floquet – Bloch theory for the operator $H(A, V)$ with periodic $B = \text{curl } A$ and V . For simplicity, throughout the subsection we assume $(A, V) \in C^\infty(\mathbf{R}^m; \mathbf{R}^{m+1})$.

In §2.4.2 we suppose that the components $A_j, j = 1, \dots, m$, of the magnetic potential A as well as the electric potential V are periodic over \mathbf{R}^m , $m \geq 2$, with common lattice Γ of periods. In this case the general theory of elliptic operators with periodic coefficients over \mathbf{R}^m is applicable (see [Shu 2, Subsect. 4.3]). However, the results of Section 2.2 show that the spectral properties of $H(A, V)$ are determined by the magnetic field $B = \text{curl } A$ and not by the magnetic potential A itself. From this point of view it is natural to assume that only B and not A is periodic. We deal with this case in §2.4.3, assuming for simplicity $m = 2$. Note that the periodicity of B (which can be considered as a scalar function if $m = 2$) does not entail the periodicity of A : – if the mean value of B is not zero, then the components of A contain a linear term. Therefore some modifications in the standard spectral theory of elliptic operators with periodic coefficients are necessary in this case.

The spectral theory of $H(A, V)$ with periodic $B = \text{curl } A$ and V is quite rich and interesting. The present brief subsection just outlines some of the most general aspects of this theory. More detailed information and a lot of recent results can be found in [Nov], [Hel.Sjö 4], [Ger.Mar.Sjö], [Gri.Moh] and [Lag.Sma]. We mention just these works since they treat the specific effects of the magnetic field, and leave out of the scope of the survey the enormous literature on the spectral theory of the operator $H(0, V)$ with periodic V .

2.4.2. Let

$$\Gamma = \left\{ X \in \mathbf{R}^m : X = \sum_{j=1}^m n_j \mathbf{a}_j, n_j \in \mathbf{Z}, j = 1, \dots, m \right\}$$

be a lattice associated with the basis $\{\mathbf{a}_j\}_{j=1}^m$ in \mathbf{R}^m , $m \geq 2$. Assume that we have

$$A(X + \gamma) = A(X), V(X + \gamma) = V(X), \forall X \in \mathbf{R}^m, \forall \gamma \in \Gamma.$$

Let $\{\mathbf{a}_j^*\}_{j=1}^m$ be the basis in \mathbf{R}^m dual to the basis $\{\mathbf{a}_j\}_{j=1}^m$, i.e the basis satisfying

$$(\mathbf{a}_i, \mathbf{a}_j^*) = 2\pi\delta_{ij}, i, j = 1, \dots, m,$$

where (\cdot, \cdot) is the inner product in \mathbf{R}^m generated by the euclidean metrics. Set

$$\Gamma^* = \{\Xi \in \mathbf{R}^m : \Xi = \sum_{j=1}^m n_j \mathbf{a}_j^*, n_j \in \mathbf{Z}, j = 1, \dots, m\}.$$

Introduce the tori $\mathbf{T} = \mathbf{R}^m/\Gamma$ and $\mathbf{T}^* = \mathbf{R}^m/\Gamma^*$ and denote respectively by Ω and Ω^* their fundamental domains.

For a fixed $\Xi \in \overline{\Omega}^*$ introduce the operator $H_{\mathbf{T}}(A - \Xi, V)$ (see §2.2.1). Since the torus \mathbf{T} is a compact manifold without boundary, and the operator $H_{\mathbf{T}}(A - \Xi, V)$, $\Xi \in \overline{\Omega}^*$, is elliptic, its spectrum is purely discrete. Denote by $\{E_l(\Xi)\}_{l \geq 1}$ the nondecreasing sequence of the eigenvalues of the operator $H_{\mathbf{T}}(A - \Xi, V)$, $\Xi \in \overline{\Omega}^*$. It is well-known that the functions E_l , $l \geq 1$, are continuous over the torus T^* .

Introduce the Hilbert space

$$\mathcal{L} := \frac{1}{\text{vol } \Omega^*} \int_{\mathbf{T}^*}^{\oplus} L^2(\mathbf{T}) d\Xi.$$

We can identify \mathcal{L} with $L^2(\Omega \times \Omega^*; dX d\Xi/\text{vol } \Omega^*)$ where dX and $d\Xi$ are the Lebesgue measures respectively over Ω and Ω^* . Introduce the operator

$$\tilde{H} := \frac{1}{\text{vol } \Omega^*} \int_{\mathbf{T}^*}^{\oplus} H_{\mathbf{T}}(A - \Xi, V) d\Xi$$

which is selfadjoint in \mathcal{L} . Define the operator $U : L^2(\mathbf{R}^m) \rightarrow \mathcal{L}$ by

$$(Uf)(X, \Xi) = \sum_{\gamma \in \Gamma} \exp\{-i\Xi \cdot (X + \gamma)\} f(X + \gamma), f \in L^2(\mathbf{R}^m), \quad (2.47)$$

It is easy to check that the operator U is isometric and, since it commutes with the multipliers by A_j , $j = 1, \dots, m$, and by V , we have

$$UH(A, V)U^* = \tilde{H}$$

(cf. e.g. [Re.Sim 4, Sect. XIII.16].) Therefore, the spectrum of $H(A, V)$ has a band structure:

$$\sigma(H(A, V)) = \bigcup_{l=1}^{\infty} \bigcup_{\Xi \in \overline{\Omega}^*} \{E_l(\Xi)\}. \quad (2.48)$$

2.4.3. Let $m = 2$. Assume for simplicity that $\mathbf{a}_j = a_j \mathbf{e}_j$, $j = 1, 2$, where $\{\mathbf{e}_j\}_{j=1,2}$ is the standard orthonormal basis in \mathbf{R}^2 and a_j , $j = 1, 2$, are positive numbers. Then we have $\Omega = (0, a_1) \times (0, a_2)$ and $\Omega^* = (0, 2\pi/a_1) \times (0, 2\pi/a_2)$. Moreover, the operator U in (2.47) can be re-written as

$$(Uf)(x, y; \xi, \eta) = \sum_{p \in \mathbf{Z}} \sum_{q \in \mathbf{Z}} \exp \{-i[\xi(x + pa_1) + \eta(y + qa_2)]\} f(x + pa_1, y + qa_2)$$

where $X = (x, y)$ and $\Xi = (\xi, \eta)$.

Assume that the functions $B \equiv \partial_x A_2 - \partial_y A_1$ and V are periodic with periods a_1 and a_2 . More precisely, we have

$$B(x + a_1, y) = B(x, y) = B(x, y + a_2), \forall (x, y) \in \mathbf{R}^2,$$

$$V(x + a_1, y) = V(x, y) = V(x, y + a_2), \forall (x, y) \in \mathbf{R}^2.$$

Introduce the flux Φ of the magnetic field B through unit cell Ω :

$$\Phi = \int_{\Omega} B(x, y) dx dy.$$

If $\Phi = 0$, we can find a periodic potential $\tilde{A} = (\tilde{A}_1, \tilde{A}_2)$ such that the operators $H(A, V)$ and $H(\tilde{A}, V)$ are unitary equivalent under some gauge transformation.

If $\Phi \neq 0$, we can introduce the magnetic potential $A_{\Phi} = (\tilde{A}_1, \Phi_0 x + \tilde{A}_2)$ with $\Phi_0 := \Phi / \text{vol } \Omega \equiv \frac{\Phi}{a_1 a_2}$ and periodic \tilde{A} such that the operators $H(A, V)$ and $H(A_{\Phi}, V)$ are unitary equivalent under some gauge transformation. For a fixed $\Xi \in \overline{\Omega}^*$ introduce the operator $H_{\mathbf{T}}^E(A_{\Phi} - \Xi, V)$ as the unique selfadjoint operator generated in $L^2(\Omega)$ by the closed quadratic form

$$\int_{\Omega} \left\{ |i\partial_x u + (\tilde{A}_1 - \xi)u|^2 + |i\partial_y u + (\tilde{A}_2 + \Phi_0 x - \eta)u|^2 \right\} dx dy$$

with domain

$$\left\{ u \in W_1^2(\Omega) : u(0, y) = e^{i\Phi_0 a_1 y} u(a_1, y), y \in (0, a_2), u(x, 0) = u(x, a_2), x \in (0, a_1) \right\}.$$

Note that the operator $H_{\mathbf{T}}^E(A_{\Phi} - \Xi, V)$ is closely connected with the operator H_M^E introduced in §2.2.4 for the case $M = \mathbf{T}$. Denote by $\{E_{l, \Phi}(\Xi)\}_{l \geq 1}$ the nondecreasing sequence of the eigenvalues of the operator $H_{\mathbf{T}}^E(A_{\Phi} - \Xi, V)$, $\Xi \in \overline{\Omega}^*$. Set

$$\tilde{H}_{\Phi} := \frac{1}{\text{vol } \Omega^*} \int_{\mathbf{T}^*}^{\oplus} H_{\mathbf{T}}^E(A - \Xi, V) d\Xi.$$

Since the operator U in (2.47) does not commute with the linear term $\Phi_0 x$ in the second component of A_{Φ} , we cannot establish the unitary equivalence of $H(A, V)$ and \tilde{H}_{Φ} . However, if $\Phi \in 2\pi\mathbf{Z}$, we can define the commuting unitary operators $t_j : L^2(\mathbf{R}^2) \rightarrow L^2(\mathbf{R}^2)$, $j = 1, 2$, by

$$(t_1 f)(x, y) := \exp(-i\Phi_0 a_1 y) f(x + a_1, y), (t_2 f)(x, y) := f(x, y + a_2).$$

Then we can put

$$(U_\Phi f)(x, y; \xi, \eta) = \sum_{p \in \mathbf{Z}} \sum_{q \in \mathbf{Z}} \exp \{-i[\xi(x + pa_1) + \eta(y + qa_2)]\} \cdot (t_1^p t_2^q f)(x, y).$$

The operator U_Φ maps unitarily $L^2(\mathbf{R}^2)$ onto \mathcal{L} , and the relation

$$U_\Phi H(A, V) U_\Phi^* = \tilde{H}_\Phi \tag{2.49}$$

is valid up to a gauge transformation. Hence, we have

$$\sigma(H(A, V)) = \bigcup_{l=1}^{\infty} \bigcup_{\Xi \in \bar{\Omega}^*} \{E_{l, \Phi}(\Xi)\}. \tag{2.50}$$

In the case where $\Phi/2\pi = j/l$, $j \in \mathbf{Z}$, $l \in \mathbf{Z}$, we notice that the operators t_1^l and t_2 commute, and can re-define the operators \tilde{H}_Φ and U_Φ replacing a_1 by la_1 . Thus, in this case analogues of (2.49) and (2.50) are valid again. Therefore, if the number $\Phi/2\pi$ is rational, the spectrum of the operator $H(A, V)$ has a band structure. On the contrary, if $\Phi/2\pi$ is irrational, no analogue of the formula (2.50) exists, and $H(A, V)$ reveals spectral properties typical for the Hamiltonians of disordered systems, e.g. the operator $H(0, V)$ with almost periodic potential V (see [Hel.Sjö 4, §9].)

3 The essential spectrum of the operator $H(A, V)$

In this section we investigate the essential spectrum of $H(A, V)$. Since $H(A, V)$ is uniformly elliptic, its essential spectrum depends only on the properties of the potentials A and V at infinity. We allocate particular attention to the specific effects due to the presence of a nonzero magnetic field.

In Subsection 3.1 we establish sufficient (resp., necessary) conditions for the resolvent compactness of the operator $H(A, V)$, i.e. conditions which imply (resp., follow from) $\sigma_{ess}(H(A, V)) = \emptyset$.

In Subsection 3.2 we introduce a class of potentials (A, V) which admit an explicit description of the essential spectrum of $H(A, V)$. At the end of this subsection we give several examples of Schrödinger operators with remarkable essential spectra. These examples enlighten many typical situations and illustrate the great variety of different kinds of $\sigma_{ess}(H(A, V))$.

3.1 Necessary and sufficient conditions for resolvent compactness

3.1.1. We begin our discussion with the elementary case when the resolvent compactness of $H(A, V)$ follows from the mere fact that $H(0, V) \equiv -\Delta + V$ has a compact resolvent.

Proposition 3.1 [Av.Her.Sim 1, Theorem 2.7] *Assume that the operator $H(0, V) \equiv -\Delta + V$ is selfadjoint in $L^2(\mathbf{R}^m)$, and its resolvent is compact. Let $A \in L^2_{loc}(\mathbf{R}^m; \mathbf{R}^m)$. Then the resolvent of the operator $H(A, V)$ is compact as well.*

Sketch of the proof. The assertion follows from Theorem 2.1 and Proposition 2.2.

Remark. It is well-known that the resolvent of $-\Delta + V$ is compact if $V(X) \rightarrow \infty$ as $|X| \rightarrow \infty$. Some necessary and sufficient conditions for the compactness of the resolvent of $H(A, V)$ are well-known (see e.g. the classical result in [Mol] concerning the one-dimensional case.)

Proposition 3.1 provides just a *sufficient* condition for the resolvent compactness of $H(A, V)$. In the following paragraph we shall see that the resolvent of $H(A, V)$ may be compact even if $V \equiv 0$.

3.1.2. In this subsection we describe some “semi-effective” criteria for the resolvent compactness of the operator $H(A, V)$.

We say that the potential (A, V) satisfies the condition \mathcal{G} if and only if there exists a measurable function p over \mathbf{R}^m such that:

- a) $p(X) \rightarrow \infty$ as $|X| \rightarrow \infty$;
- b) the estimate

$$(pu, u) \leq C((H(A, V)u, u) + \|u\|^2), \forall u \in C_0^\infty(\mathbf{R}^m),$$

holds for some constant $C > 0$.

Let $\Omega \subset \mathbf{R}^m$ be an open set. Introduce the quantity

$$e_{A, V}(\Omega) := \inf_{u \in C_0^\infty(\Omega), \|u\|=1} (H(A, V)u, u).$$

Denote by $\mathbf{B}(X; R)$ the ball $\{Y \in \mathbf{R}^m : |X - Y| < R\}$. Set $\tilde{\mathbf{B}}_R = \mathbf{R}^m \setminus \overline{\mathbf{B}(0, R)}$.

Theorem 3.1 [Iwa 4, Main Theorem] *Assume that (2.1) holds. Then the following four conditions are equivalent:*

- i) *the resolvent of $H(A, V)$ is compact;*
- ii) *$e_{A, V}(\tilde{\mathbf{B}}_R) \rightarrow \infty$ as $R \rightarrow \infty$;*
- iii) *$e_{A, V}(\mathbf{B}(X; 1)) \rightarrow \infty$ as $|X| \rightarrow \infty$;*
- iv) *the potential (A, V) satisfies the condition \mathcal{G} .*

Sketch of the proof. The condition iii) follows trivially from ii), and ii), on its turn, follows easily from iv). Moreover, we can establish the equivalence of i) and ii) if we note that the lower bound of $\sigma_{ess}H(A, V)$ is given, as in the case $A \equiv 0$, by the Persson’s formula

$$\inf \sigma_{ess} H(A, V) = \lim_{R \rightarrow \infty} e_{A, V}(\tilde{\mathbf{B}}_R).$$

(see [Per].) In order to verify this formula, take into account the ellipticity of $H(A, V)$ and note that the validity of the inequality

$$\|(H(A, V) - \lambda)u\| \geq \delta \|u\|, \forall u \in C_0^\infty(\tilde{\mathbf{B}}_R),$$

for some $\lambda \in \mathbf{R}$, $R > 0$, $\delta > 0$ entails $\lambda \notin \sigma_{ess} H(A, V)$ (cf. [Moh 1, Lemma 2.1].)

Finally, in order to check that iii) entails iv) introduce a partition of unity $\{\varphi_\alpha\}_{\alpha \in \mathbf{Z}^m}$ satisfying $\sum_{\alpha \in \mathbf{Z}^m} \varphi_\alpha^2(X) \equiv 1$, $\text{supp } \varphi_\alpha \subset \mathbf{B}(\alpha/\sqrt{m}; 1)$ and $\sum_{\alpha \in \mathbf{Z}^m} |\nabla \varphi_\alpha|^2 \in L^\infty(\mathbf{R}^m)$ (see [Iwa 4, p.363] for the explicit construction of such a partition.) Then the operator $H(A, V)$ satisfies the condition \mathcal{G} with $p(X) := \sum_{\alpha \in \mathbf{Z}^m} e_{A, V}(\mathbf{B}(\alpha/\sqrt{m}; 1)) \varphi_\alpha^2(X)$.

3.1.3. Until the end of the subsection we suppose that $A \in C^1(\mathbf{R}^m; \mathbf{R}^m)$, $V \in L_{loc}^2(\mathbf{R}^m; \mathbf{R})$, and the multiplier by $V_- \equiv \min(V, 0)$ is Δ -bounded with relative bound smaller than one; hence, in particular, the assumptions of Theorem 2.2 hold. Moreover, it is easy to check that Theorem 3.1 remains valid under these hypotheses.

In this paragraph we establish some sufficient conditions for the resolvent compactness of the operator $H(A, V)$.

The following corollary can be deduced from the implication iv) \Rightarrow i) in Theorem 3.1. It was established partially by A. Dufresnoy in [Duf], and later was obtained in a more complete form by A. Iwatsuka in [Iwa 4].

Corollary 3.1 *Assume that the magnetic field B is C^2 -smooth and, moreover, we have*

$$|B(X)| \rightarrow \infty \text{ as } |X| \rightarrow \infty, \quad (3.1)$$

$$|\nabla B(X)|/|B(X)|^2 \rightarrow 0 \text{ as } |X| \rightarrow \infty, \quad (3.2)$$

(see (1.3).) *Then the resolvent of $H(A, 0)$ is compact.*

Sketch of the proof. We shall prove the corollary under a stronger assumption than (3.2); namely, we suppose that the estimate

$$|\nabla B(X)| \leq C(1 + |B(X)|)^\delta \quad (3.3)$$

holds for some $\delta \in [0, 2)$ and $C > 0$. In this case the Riemannian metric $g_X(z) := |B(X)|^{2\delta-2}|z|^2$ is temperate (see [Den]), and we can find a weight function $p(X) \geq 1$ and a constant $C > 1$ such that we have

$$C^{-1}p(X) \leq (1 + |B(X)|)^{1-\delta} \leq Cp(X), |\nabla p(X)| \leq C. \quad (3.4)$$

Let $u \in C_0^\infty(\mathbf{R}^m)$. Then the identity

$$\begin{aligned} \|m^{-1/2}[L_i; L_j]u\|^2 &= (L_j u, m^{-1}[L_i; L_j]L_i u) - (L_i u, m^{-1}[L_i; L_j]L_j u) + \\ &\quad (L_j u, [L_i; m^{-1}[L_i; L_j]]u) - (L_i u, [L_j; m^{-1}[L_i; L_j]]u) \end{aligned}$$

holds for the operators L_i , $i = 1, \dots, m$, defined in (2.2), $m(X) := p(X)^{1/(1-\delta)}$ if $\delta \in [0, 3/2]$, and $m(X) := p(X)^{-(2-2\delta)/(1-\delta)}$ if $\delta \in [3/2, 2)$ (cf. the demonstration of [Hel.Moh, Theorem 1.1].)

Since the operator $m^{-1}[L_i; L_j]$ coincides with the multiplier by a bounded function whose gradient norm is estimated from above by $Cm(X)^{-1/2}(1 + |B(X)|)$, we easily get

$$(m^{-1/2}|B|u, u) \leq C((H(A, 0)u, u) + \|u\|^2), \forall u \in C_0^\infty(\mathbf{R}^m). \quad (3.5)$$

The estimate (3.5) allows us to apply Theorem 3.1, and to complete the proof in the case where B satisfies (3.3).

For the case of general B satisfying (3.2), see [Iwa 4, Theorem 6.1]; here we only note that if $m = 2$, then condition (3.2) is not necessary for the proof since in this case we have $[L_1; L_2] = \pm|B(X)|$.

The condition (3.1) is neither necessary nor sufficient for the resolvent compactness. The works [Duf] and [Iwa 4] contain examples where (3.1) is valid but the resolvent of $H(A, 0)$ is not compact. Here we give the example from [Iwa 4, Assertion 7.1].

Example. Let $m \geq 3$, $g(X) := X_m \log(\log(|X|^2 + 2))$, and $A_1(X) := \cos g(X)$, $A_2(X) := \sin g(X)$, $A_j(X) := 0$, $j \geq 3$. Then B satisfies (3.1). However, A is bounded and the resolvent of $H(A, 0)$ cannot be compact since we have

$$(H(A, 0)u, u) \leq c \int_{\mathbf{R}^m} (|\nabla u|^2 + |u|^2) dX, \forall u \in C_0^\infty(\mathbf{R}^m),$$

for some constant $c < \infty$ and, therefore, $\inf \sigma_{ess}(H(A, 0)) \leq c$.

3.1.4. The following theorem contains a necessary condition for the resolvent compactness of $H(A, V)$.

Theorem 3.2 [Iwa 4, Theorem 5.2] *Assume that the magnetic potential A is C^2 -smooth. Then the compactness of the resolvent of $H(A, V)$ implies*

$$\int_{\mathbf{B}(X; \delta)} (|B(Y)|^2 + V_+(Y)) dY \rightarrow \infty \text{ as } |X| \rightarrow \infty, \quad (3.6)$$

for some $\delta > 0$.

Proof. We shall prove the theorem by contradiction. Assume that (3.6) is false. Then there exist numbers $\delta > 0$ and $C > 0$ such that

$$\int_{\mathbf{B}(Z^j; \delta)} (|B(Y)|^2 + V_+(Y)) dY \leq C, \forall j \geq 0, \quad (3.7)$$

where $\{Z^j\}_{j \geq 0}$ is a sequence of points in \mathbf{R}^m satisfying $Z^j \rightarrow \infty$ as $j \rightarrow \infty$.

For any fixed $Y \in \mathbf{R}^m$ consider the magnetic potential $A(Y; X)$ with components

$$A_1(Y; X) = 0,$$

$$A_k(Y; X) = \sum_{i=1}^{k-1} \int_{Y_i}^{X_i} B_{k,i}(Y_1, \dots, Y_{i-1}, t, X_{i+1}, \dots, X_m) dt, k = 2, \dots, m;$$

the potential $A(Y; X)$ is gauge-equivalent to $A(X)$ (see [Hel.Moh]). Note that if $f(X)$ is a continuous function over \mathbf{R}^m , $m > 1$, satisfying the estimate $\int_{|X| < \delta} |f(X)|^2 dX \leq C$, then there exists a number $Y_1 \in \mathbf{R}$ such that $|Y_1| \leq \delta/\sqrt{2}$, and

$$\int_{|X'| < \delta/\sqrt{2}} |f(Y_1, X')|^2 dX' \leq \delta^{-1} C / \sqrt{2}.$$

This property together with (3.7) shows that one can find points Y^j in $\mathbf{B}(Z^j; \delta/\sqrt{2})$ for which the estimate

$$\int_{|X'| < \delta 2^{-m/2}} |B(Y_1^j, \dots, Y_{i-1}^j, Y_i^j + X'_i, \dots, Y_m^j + X'_m)|^2 dX' \leq C_0 \quad (3.8)$$

holds for each $i = 1, \dots, m-1$, and $j \geq 0$, with a constant C_0 independent of δ and C . Set $A^j(X) := A(Y^j; X)$.

It follows from (3.7) and (3.8) that there exists a functional sequence $\{u_j\}_{j \geq 0}$ such that $\|u_j\| = 1$, $u_j \in C_0^\infty(\mathbf{B}(Y^j; \delta 2^{-m/2}))$, and the sequence $((H(A_j, V_+)u_j, u_j))_{j \geq 0}$ is bounded. Making a gauge transformation, we find that there exists a functional sequence $\{v_j\}_{j \geq 0}$, $\|v_j\| = 1$, $v_j \in C_0^\infty(\mathbf{B}(Z^\delta))$, such that the sequence

$$((H(A, V_+)v_j, v_j))_{j \geq 0}$$

is bounded.

The last property together with the Persson's formula shows that the resolvent of the operator $H(A, V_+)$ is not compact.

Since V_- is $H(A, V_+)$ -bounded with relative bound smaller than one (see Proposition 2.1), the resolvent of $H(A, V)$ is not compact too.

3.1.5. In the case where V and the components of A are polynomials, there exists a necessary and sufficient condition for the resolvent compactness of $H(A, V)$.

In the sequel we shall use the standard notations $D_j := -i\partial_j$, $j = 1, \dots, m$, and $D^\alpha := D_1^{\alpha_1} \dots D_m^{\alpha_m}$ for any multiindex $\alpha \in \mathbf{N}^m$.

Theorem 3.3 *Let $A_j(X)$, $j = 1, \dots, m$, and $V(X)$ be polynomials in X . Assume that there is a constant $C > 0$ such that $V(X) \geq -C$, $\forall X \in \mathbf{R}^m$. Then the resolvent of the operator $H(A, V)$ is compact if and only if*

$$m(X) := 1 + \sum_{\alpha} (|D^{\alpha}V(X)| + |D^{\alpha}B(X)|) \rightarrow \infty \text{ as } |X| \rightarrow \infty. \quad (3.9)$$

Proof. If $V(X)$ is a sum of squares of polynomials, the sufficient condition is known from the Lie-group theory (cf. [Hel.Nou].) In the general case the sufficient condition follows from the results of [Hel.Moh] and [Moh.Nou] (see Theorems 3.4 – 3.5 below.) In order to see that (3.9) is a necessary condition, it suffices to employ the necessary condition (3.6) and the fact that there exist constants $C_k > 1$ such that

$$C_k^{-1} |||P||| \leq \int_{|X|<1} |P(X)| dX \leq C_k |||P|||, \forall P \in \mathbf{P}_k,$$

where \mathbf{P}_k is the space of the polynomials whose degree does not exceed k , and $|||P||| := \sum_{\alpha} |D^{\alpha}P(X)|_{|X=0}$.

In the following two theorems we extend Theorem 3.3 to the case where V and the components of A behave like polynomials; these results can be found in [Hel.Moh] and [Moh.Nou].

Theorem 3.4 *Let $A \in C^2(\mathbf{R}^m; \mathbf{R}^m)$ and $V(X) \geq 0$. Assume that there exists such an integer $r \geq 0$ that we have*

$$V \in C^{r+2}(\mathbf{R}^m), B_{i,j} \in C^{r+1}(\mathbf{R}^m), 1 \leq i, j \leq m. \quad (3.10)$$

Suppose, in addition, that the estimate

$$\sum_{|\alpha|=r+2} |D^{\alpha}V(X)| + \sum_{|\beta|=r+1} |D^{\beta}B(X)| \leq Cm(X) \quad (3.11)$$

holds with

$$m(X) := 1 + \sum_{|\alpha| \leq r+1} |D^{\alpha}V(X)| + \sum_{|\beta| \leq r} |D^{\beta}B(X)|$$

and a constant $C > 0$. Then the resolvent of $H(A, V)$ is compact if and only if

$$m(X) \rightarrow \infty \text{ as } |X| \rightarrow \infty. \quad (3.12)$$

Remark. We can replace $m(X)$ in (3.11) by $m^{\delta}(X)$. The optimal power seems to be $\delta_r = 1 + (2^{r+2} - 3)^{-1}$ (see Corollary 3.1.) When $\delta < \delta_r$ and $m(X) \rightarrow \infty$ as $|X| \rightarrow \infty$, one can prove in the same manner as above that the resolvent of $H(A, V)$ is compact (see [Mef].)

Theorem 3.5 *Let $A \in C^2(\mathbf{R}^m)^m$, and let $V(X)$ be in the form*

$$V(X) = V_0(X) + \sum_{j=1}^p V_j^2(X), V_0(X) \in C^1(\mathbf{R}^m; \mathbf{R}_+). \quad (3.13)$$

Assume that there exists such an integer $r \geq 0$ that we have

$$V_j \in C^{r+2}(\mathbf{R}^m), \forall j = 1, \dots, p, B_{i,j} \in C^{r+1}(\mathbf{R}^m), 1 \leq i, j \leq m. \quad (3.14)$$

Suppose, in addition, that for some constant $C > 0$ the estimates

$$\begin{cases} |\nabla V_0(X)| \leq C(1 + |V_0(X)|), \\ \sum_{j=1}^p \sum_{|\alpha|=r+2} |D^\alpha V_j(X)| + \sum_{|\beta|=r+1} |D^\beta B(X)| \leq M(X), \end{cases} \quad (3.15)$$

hold with

$$M(X) := 1 + \sum_{j=1}^p \sum_{|\alpha|=r+1} |D^\alpha V_j(X)| + \sum_{|\beta|=r} |D^\beta B(X)|.$$

Then the resolvent of $H(A, V)$ is compact if and only if

$$V_0(X) + M(X) \rightarrow \infty \text{ as } |X| \rightarrow \infty. \quad (3.16)$$

Proof of Theorems 3.4 – 3.5. In order to see that (3.12) (resp. (3.16)) is a necessary condition for the resolvent compactness of $H(A, V)$, it suffices to notice that the weight $m(X)$ (resp. $M(X)$) is temperate; in other words, there exist positive numbers η and C such that the inequality $|X - Y| < \eta$ entails

$$C^{-1}m(Y) \leq m(X) \leq Cm(Y) \quad (3.17)$$

(resp., $C^{-1}M(Y) \leq M(X) \leq CM(Y)$.) Therefore, if (3.11) (resp., (3.15)) holds and (3.12) (resp., (3.16)) is not valid, then the condition (3.6) cannot be fulfilled.

To check that the condition (3.16) is sufficient, apply the estimate

$$(M^{2-r}u, u) \leq C((H(A, V)u, u) + \|u\|^2), \forall u \in C_0^\infty(\mathbf{R}^m), C > 0, \quad (3.18)$$

which can be verified if one employs the commutators as in the proof of Corollary 3.1 (see also [Hel.Moh].)

Theorem 3.1 together with (3.16) and (3.18) allows us to complete the proof of Theorem 3.4.

In order to check that the condition (3.12) is sufficient, we have to regularize $m(X)$. Because of (3.17), one can find a weight $m^*(X) \in C^\infty$ which satisfies $C^{-1}m(Y) \leq m^*(X) \leq Cm(Y)$ and $|D^\alpha m^*(X)| \leq C_\alpha m^*(X)$ for each multiindex α and some constants C_α . Set $V_1(X) := \varepsilon V(X)(m^*(X))^{-1/2}$ where $\varepsilon > 0$ is chosen so small that we have $\mathcal{W}(X) := V_1^2(X) \leq V(X)$. Applying (3.18) with respect to $(H(A, \mathcal{W})u, u)$, we get

$$(m^{2-r}u, u) \leq C((H(A, \mathcal{W})u, u) + \|u\|^2) \leq C((H(A, V)u, u) + \|u\|^2), \forall u \in C_0^\infty(\mathbf{R}^m).$$

This estimate allows to complete the proof of Theorem 3.5.

Theorem 3.5 can be found in [Hel.Moh] and Theorem 3.4 can be found in [Moh.Nou]; note that the latter work contains estimates which are more precise than (3.18).

Theorems 3.4 – 3.5 show that the condition (3.1) is not necessary for the resolvent compactness: if $m = 2$ and $A_1(X) = X_1X_2^2$, $A_2(X) = 0$, then the resolvent of $H(A, 0)$ is compact although the condition (3.1) is violated.

3.2 Characterization of the essential spectrum

3.2.1. Let $A \in C^2(\mathbf{R}^m; \mathbf{R}^m)$. Assume that (3.13) and (3.14) hold and, moreover, we have

$$B_{i,j} \in C^{r+3}(\mathbf{R}^m), 1 \leq i, j \leq m. \quad (3.19)$$

Let φ be a temperate weight over \mathbf{R}^m satisfying

$$\begin{cases} \varphi(X) \geq 1, \varphi(X) \rightarrow \infty \text{ as } |X| \rightarrow \infty, \\ |X - Y| \leq \eta\varphi(X) \Rightarrow C_0^{-1}\varphi(Y) \leq \varphi(X) \leq C_0\varphi(Y), \end{cases} \quad (3.20)$$

for some $\eta > 0$ and $C_0 > 0$. Then, instead of (3.15), we assume that there exists a constant $C > 0$ such that we have

$$|\nabla V_0(X)| + \sum_{j=1}^p \sum_{|\alpha|=r+2} |D^\alpha V_j(X)| + \sum_{|\beta|=r+1}^{r+3} \varphi^{|\beta|-r-1} |D^\beta B(X)| \leq C/\varphi(X). \quad (3.21)$$

Let the real numbers $V_{z_\infty, j, \alpha}$ and the matrices $B_{z_\infty, \alpha}$ be those ones for which there exists a sequence z_∞ of points $Y_\nu \in \mathbf{R}^m$, $\nu \leq 1$, satisfying

$$|Y_\nu| \rightarrow \infty \text{ as } \nu \rightarrow \infty,$$

$$\partial_X^\alpha V_j(Y_\nu) \rightarrow V_{z_\infty, j, \alpha} \text{ as } \nu \rightarrow \infty,$$

and

$$\partial_X^\alpha B(Y_\nu) \rightarrow B_{z_\infty, \alpha} \text{ as } \nu \rightarrow \infty.$$

Set

$$\begin{aligned}
A_{z_\infty}(X) &:= \sum_{|\alpha| \leq r} X^\alpha (\alpha! (2 + |\alpha|))^{-1} (B_{z_\infty, \alpha} \cdot X), \\
V_{z_\infty}(X) &:= V_{z_\infty, 0, 0} + \sum_{j=1}^p \left(\sum_{|\alpha| \leq r+1} (\alpha!)^{-1} X^\alpha V_{z_\infty, j, \alpha} \right)^2, \\
S_\infty &:= \bigcup_{z_\infty} \sigma(H(A_{z_\infty}, V_{z_\infty})).
\end{aligned}$$

Theorem 3.6 [Hel.Moh, Theorem 1.5] *Under the preceding hypotheses we have*

$$\sigma_{ess}(H(A, V)) = \overline{S_\infty}.$$

The proof of the theorem can be found in the original work [Hel.Moh, pp. 102-110].

3.2.2. In this paragraph we present some perturbation-theory results concerning the essential spectrum of $H(A, V)$. In particular, we deal with the case $(A, V) = (A^0, V^0) + (A^1, V^1)$ where (A^1, V^1) decays at infinity in a certain sense, and show that $\sigma_{ess}(H(A, V)) = \sigma_{ess}(H(A^0, V^0))$.

Corollary 3.2 *Let the potential (A, V_+) satisfy (2.1). Let \mathcal{W} be the same as in Proposition 2.2. Assume that the operator $\mathcal{W}(-\Delta+1)^{-1}$ (resp. $|\mathcal{W}|^{1/2}(H_++1)^{-n/2}$, $n \in \mathbf{N}_+$) is compact. Then we have*

$$\sigma_{ess}(H_+ + \mathcal{W}) = \sigma_{ess}(H_+) \tag{3.22}$$

where $H_+ + \mathcal{W}$ is the operator sum (resp., the form sum) of H_+ and \mathcal{W} .

Proof. The equality (3.22) follows from Proposition 2.2 and the invariance of the essential spectrum under relatively compact perturbations (see [Re.Sim 4, Sect. XIII.4, Corollaries 2 and 4 (ii)].)

Corollary 3.3 *Let $B = \text{curl } A$ satisfy (2.26). Assume that the operator $|V|^{1/2}(-\Delta + 1)^{-n/2}$ is compact for some $n \in \mathbf{N}_+$.*

i) If $k \equiv \dim \text{Ker } B > 0$, then $\sigma_{ess}(H(A, V))$ coincides with the set $[\Lambda_1, \infty)$.

ii) If $k = 0$, then we have $\sigma_{ess}(H(A, V)) = \bigcup_{q=1}^\infty \{\Lambda_q\}$.

(Here $H(A, V)$ is the form sum of $H(A, 0)$ and the multiplier by V .)

Proof. The corollary follows from Propositions 2.5 and 2.2 (ii).

Theorem 3.7 *Let $A \in C^\infty(\mathbf{R}^m; \mathbf{R}^m)$ and $|\text{curl } A| \rightarrow 0$ as $|X| \rightarrow \infty$. Suppose that V is $-\Delta$ -form-compact. Then we have $\sigma_{ess}(H(A, V)) = \sigma(H(A, 0)) = [0, \infty)$.*

Proof. Since V is $-\Delta$ -form-compact, Proposition 2.2 and the invariance of the essential spectrum under relatively compact perturbations entails $\sigma_{ess}(H(A, V)) = \sigma_{ess}(H(A, 0))$.

Let $A(Y; X)$ be the magnetic potential defined in the proof of Theorem 3.2. Choose sufficiently small $\varepsilon > 0$ such that $|B(X)| \leq \varepsilon^2$ if $|X| > R$ and $R > \varepsilon^{-2}$. If $|Y| > 2R$, then we have $|A(Y; X)| \leq C\varepsilon$ for each $X \in \mathbf{B}(Y; 1/\varepsilon)$. Denote by $\{\lambda_{j,\varepsilon}\}_{j \geq 1}$ be the nondecreasing sequence of the eigenvalues of the Dirichlet operator $H_{\mathbf{B}(Y; 1/\varepsilon)}^D$; note that we have $\lambda_{j,\varepsilon} = \lambda_{j,1}\varepsilon^2$, $j \geq 1$. Fix an eigenvalue $\lambda_{j,\varepsilon} < \varepsilon^{-1/2}$ and denote by u_ε any fixed normalized eigenfunction corresponding to $\lambda_{j,\varepsilon}$. Then, using the gauge $A(Y; X)$, we have

$$\|(H(A, 0) - \lambda_{j,\varepsilon})\chi u_\varepsilon\| \leq C\varepsilon^{1/2}$$

where $\chi \in C_0^\infty(\mathbf{R}^m)$ is supported on the ball $\mathbf{B}(Y; 1/\varepsilon)$, and equals identically one on $\mathbf{B}(Y; 1/2\varepsilon)$. Hence, there exists a nonzero $v \in C_0^\infty(\mathbf{R}^m)$ such that

$$\|(H(A, 0) - \lambda_{j,\varepsilon})v\| \leq C\varepsilon^{1/2}\|v\|.$$

Therefore, we have

$$\sigma(H(A, 0)) \cap (-C\varepsilon^{1/2} + \lambda_{j,\varepsilon}, \lambda_{j,\varepsilon} + C\varepsilon^{1/2}) \neq \emptyset.$$

Since $\sigma(H(A, 0))$ is a closed set and $\varepsilon > 0$ can be chosen arbitrarily small, we find that $[0, \infty) \subset \sigma(H(A, 0))$. Finally, $H(A, 0) \geq 0$ entails $\sigma(H(A, 0)) = [0, \infty)$.

In [Cy.Fr.Ki.Sim, Theorem 6.1] the above assertion is proved for the cases $m = 1, 2$.

3.2.3. In this paragraph we offer several examples of essential spectra of $H(A, V)$ which are applications of Theorem 3.6.

Example 1. [Hel.Moh, p.110] Let

$$m = 1, A = 0, V(X) = (X^2 + 1)^{1/2}(\cos(X^2 + 1))^{1/4}.$$

Theorem 3.6 allows us to conclude that $\sigma_{ess}(H(A, 0)) = \{\frac{1}{2} + j : j \in \mathbf{N}\}$. This is an example of a Schrödinger operator without magnetic field and purely point nonempty essential spectrum.

Example 2. [Hel.Moh, p.111] Let

$$m = 2, A_1(X) = 0, A_2(X) = \mu X_1, V(X) = \lambda \cos(|X|^{1/2}),$$

λ and μ being positive constants. Theorem 3.6 shows that $\sigma_{ess}(H(A, V))$ consists of bands of constant length:

$$\sigma_{ess}(H(A, V)) = \bigcup_{j \geq 0} [-\lambda + \mu(1 + 2j), \mu(1 + 2j) + \lambda].$$

Example 3. ([Mil.Sim] or [Cy.Fr.Ki.Sim, Sect. 6.2]) Let

$$m = 2, A_1(X) = \delta X_2(1 + |X|^2)^{-\gamma}, A_2(X) = -\delta X_1(1 + |X|^2)^{-\gamma},$$

δ and γ being positive constants. Then we have:

- i) if $\gamma > 1/2$, then $\sigma(H(A, V)) = \sigma_{ac}(H(A, V)) = [0, \infty)$;
- ii) if $\gamma < 1/2$, then $\sigma(H(A, V)) = \sigma_{pp}(H(A, V)) = [0, \infty)$;
- iii) if $\gamma = 1/2$, then $\sigma_{ess}(H(A, V)) = [0, \infty)$, $\sigma_{pp}(H(A, V)) = [0, \delta^2]$ and, finally, $\sigma_{ac}(H(A, V)) = [\delta^2, \infty)$.

In order to verify i) - iii), it is necessary to write $H(A, 0)$ in polar coordinates (ϱ, φ) , and then to expand the function $u(\varrho, \varphi)$ into a Fourier series with respect to the eigenfunctions of the angular-momentum operator $-i\partial/\partial\varphi$. Thus we find that the operator $H(A, 0)$ is unitarily equivalent to $\sum_{l \in \mathbf{Z}}^{\oplus} H^{(l)}$ where the operators

$$H^{(l)} = -\frac{1}{\varrho} \frac{d}{d\varrho} \varrho \frac{d}{d\varrho} + \frac{l^2}{\varrho^2} + \frac{\delta^2 \varrho^2}{(1 + \varrho^2)^{2\gamma}} + \frac{2l\delta}{(1 + \varrho^2)^\gamma}, l \in \mathbf{Z},$$

are defined originally on $C_0^\infty(0, \infty)$ and then are extended by Friedrichs to operators selfadjoint in $L^2[(0, \infty); \varrho d\varrho]$. In the case $\gamma < 1/2$ the spectrum of each $H^{(l)}$ is purely discrete; if $\gamma > 1/2$ the spectrum of each $H^{(l)}$ is purely absolutely continuous; if $\gamma = 1/2$, the spectrum of each $H^{(l)}$ is purely discrete on $[0, \delta^2)$ and purely absolutely continuous on $[\delta^2, \infty)$. On the other hand, Theorem 3.6 (or Theorem 3.7) implies $\sigma_{ess}(H(A, 0)) = \sigma(H(A, 0)) = [0, \infty)$. Thus we obtain the needed result.

Example 4. ([Iwa 3] or [Cy.Fr.Ki.Sim, Subsect. 6.5]) Let

$$m = 2, V(X) = 0, A_1(X) = 0, A_2(X) = a(X_1) \equiv \int_0^{X_1} b(t) dt,$$

where b is a continuous increasing function having two different positive limits at $\pm\infty$: $b(t) \rightarrow b_\pm$ as $t \rightarrow \pm\infty$ with $0 < b_- < b_+$.

In this case the spectrum of $H(A, 0)$ is purely absolutely continuous and has band structure:

$$\sigma(H(A, 0)) = \sigma_{ac}(H(A, 0)) = \bigcup_{q=0}^{\infty} [(2q+1)b_-, (2q+1)b_+]. \quad (3.23)$$

In order to establish this result, one has to perform a partial Fourier transform with respect to X_1 and to notice that the operator $H(A, 0)$ is unitarily equivalent to the direct integral $\int_{\mathbf{R}}^{\oplus} h(\xi) d\xi$, where $h(\xi) = -d^2/dt^2 + (\xi - a(t))^2$. The resolvent of $h(\xi)$ is compact for any fixed $\xi \in \mathbf{R}$, and for each integer k , the k -th eigenvalue of $h(\xi)$ is an

analytic function which is not constant. Then the absolute continuity of the spectrum of $H(A, 0)$ follows from [Re.Sim 4, Theorem XIII.86]. For the proof of (3.23), see [Iwa 3, p.400].

4 Asymptotic behaviour of the eigenvalues of the operator $H(A, V)$ near their accumulation points

This section contains asymptotic results describing the behaviour of the quantity $N(\lambda; H(A, V))$ as $\lambda \rightarrow \infty$, if the resolvent of $H(A, V)$ is compact, or as λ approaches a given essential-spectrum tip, if $\sigma_{ess}(H(A, V))$ is not empty.

In Subsection 4.1 we consider the former case. Here we present a result due to Y. Colin de Verdière (see [CdV]) containing the first asymptotic term of $N(\lambda; H(A, 0))$ as $\lambda \rightarrow \infty$. Besides, we describe briefly the results of H. Matsumoto (see [Mat 2]) concerning the asymptotics of $N(\lambda; H(A, V))$ as $\lambda \rightarrow \infty$.

In Subsections 4.2 - 4.3 we deal with the case of constant magnetic field and electric potentials which decay at infinity, and investigate the eigenvalue asymptotics near the essential-spectrum tips. Since the essential spectrum of $H(A, V)$ has quite a different structure in the cases $k \equiv \dim \text{Ker } B > 0$ and $k = 0$ (see Corollary 3.3), we consider these two cases separately: the former case is studied in Subsection 4.2, while the latter one is treated in Subsection 4.3.

Finally, in Subsection 4.4 we consider quite general potentials (A, V) , and do not establish precise asymptotic formulae but just two-sided estimates of the counting function $N(\lambda; H(A, V))$ following [Moh 1] and [Moh.Nou].

4.1 Eigenvalue asymptotics for Hamiltonians $H(A, V)$ with compact resolvent

4.1.1. In this paragraph we suppose $V \equiv 0$. Assume that the magnetic field $B = \text{curl } A$ satisfies the following properties:

- a) $|B(X)| \rightarrow \infty$ as $|X| \rightarrow \infty$;
- b) there exists a constant $c > 0$ such that the estimate $|B(X)| \leq c|B(X')|$ holds for each pair X, X' satisfying $|X - X'| < 1$ with $|X|$ large enough;
- c) $M(X) := \max_{|\beta|=2} \sup_{|X-X'|\leq 1} |D^\beta A(X')| = o(|B(X)|^{3/2})$ as $|X| \rightarrow \infty$.

Note that if the magnetic field B satisfies the conditions a) and c), then Corollary 3.1 implies the resolvent compactness of $H(A, 0)$.

Set

$$\nu_1(\lambda) := \int_{\mathbf{R}^m} \Theta(\lambda; B(X)) dX, \lambda \in \mathbf{R},$$

(see §2.3.3.)

The condition a) guarantees that $\nu_1(\lambda)$ is finite for each λ . Set

$$\mathrm{Tr}^+ B(X) = \sum_{j=1}^{d(X)} b_j(X)$$

where $2d(X) = \mathrm{rank} B(X)$ and $b_j(X)$, $j = 1, \dots, d(X)$, are the strictly positive eigenvalues of the matrix $iB(X)$. Put

$$\tilde{\Psi}_B(\lambda) = \mathrm{vol}\{X \in \mathbf{R}^m : \mathrm{Tr}^+ B(X) < \lambda\}. \quad (4.1)$$

Then the estimates

$$c^{-1}\lambda^{m/2}\tilde{\Psi}_B(\delta\lambda) \leq \nu_1(\lambda) \leq c\lambda^{m/2}\tilde{\Psi}_B(\lambda) \quad (4.2)$$

hold for each $\lambda \in \mathbf{R}$, $\delta \in [0, 1)$ and some constant $c > 1$.

Theorem 4.1 [CdV, Theorem 4.1] *Assume that the magnetic field $B = \mathrm{curl} A$ satisfies the conditions a) - c). Then we have*

$$N(\lambda; H(A, 0)) \leq \nu_1((1 + \varepsilon)\lambda)(1 + o(1)), \lambda \rightarrow \infty, \forall \varepsilon > 0, \quad (4.3)$$

$$N(\lambda; H(A, 0)) \geq \nu_1((1 - \varepsilon)\lambda)(1 + o(1)), \lambda \rightarrow \infty, \forall \varepsilon \in (0, 1). \quad (4.4)$$

Let $\phi(\lambda) : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be a nondecreasing (resp., nonincreasing) function. We shall say that ϕ satisfies the condition \mathcal{T}_0 if and only if we have

$$\lim_{\varepsilon \downarrow 0} \limsup_{\lambda \rightarrow \infty} \phi((1 + \varepsilon)\lambda)/\phi(\lambda) = 1,$$

(respectively,

$$\lim_{\varepsilon \downarrow 0} \limsup_{\lambda \downarrow 0} \phi((1 - \varepsilon)\lambda)/\phi(\lambda) = 1.)$$

A sufficient condition that the function ϕ satisfies \mathcal{T}_0 is the asymptotic relation

$$\phi(\lambda) = C\lambda^l |\log \lambda|^p (1 + O(1)), C > 0,$$

with $l > 0$, $p \in \mathbf{R}$, or $l = 0$, $p > 0$ as $\lambda \rightarrow \infty$, if ϕ is nondecreasing, or with $l < 0$, $p \in \mathbf{R}$, or $l = 0$, $p > 0$, as $\lambda \downarrow 0$, if ϕ is nonincreasing.

Note that if the function $\nu_1(\lambda)$ satisfies the condition \mathcal{T}_0 , then the estimates (4.3) – (4.4) can be united into a single asymptotic

$$N(\lambda; H(A, 0)) = \nu_1(\lambda)(1 + o(1)), \lambda \rightarrow \infty.$$

The result of Theorem 4.1 for $m = 3$ has been obtained independently and in a different way in [Tam 1, Theorem 1].

In the proof of Theorem 4.1 we use the following technical result.

Lemma 4.1 [CdV, Lemma 5.2] *Assume that B satisfies the conditions a) - c). Fix $\varepsilon > 0$. Then there exist a pavement of \mathbf{R}^m by cubes $\{q_i\}_{i \geq 0}$ of side-lengths r_i , and numbers $r_{0,i} \in (0, r_i/2]$, $i \geq 1$, such that if $M_i := \max_{|\beta|=2} \sup_{X \in q_i} |D^\beta A(X)|$, $i \geq 1$, then the following inequalities are satisfied:*

- (i) $r_i^2 M_i \leq \varepsilon \sup_{X \in q_i} |B(X)|^{1/2}$, $i \geq 1$;
- (ii) $r_{0,i}^{-2} \leq 4 \left(\varepsilon \sup_{X \in q_i} |B(X)| + \varepsilon^{-1} \right)$, $i \geq 1$.

Proof of (4.4). Put $\Omega = \mathbf{R}^m \setminus q_0$. The minimax principle implies

$$N(\lambda; H(A, 0)) \geq N(\lambda; H_\Omega^D(A, 0)) + N(\lambda; H_{q_0}^D(A, 0)). \quad (4.5)$$

For each $i \geq 1$ fix an arbitrary point $X_i \in q_i$. Set $A^{(i)} := B(X_i)(X - X_i)/2$. Applying the Taylor expansion of A at X_i , we obtain

$$h_{q_i}[u; A^{(i)}, 0] \geq (1 - \varepsilon_1)h_{q_i}[u; A, 0] - (\varepsilon_1^{-1} - 1)M_i^2 r_i^4 \int_{q_i} |u|^2 dX, \forall \varepsilon_1 \in (0, 1). \quad (4.6)$$

Fix $\varepsilon \in (0, 1/2)$ so that $\varepsilon \geq \varepsilon_1$ and $(\varepsilon_1^{-1} - 1)\varepsilon^2 \leq \varepsilon_1$. Making use of the property (i), we find that (4.6) entails the estimate

$$(1 - \varepsilon_1)h_{q_i}[u; A, 0] \leq h_{q_i}[u; A^{(i)}, 0] + \varepsilon_1 |B(X_i)| \int_{q_i} |u|^2 dX$$

which, combined with an elementary variational argument, yields

$$N(\lambda; H_\Omega^D(A, 0)) \geq \sum_{i \geq 1} N((1 - \varepsilon_1)\lambda - \varepsilon_1 |B(X_i)|; H_{q_i}^D(A^{(i)}, 0)). \quad (4.7)$$

Taking into account (2.43) and the property (ii), we obtain

$$N((1 - \varepsilon_1)\lambda - \varepsilon_1 |B(X_i)|; H_{q_i}^D(A^{(i)}, 0)) \geq$$

$$(r_i - r_{0,i})^m \Theta((1 - \varepsilon_1)\lambda - \varepsilon_1(1 + 4C)|B(X_i)| - 4C/\varepsilon; B(X_i)), \forall i \geq 1, \quad (4.8)$$

where C is the same constant as in (2.43). Recalling the explicit expression for Θ , and making use of the equivalence relation $|B(X_i)| \asymp \text{Tr}^+ B(X_i)$, we find that the inequality

$$(r_i - r_{0,i})^m \Theta((1 - \varepsilon_1)\lambda - \varepsilon_1(1 + 4C)|B(X_i)| - 4C/\varepsilon; B(X_i)) \geq$$

$$(1 - C'\varepsilon_2)r_i^m \Theta((1 - C'\varepsilon_2)\lambda; B(X_i)), \forall i \geq 1, \quad (4.9)$$

holds for every given sufficiently small $\varepsilon_2 > 0$, appropriate ε and ε_1 , and a constant C' independent of $\lambda \gg 1$, ε_2 and $i \geq 1$. Now, choose X_i so that the function $\tilde{\Theta}(X) := \Theta((1 - C'\varepsilon_2)\lambda; B(X))$ takes its maximum value on q_i at the point X_i . Combining (4.7) – (4.9), we get

$$N(\lambda; H_\Omega^D(A, 0)) \geq (1 - C'\varepsilon_2) \int_\Omega \Theta((1 - C'\varepsilon_2)\lambda; B(X)) dX. \quad (4.10)$$

Finally, note that the well-known asymptotics

$$N(\lambda; H_q^D(A, 0)) = \omega_m \text{vol}\{q_0\} \lambda^{m/2} (1 + o(1)), \lambda \rightarrow \infty,$$

together with (4.2) entail

$$N(\lambda; H_q^D(A, 0)) = o(\nu_1(\lambda)), \lambda \rightarrow \infty. \quad (4.11)$$

Combining (4.5), (4.10) and (4.11), we come to (4.4).

Sketch of the proof of (4.3). Denote by \tilde{q}_i , $i \geq 0$, the cubes in \mathbf{R}^m whose centres are the same as for q_i but whose side lengths equal $r_i + r_{0,i}$. Introduce a family of functions $\varphi_i \in C_0^\infty(\tilde{q}_i)$ such that $\sum_{i \geq 0} \varphi_i^2 \equiv 1$ and $|\nabla \varphi_i(X)| \leq \tilde{C}/r_{0,i}$ (cf. Lemma 2.5). Then the minimax principle yields

$$N(\lambda; H(A, 0)) \leq \sum_{i \geq 0} N(\lambda - \tilde{C}^2 r_{0,i}^{-2}; H_{\tilde{q}_i}^D(A, 0)).$$

Further the proof of (4.3) is quite similar to the one for (4.4).

4.1.2. In this paragraph we describe briefly the results of [Mat 2] which can be regarded as typical examples of the asymptotic behaviour of $N(\lambda; H(A, V))$ as $\lambda \rightarrow \infty$ in the case where $H(A, V)$ has a compact resolvent and both A and V do not vanish identically. The methods applied in [Mat 2] (as well as the ones in the preceding related works [Ode] and [Mat 1]) are probabilistic, and the corresponding results do not contain explicitly the asymptotics of $N(\lambda; H(A, V))$ as $\lambda \rightarrow \infty$, but rather the asymptotics of $\text{Tr}(\exp -tH(A, V))$ as $t \downarrow 0$. However, it is well-known that if $N_j(\lambda)$, $j = 1, 2$, $\lambda \in \mathbf{R}$, are two nondecreasing functions with lower-bounded supports, then the asymptotic relation

$$\int_{\mathbf{R}} e^{-t\lambda} dN_1(\lambda) = \int_{\mathbf{R}} e^{-t\lambda} dN_2(\lambda)(1 + o(1)), t \downarrow 0,$$

implies

$$N_1(\lambda) = N_2(\lambda)(1 + o(1)), \lambda \rightarrow \infty,$$

provided that the hypotheses of Karamata Tauberian theorem are satisfied (see [Shu 1, Problem 14.2].)

In order to describe the results of [Mat 2], it is convenient to introduce here some classes of functions over \mathbf{R}^m having power-like behaviour at infinity. These classes will be used quite often in the sequel.

Let F be a real-valued function over \mathbf{R}^m . We shall write $F \in \mathcal{D}_{\alpha,l}$, $\alpha \in \mathbf{R}$, $l \in \mathbf{N}_+$, if and only if $F \in C^l(\mathbf{R}^m)$ and for each multiindex γ , such that $|\gamma| \leq l$, we have

$$|D^\gamma F(X)| \leq c_\gamma \langle X \rangle^{\alpha-|\gamma|}, X \in \mathbf{R}^m,$$

where $\langle X \rangle := (1 + |X|^2)^{1/2}$, and c_γ are some positive constants. We shall write $F \in \mathcal{D}_{\alpha,l}^+$ if and only if $F \in \mathcal{D}_{\alpha,l}$, and, moreover, we have

$$C \langle X \rangle^\alpha \leq F(X), |X| > R, R > 0, C > 0.$$

Finally, we shall write $F \in \mathcal{D}_{\alpha,l}^{++}$ if and only if $F \in \mathcal{D}_{\alpha,l}^+$ and, moreover, we have

$$C \langle X \rangle^\alpha \leq |X \cdot \nabla F(X)|, |X| > R, R > 0, C > 0.$$

Set

$$\mathcal{D}_{\alpha,\infty} := \bigcap_{l=1}^{\infty} \mathcal{D}_{\alpha,l}, \mathcal{D}_{\alpha,\infty}^+ := \bigcap_{l=1}^{\infty} \mathcal{D}_{\alpha,l}^+, \mathcal{D}_{\alpha,\infty}^{++} := \bigcap_{l=1}^{\infty} \mathcal{D}_{\alpha,l}^{++}.$$

If the value of the index l in the notations $\mathcal{D}_{\alpha,l}$, $\mathcal{D}_{\alpha,l}^+$ and $\mathcal{D}_{\alpha,l}^{++}$ is of no importance, it will be sometimes omitted.

Theorem 4.2 [Mat 2, Theorem 1] *Let $V \in \mathcal{D}_{\alpha,1}^+$, $\alpha > 0$. Assume $A \in C^2(\mathbf{R}^m; \mathbf{R}^m)$, $|D^\gamma A(X)| = o(V(X))$ as $|X| \rightarrow \infty$ for $|\gamma| = 1, 2$. Then we have*

$$\text{Tr} \exp(-tH(A, V)) = \text{Tr} \exp(-tH(0, V))(1 + o(1)), t \downarrow 0.$$

Under the hypotheses of Theorem 4.2 we have

$$N(\lambda; H(0, V)) = (2\pi)^{-m} \left\{ (X, \Xi) \in T^*\mathbf{R}^m : |\Xi|^2 + V(X) < \lambda \right\} (1 + o(1)), \lambda \rightarrow \infty.$$

Thus, if these hypotheses hold together with the natural assumptions allowing to apply the Karamata Tauberian theorem, then Conjecture 1.2 is valid.

Theorem 4.3 [Mat 2, Theorem 2] *Let $B = \text{curl } A$, $|B(X)| \rightarrow \infty$, $|D^\gamma A(X)| = o(|B(X)|^{3/2})$, $|\gamma| = 2$, as $|X| \rightarrow \infty$. Assume that the function $\tilde{\Psi}_B(\lambda)$ defined in (4.1) satisfies the inequality*

$$\tilde{\Psi}(2\lambda) \leq C\tilde{\Psi}(\lambda), C > 0,$$

for sufficiently large $\lambda > 0$. Moreover, suppose that $V \in L^2_{loc}(\mathbf{R}^m; \mathbf{R})$ satisfies the estimates

$$V(X) \geq -C, C \geq 0,$$

$$V(X) = o(|B(X)|) \text{ as } |X| \rightarrow \infty.$$

Then we have

$$\text{Tr} \exp(-tH(A, V)) = \text{Tr} \exp(-tH(A, 0))(1 + o(1)), t \downarrow 0.$$

Under the assumptions of Theorem 4.3 the asymptotics of $N(\lambda; H(A, 0))$ is given essentially by (4.3) – (4.4). Hence, in this case Conjecture 1.2 is not valid any more, and the classical phase-space volume

$$(2\pi)^{-m} \text{vol} \left\{ (X, \Xi) \in T^*\mathbf{R}^m : |\Xi - A(X)|^2 + V(X) < \lambda \right\}$$

must be replaced by the function $\nu_1(\lambda)$.

4.2 Eigenvalue asymptotics for Hamiltonians $H(A, V)$ with constant magnetic field B and electric potential which decays at infinity. The case $\dim \text{Ker } B > 0$

4.2.1. Throughout this and the following subsection we assume that the conditions (2.26) are fulfilled. Besides, we shall impose such restrictions on the electric potential V that imply, in particular, the $-\Delta$ -form-compactness of V and, hence, the validity of Corollary 3.3.

In this subsection we assume in addition $k \equiv \dim \text{Ker } B > 0$. Recall that in this case we have $\sigma_{ess}(H(A, V)) = [\Lambda_1, \infty)$ (see Corollary 3.3.)

Set

$$\mathbf{N}_1^-(\lambda) = N(\Lambda_1 - \lambda; H(A, V)), \lambda > 0. \quad (4.12)$$

Theorem 4.4 [Rai 2, Theorem 2.2] *Let $k > 0$. Assume that $-V \in \mathcal{D}_{-\alpha, 1}^+$ with $\alpha \in (0, 2)$. For $\lambda > 0$ put*

$$\nu_2(\lambda) = \omega_k b_1 \dots b_d \int_{\mathbf{R}^m} (V(X) + \lambda)_-^{k/2} dX / (2\pi)^{d+k}.$$

Then we have

$$\mathbf{N}_1^-(\lambda) = \nu_2(\lambda)(1 + o(1)), \lambda \downarrow 0. \quad (4.13)$$

Remark. The assumptions of Theorem 4.4 entail the asymptotic estimate

$$\nu_2(\lambda) \asymp \lambda^{k/2 - m/\alpha}, \lambda \downarrow 0.$$

Moreover, we have

$$\nu_2(\lambda) = \int_{\mathbf{R}^m} \Theta(-V(X) + \Lambda_1 - \lambda) dX(1 + o(1)), \lambda \downarrow 0.$$

We omit the proof of Theorem 4.4 since it is quite the same as the one for [Tam 2, Theorem 1 (i)], and is based on Theorem 2.7. Besides, a pavement of \mathbf{R}^m similar to the one introduced in [Roz] is utilized; we shall use this construction several times in Section 5.

Let $\phi : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be a nonincreasing function which is differentiable on the interval $(0, \lambda_0]$ for some $\lambda_0 > 0$. We say that ϕ satisfies the condition \mathcal{T} if and only if the inequalities

$$c_1\phi(\lambda) \leq -\lambda\phi'(\lambda) \leq c_2\phi(\lambda), \forall \lambda \in (0, \lambda_0],$$

are valid for some positive constants c_1 and c_2 . Note that a function ϕ satisfies the condition \mathcal{T} , then it satisfies the condition \mathcal{T}_0 too.

Let F be a real function over \mathbf{R}^m such that $F(X) \rightarrow 0$ as $|X| \rightarrow \infty$. For $\lambda > 0$ put

$$\Phi_F(\lambda) := \text{vol} \{X \in \mathbf{R}^m : F(X) > \lambda\}.$$

Then the inclusion $F \in \mathcal{D}_{-\alpha, 1}^{++}$, $\alpha > 0$ is a sufficient condition that the function $\Phi_F(\lambda)$ satisfies \mathcal{T} .

Theorem 4.5 [Rai 2, Theorem 2.4 i)-ii)] *Let $k = 1, 2$. Assume that $-V \in \mathcal{D}_{-\alpha, \infty}^{++}$ with $\alpha > 2$. For $\lambda > 0$ put*

$$\nu(\lambda) = (2\pi)^{-d} b_1 \dots b_d \text{vol} \left\{ (x, y) \in \mathbf{R}^{2d} \equiv \mathbf{R}^m \ominus \text{Ker } B : \int_{\mathbf{R}^k} V(x, y, z) dz < -\lambda \right\},$$

and assume that the function $\nu(\lambda)$ satisfies the condition \mathcal{T} .

(i) *Let $k = 1$. For $\lambda > 0$ put $\nu_{3,1}(\lambda) = \nu(2\lambda^{1/2})$. Then we have*

$$\mathbf{N}_1^-(\lambda) = \nu_{3,1}(\lambda)(1 + o(1)), \lambda \downarrow 0. \quad (4.14)$$

(ii) Let $k = 2$. For $\lambda \in (0, 1)$ put $\nu_{3,2}(\lambda) = \nu(4\pi|\log \lambda|^{-1})$. Then we have

$$\mathbf{N}_1^-(\lambda) = \nu_{3,2}(\lambda)(1 + o(1)), \lambda \downarrow 0. \quad (4.15)$$

Remark. Under the hypotheses of Theorem 4.5 we have

$$\nu_{3,1}(\lambda) \asymp \lambda^{-d/(\alpha-1)}, \lambda \downarrow 0,$$

and

$$\nu_{3,2}(\lambda) \asymp |\log \lambda|^{2d/(\alpha-2)}, \lambda \downarrow 0.$$

Theorem 4.6 [Rai 2, Theorem 2.4 iii)] *Let $k \geq 3$. Set $V_\pm := \max\{\pm V, 0\}$. Assume that V_+ is $-\Delta$ -form-compact and $\langle X \rangle^\alpha V_- \in L^\infty(\mathbf{R}^m)$ for some $\alpha > 2$. Then the asymptotic estimate*

$$\mathbf{N}_1^-(\lambda) = O(1), \lambda \downarrow 0, \quad (4.16)$$

holds, i.e. the isolated eigenvalues of the operator $H(A, V)$ do not accumulate to its essential-spectrum lower bound.

Remark. In the proof of Theorem 4.6 we assume $V_+ \equiv 0$ without any loss of generality.

The proofs of Theorems 4.5 and 4.6 are contained in the following five paragraphs. In these proofs we apply an approach which combines and develops the methods used in [Sob 1] and [Tam 2] where the result of Theorem 4.5 (i) for the case $m = 3$ (i.e. $d = 1$ and $k = 1$) has been established.

In the formulations of Theorems 4.5 and 4.6 we do not seek for maximal generality of the assumptions in respect to the behaviour of V at infinity, and its local smoothness. In these two theorems, our main goal is to point out, clearly enough, to the specific effects which are due to the presence of the nonvanishing constant magnetic field B , and to outline the dependence of these effects on the decay rate α of the potential V , and the value k of the deficiency index of B . One of the most striking differences in comparison with the case $B \equiv 0$ is the fact that the eigenvalues of $H(A, V)$ *do accumulate* to the essential-spectrum lower-bound Λ_1 even if V decays rapidly at infinity (e.g. if $V \in \mathcal{D}_{-\alpha}$ with $\alpha > 2$) provided that $k = 1, 2$. The deep and inconspicuous reason for this effect is the fact that the operator $-\Delta + \lambda\mathcal{W}$ considered in $L^2(\mathbf{R}^k)$, $k = 1, 2$, has at least one negative eigenvalue for each (arbitrarily small λ) if we have $\int_{\mathbf{R}^k} \mathcal{W} dX < 0$. On the contrary, if $k \geq 3$ and $\mathcal{W} \in L^{k/2}(\mathbf{R}^k)$, the operator $-\Delta + \lambda\mathcal{W}$ is nonnegative for sufficiently small $\lambda > 0$ even if $\mathcal{W} < 0$ almost everywhere. This is the intrinsic reason

for the fact that the isolated eigenvalues of $H(A, V)$ do not accumulate to Λ_1 in the case of rapidly decaying potential and $k \geq 3$.

At the end of this subsection we would mention some of the possible extensions of Theorems 4.4 and 4.5.

(i) Let the potential V satisfy the hypotheses of Theorem 4.4 or 4.5. Assume that $V_1 \in L^{m/2}(\mathbf{R}^m; \mathbf{R})$ and $\text{supp } V_1$ is compact. Then the asymptotics (4.13) - (4.15) remain valid if we substitute $H(A, V)$ for $H(A, V + V_1)$ (cf. [Tam 2, Sect. 5].)

(ii) In [Sob 2] - [Sob 4] assertions similar to Theorem 4.5 (i) for the case $m = 3$ have been established without the assumption that V is nonpositive outside a given compact.

(iii) Finally, in [Tam 3] and [Vul.Zhi] analogues of Theorem 4.4 and Theorem 4.5 (i), $m = 3$, for the multiparticle Schrödinger operator can be found.

4.2.2. In this section we assume that the hypotheses of any Theorem 4.5 or Theorem 4.6 are valid.

Set

$$\mathcal{H}_{2,1} = \int_{\mathbf{R}^d}^{\oplus} h_2 dy \quad (4.17)$$

(see (2.35).)

Define the operator $h_4 = h_4(z)$, $z \in \mathbf{R}^k$, as a Ψ DO with Weyl symbol

$$-\frac{1}{(2\pi)^d} \int_{\mathbf{R}^{3d}} f_{1,1}(x) f_{1,1}(x') \exp \{i\xi \cdot (x - x')\} V_{\mathbf{b}}\left(\frac{1}{2}(x + x') - \eta, y - \xi, z\right) dx dx' d\xi \quad (4.18)$$

(see (2.29) - (2.30), and the definition of the operator h_3 in §2.3.2.) Evidently, $h_4(z)$ is selfadjoint and bounded on $L^2(\mathbf{R}^d)$ for each $z \in \mathbf{R}^k$.

In (4.18) change the variables $x - x' = s$, $(x + x')/2 = t$, and calculate the integral with respect to s explicitly taking into account (2.30) with $q = 1$. Next, change the variables $t - \eta = -\eta'$, $y - \xi = y'$. Bearing in mind the connection between the Weyl and the antiwick symbol of a given Ψ DO (see (2.33)), we find that $h_4(z)$ is a Ψ DO with antiwick symbol $-V_{\mathbf{b}}(-\eta, y; z)$ depending on the parameter $z \in \mathbf{R}^k$.

Lemma 4.2 a) *Under the hypotheses of Theorem 4.5 the operator inequalities*

$$h_4(z) \leq c_1^+ (h_1 + |z|^2)^{-\alpha/2}, z \in \mathbf{R}^k, \quad (4.19)$$

$$h_4(z) \geq c_1^- (h_1 + |z|^2)^{-\alpha/2}, z \in \mathbf{R}^k, \quad (4.20)$$

hold with some positive constants c_1^+ and c_1^- .

b) *Under the hypotheses of Theorem 4.6 the estimate (4.19) is valid.*

Proof. For definiteness we prove the first assertion. Let $z \in \mathbf{R}^k$, $\alpha > 0$ and $\mu_0 > \Lambda_1$. Then the antiwick symbol $\mathbf{r}_{\alpha/2}(x, \xi; |z|^2 + \mu_0)$ of the operator $(h_1 + \mu_0 + |z|^2)^{-\alpha/2}$ admits the estimates

$$\pm \mathbf{r}_{\alpha/2}(x, \xi; |z|^2 + \mu_0) \leq \pm c_2^\pm (|x|^2 + |\xi|^2 + |z|^2 + 1)^{-\alpha/2}, (x, \xi, z) \in \mathbf{R}^m, 0 < c_2^\pm < \infty,$$

(see (2.31).) Making use of the inclusion $-V \in \mathcal{D}_{-\alpha}$, we obtain the estimates

$$\mp V_{\mathbf{b}}(-\xi, x; z) \leq \pm \tilde{c}_2^\pm (|x|^2 + |\xi|^2 + |z|^2 + 1)^{-\alpha/2}, (x, \xi, z) \in \mathbf{R}^m, 0 < \tilde{c}_2^\pm < \infty.$$

Since any Ψ DO with nonnegative antiwick symbol is a nonnegative operator (see [Shu 1, Proposition 24.1]), and the operator h_1 is strictly positively definite, we come to (4.19) - (4.20).

Set

$$\mathcal{H}_{2,2} = \int_{\mathbf{R}^k}^\oplus h_4(z) dz.$$

Define the operator

$$\mathcal{H}_2(\varepsilon) = \mathcal{H}_{2,1} - (1 + \varepsilon)\mathcal{H}_{2,2}, \varepsilon > -1,$$

which is selfadjoint and bounded in $L^2(\mathbf{R}^{k+d})$.

Proposition 4.1 (cf. [Tam 2, Subsects. 3.2 and 4.1]) *We have*

$$N(\Lambda_1 - \lambda; \mathcal{H}_1) \leq N(-\lambda; \mathcal{H}_2(\varepsilon)) + O(1), \lambda \downarrow 0, \forall \varepsilon > 0, \quad (4.21)$$

$$N(\Lambda_1 - \lambda; \mathcal{H}_1) \geq N(-\lambda; \mathcal{H}_2(-\varepsilon)) + O(1), \lambda \downarrow 0, \forall \varepsilon \in (0, 1). \quad (4.22)$$

Sketch of the proof. We shall outline only the proof of (4.21) since the proof of (4.22) is completely analogous.

Define the orthogonal projection $P_1 : L^2(\mathbf{R}^m) \rightarrow L^2(\mathbf{R}^m)$ by

$$(P_1 u)(x, y, z) = f_{1,1}(x) \int_{\mathbf{R}^d} u(x', y, z) f_{1,1}(x') dx', u \in L^2(\mathbf{R}^m),$$

and set $Q_1 = \text{Id} - P_1$. In view of $V < 0$, we have the inequality

$$\mathcal{H}_1 \geq P_1(\mathcal{H}_{1,1} + \mathcal{H}_{1,2} + (1 + \varepsilon)\mathcal{H}_{1,3})P_1 + Q_1(\mathcal{H}_{1,1} + \mathcal{H}_{1,2} + (1 + \varepsilon^{-1})\mathcal{H}_{1,3})Q_1, \forall \varepsilon > 0.$$

Note that the operator at the right-hand side is unitarily equivalent to the orthogonal sum $(\Lambda_1 + \mathcal{H}_{2,2}(\varepsilon)) \oplus \hat{\mathcal{H}}_2(\varepsilon)$ where $\hat{\mathcal{H}}_2(\varepsilon)$ is a selfadjoint operator such that $\inf \sigma_{ess}(\hat{\mathcal{H}}_2(\varepsilon)) = \Lambda_2, \forall \varepsilon > 0$. Hence, we have

$$N(\Lambda_1 - \lambda; \mathcal{H}_1) \leq N(-\lambda; \mathcal{H}_2(\varepsilon)) + N(\Lambda_1 - \lambda; \hat{\mathcal{H}}_2(\varepsilon)), \forall \varepsilon > 0,$$

which entails (4.21).

4.2.3. In this and the following three paragraphs we assume that the hypotheses of Theorem 4.5 hold; in particular, we suppose $k = 1, 2$.

Set

$$\mathcal{R}(\lambda) \equiv \mathcal{R}^{(k)}(\lambda) := (\mathcal{H}_{1,2} + \lambda)^{-1}, \lambda > 0, k = 1, 2,$$

(see (2.36).) The Birman-Schwinger principle yields

$$N(-\lambda; \mathcal{H}_2(\varepsilon)) = n((1 + \varepsilon)^{-1}; \mathcal{R}(\lambda)^{1/2} \mathcal{H}_{2,2} \mathcal{R}(\lambda)^{1/2}), \lambda > 0, \varepsilon \in (-1, 1). \quad (4.23)$$

Set

$$\mathcal{M}(\lambda) \equiv \mathcal{M}^{(k)}(\lambda) = \mathcal{H}_{2,2}^{1/2} \mathcal{R}^{(k)}(\lambda) \mathcal{H}_{2,2}^{1/2}, k = 1, 2.$$

Since the nonzero singular numbers of the operators $\mathcal{H}_{2,2}^{1/2} \mathcal{R}(\lambda)^{1/2}$ and $\mathcal{R}(\lambda)^{1/2} \mathcal{H}_{2,2}^{1/2}$ coincide together with the multiplicities, we have

$$n((1 + \varepsilon)^{-1}; \mathcal{R}(\lambda)^{1/2} \mathcal{H}_{2,2} \mathcal{R}(\lambda)^{1/2}) = n((1 + \varepsilon)^{-1}; \mathcal{M}(\lambda)), \lambda > 0, \varepsilon \in (-1, 1). \quad (4.24)$$

Next, we obtain suitable representations for the operators $\mathcal{R}^{(k)}(\lambda), k = 1, 2$. They are inspired by the standard regularization procedure concerning the distribution $|\zeta|^{-2}$ defined over $\mathbf{R}^k, k = 1, 2$.

Denote by $\mathfrak{r}(\lambda) \equiv \mathfrak{r}^{(k)}(\lambda)$ the operator $(\mathfrak{h}_2 + \lambda)^{-1}, \lambda > 0$, defined on $L^2(\mathbf{R}^k), k = 1, 2$ (see (2.35).) Then the kernel $r(z, z'; \lambda) \equiv r^{(k)}(z, z'; \lambda)$ of $\mathfrak{r}(\lambda)$ can be written in the form

$$r^{(k)}(z, z'; \lambda) = \frac{1}{(2\pi)^k} \int_{\mathbf{R}^k} \frac{e^{i\zeta \cdot (z-z')}}{|\zeta|^2 + \lambda} d\zeta, k = 1, 2, \lambda > 0.$$

If $k = 2$, the integral should be understood in the distributional sense.

Moreover, we have

$$\int_{\mathbf{R}^d}^{\oplus} \mathfrak{r}(\lambda) dy = \mathcal{R}(\lambda).$$

Set

$$r_1^{(1)}(z, z'; \lambda) \equiv r_1^{(1)}(\lambda) := \frac{1}{2\pi} \int_{\mathbf{R}} \frac{d\zeta}{\zeta^2 + \lambda} \equiv \frac{1}{2} \lambda^{-1/2}, \lambda > 0,$$

$$r_2^{(1)}(z, z'; \lambda) := \frac{1}{2\pi} \int_{\mathbf{R}} \frac{e^{i\zeta(z-z')} - 1}{\zeta^2 + \lambda} d\zeta \equiv -\frac{2}{\pi} \int_0^\infty \frac{(\sin |z - z'| \varrho/2)^2}{\varrho^2 + \lambda} d\varrho, \lambda \geq 0.$$

The following three integrals are taken over subsets of \mathbf{R}^2 .

$$r_1^{(2)}(z, z'; \lambda) \equiv r_1^{(2)}(\lambda) := \frac{1}{(2\pi)^2} \int_{|\zeta| < 1} \frac{d\zeta}{|\zeta|^2 + \lambda} \equiv \frac{1}{4\pi} \log [(1 + \lambda)/\lambda], \lambda > 0,$$

$$r_{2,1}^{(2)}(z, z'; \lambda) := \frac{1}{(2\pi)^2} \int_{|\zeta| < 1} \frac{e^{i\zeta \cdot (z-z')} - 1}{|\zeta|^2 + \lambda} d\zeta \equiv$$

$$-\frac{1}{\pi^2} \int_0^1 \int_0^\pi \frac{[\sin(\cos \varphi |z - z'| \varrho/2)]^2}{\varrho^2 + \lambda} \varrho d\varrho d\varphi, \lambda \geq 0,$$

$$r_{2,2}^{(2)}(z, z'; \lambda) := \frac{1}{(2\pi)^2} \int_{|\zeta| > 1} \frac{e^{i\zeta \cdot (z-z')}}{|\zeta|^2 + \lambda} d\zeta, \lambda \geq 0,$$

the last integral being understood in the distributional sense. Put

$$r_2^{(2)}(z, z'; \lambda) := \sum_{l=1,2} r_{2,l}^{(2)}(z, z'; \lambda), \lambda \geq 0.$$

Then we have

$$r^{(k)}(z, z'; \lambda) = \sum_{j=1,2} r_j^{(k)}(z, z'; \lambda), k = 1, 2.$$

For a given Landau level $\Lambda_q, q \geq 1$, set

$$\chi_q(z) = \Lambda_q + |z|^2, z \in \mathbf{R}^k, k = 1, 2.$$

Let $G_q^{(k)}(\lambda) : L^2(\mathbf{R}^k) \rightarrow L^2(\mathbf{R}^k)$ be the integral operator with kernel

$$\chi_q(z)^{-\alpha/4} r_2^{(k)}(z, z'; \lambda) \chi_q(z')^{-\alpha/4}, \lambda > 0, \alpha > 2, k = 1, 2.$$

Lemma 4.3 *Let $k = 1$. Then the estimates*

$$\|G_q^{(1)}(\lambda)\|_2^2 \leq c_3 \lambda^{-(1-\varepsilon)} \Lambda_q^{-(\alpha-1-\varepsilon)}, \lambda > 0, q \geq 1, \quad (4.25)$$

are valid for $\varepsilon = \min\{1, \alpha_0\}$ where $\alpha_0 > 0$ is any fixed number smaller than $(\alpha - 1)/2$, and the quantity $c_3 = c_3(\varepsilon)$ is independent of λ and q .

Proof. First of all note the elementary estimate

$$|r_2^{(1)}(z, z'; \lambda)| \leq \frac{2}{\pi} \int_0^\infty \varrho^{-2} (\sin [|z - z'| \varrho / 2])^2 d\varrho = c'_4 |z - z'| \quad (4.26)$$

where c'_4 does not depend on z, z' and λ . Further, apply the Hölder inequality in order to verify that the estimates

$$|r_2^{(1)}(z, z'; \lambda)| \leq \frac{2}{\pi} \left(\int_0^\infty (\varrho^2 + \lambda)^{-p/2} d\varrho \right)^{1/p} \left(\int_0^\infty \varrho^{-p'} (\sin [|z - z'| \varrho / 2])^{2p'} d\varrho \right)^{1/p'} =$$

$$c''_4 \lambda^{-(p-1)/2p} |z - z'|^{1/p}, \forall p > 1, p' = p/(p-1), \quad (4.27)$$

are valid with c''_4 which is again independent of z, z' and λ . Combining (4.26) with (4.27), and setting $\varepsilon = 1/p$, we find that the estimate

$$|r_2^{(1)}(z, z'; \lambda)| \leq c_4 \lambda^{-(1-\varepsilon)/2} |z - z'|^\varepsilon \quad (4.28)$$

is valid for each $\varepsilon \in (0, 1]$ and a constant c_4 independent of z, z' and λ . Making use of (4.28), we obtain the inequality

$$\|G_q^{(1)}(\lambda)\|_2^2 \leq c_4^2 \lambda^{-(1-\varepsilon)} \Lambda_q^{-(\alpha-1-\varepsilon)} \int_{\mathbf{R}^2} |z - z'|^{2\varepsilon} (1 + z^2)^{-\alpha/2} (1 + z'^2)^{-\alpha/2} dz dz'. \quad (4.29)$$

Since the integral at the right-hand side of (4.29) is finite for $2\varepsilon - \alpha < -1$, we come to (4.25).

Lemma 4.4 [Rai 2, Lemma 4.3] *Let $k = 2$. Then the estimate*

$$\|G_q^{(2)}(\lambda)\|_2^2 \leq c_5 \Lambda_q^{-\delta}, \lambda \geq 0, q \geq 1, \quad (4.30)$$

holds for any $\delta \in (0, \alpha - 2)$ and a quantity $c_5 = c_5(\delta)$ independent of $\lambda \geq 0$ and $q \geq 1$.

Proof. Obviously, we have

$$\|G_q^{(2)}(\lambda)\|_2^2 \leq 2 \sum_{i=1,2} \int_{\mathbf{R}^4} \chi_q(z)^{-\alpha/2} \chi_q(z')^{-\alpha/2} r_{2,i}^{(2)}(z, z'; \lambda)^2 dz dz'.$$

Applying the elementary inequality

$$r_{2,2}^{(2)}(z, z'; 0)^2 \leq c'_6 \{1 + \log(1 + |z - z'|)\}^2 \leq c_6(\varepsilon) \chi_q(z)^{\varepsilon/2} \chi_q(z')^{\varepsilon/2}, \forall \varepsilon > 0, \forall q \leq 1,$$

with constants c'_6 and c_6 which are independent of λ and q , we get

$$\int_{\mathbf{R}^4} \chi_q(z)^{-\alpha/2} \chi_q(z')^{-\alpha/2} r_{2,2}^{(2)}(z, z'; \lambda)^2 dz dz' \leq$$

$$c'_7 \left(\int_{\mathbf{R}^2} \chi_q(z)^{-(\alpha-\varepsilon)/2} dz \right)^2 \leq c_7 \Lambda_q^{-(\alpha-2-\varepsilon)}, \forall \varepsilon \in (0, \alpha - 2),$$

where c'_7 and c_7 are independent of λ and q . Further, applying the general result of [Bir.Sol 2, Theorem 6.3 a)] we obtain the estimates

$$\int_{\mathbf{R}^4} \chi_q(z)^{-\alpha/2} \chi_q(z')^{-\alpha/2} r_{2,2}^{(2)}(z, z'; \lambda)^2 dz dz' \leq$$

$$c'_8 \int_{\mathbf{R}^2} \chi_q(z)^{-\alpha} dz \int_{|\zeta|>1} (|\zeta|^2 + \lambda)^{-2} d\zeta \leq c_8 \Lambda_q^{-(\alpha-1)},$$

where c'_8 and c_8 are independent of λ and q .

Now we are in position to conclude the validity of (4.30).

4.2.4. Define the operators $\mathcal{R}_j^{(k)}(\lambda)$, $\lambda > 0$, $j = 1, 2$, $k = 1, 2$, by

$$(\mathcal{R}_j^{(k)}(\lambda)u)(y, z) = \int_{\mathbf{R}^k} r_j^{(k)}(z, z'; \lambda) u(y, z') dz', (y, z) \in \mathbf{R}^{d+k}.$$

Set $\mathcal{M}_j^{(k)}(\lambda) = \mathcal{H}_{2,2}^{1/2} \mathcal{R}_j^{(k)}(\lambda) \mathcal{H}_{2,2}^{1/2}$, $j = 1, 2$, $k = 1, 2$. As will be seen from Lemma 4.6 and Corollary 4.1 below, the operators $\mathcal{M}_j^{(k)}(\lambda)$, $j = 1, 2$, $\lambda > 0$, are compact in $L^2(\mathbf{R}^{d+k})$, $k = 1, 2$. The fact that $\mathcal{M}_1^{(k)}(\lambda)$ is well-defined on $L^2(\mathbf{R}^{d+k})$, $k = 1, 2$, is implied also by the following simple lemma.

Lemma 4.5 *Assume that the hypotheses of Theorem 4.5 hold (hence, $k = 1, 2$.)*

a) *Let $w \in L^2(\mathbf{R}^{d+k})$. Set $u(y) = \int_{\mathbf{R}^{d+k}} (\mathcal{H}_{2,2}^{1/2} w)(y, z) dz$. Then we have $u \in L^2(\mathbf{R}^d)$.*

b) *Let $u \in L^2(\mathbf{R}^d)$. Set $w(y, z) = (h_4^{1/2}(z)u)(y)$. Then we have $w \in L^2(\mathbf{R}^{d+k})$.*

Proof. Since $\alpha > 2$ and $k = 1, 2$, the estimate (4.19) entails $\|h_4^{1/2}(z)\|_\infty \in L^2(\mathbf{R}_z^k)$ which guarantees the validity of the desired results.

Since we have $\mathcal{M}^{(k)}(\lambda) = \sum_{j=1,2} \mathcal{M}_j^{(k)}(\lambda)$, the Weyl – Ky Fan inequalities for the eigenvalues of compact selfadjoint operators (see e.g. [Bir.Sol 1, p.16]), imply that the estimates

$$\pm n(\mu; \mathcal{M}^{(k)}(\lambda)) \leq \pm n(\mu(1 \mp \tau); \mathcal{M}_1^{(k)}(\lambda)) + n(\mu\tau; \pm \mathcal{M}_2^{(k)}(\lambda)), \lambda > 0, \quad (4.31)$$

hold for each $\mu > 0$ and $\tau \in (0, 1)$.

Lemma 4.6 *Assume that the hypotheses of Theorem 4.5 hold.*

a) *If $k = 1$, then we have*

$$n(\delta; \pm \mathcal{M}_2^{(1)}(\lambda)) = o(\lambda^{-d/(\alpha-1)}), \lambda \downarrow 0, \forall \delta > 0. \quad (4.32)$$

b) *If $k = 2$, then we have*

$$n(\delta; \pm \mathcal{M}_2^{(2)}(\lambda)) = 0(1), \lambda \downarrow 0, \forall \delta > 0. \quad (4.33)$$

Proof. Set $\tilde{\mathcal{H}} := \int_{\mathbf{R}^k}^{\oplus} (h_1 + |z|^2)^{\alpha/2} dz$, $k \geq 1$. It follows from (4.19) - (4.20) that the operator $\mathcal{H}_{2,2}$ (and, hence, $\mathcal{H}_{2,2}^{1/2}$) is invertible, $D(\mathcal{H}_{2,2}^{-1}) = D(\tilde{\mathcal{H}})$, and, moreover, we have

$$\mathcal{H}_{2,2}^{-1} \leq (c_1^-)^{-1} \tilde{\mathcal{H}}, \quad (4.34)$$

$$\mathcal{H}_{2,2}^{-1} \geq (c_1^+)^{-1} \tilde{\mathcal{H}}, \quad (4.35)$$

Denote by $\tilde{\mathcal{M}} \equiv \tilde{\mathcal{M}}^{(k)}(\lambda)$, $k = 1, 2$, the operators generated by the quadratic-forms' ratio

$$(\mathcal{R}^{(k)}(\lambda)u, u) / (\tilde{\mathcal{H}}u, u), u \in D(\tilde{\mathcal{H}}), \quad (4.36)$$

where (\cdot, \cdot) is the scalar product in $L^2(\mathbf{R}^{d+k})$. The inequality (4.35) combined with standard variational arguments implies

$$n(\delta; \pm \mathcal{M}^{(k)}(\lambda)) \leq n(\delta/c_1^+; \pm \tilde{\mathcal{M}}^{(k)}(\lambda)), k = 1, 2, \forall \delta > 0. \quad (4.37)$$

Decomposing the trial function u in (4.36) in a series with respect to the eigenfunctions of the operator h_1 , we easily get

$$\begin{aligned} n(\delta; \pm \tilde{\mathcal{M}}^{(k)}(\lambda)) &= \sum_{q=1}^{\infty} \kappa_q n(\delta; \pm G_q^{(k)}(\lambda)) \leq \\ &\sum_{q=1}^{\infty} \kappa_q \text{ent} \{ \delta^{-2} \|G_q^{(k)}(\lambda)\|_2^2 \}, \forall \delta > 0, k = 1, 2, \end{aligned} \quad (4.38)$$

where $\text{ent}(t)$ denotes the nearest to t , and strictly less than t integer. Now we prove (4.32). The estimate (4.25) entails

$$\begin{aligned} &\sum_{q=1}^{\infty} \kappa_q \text{ent} \{ \delta^{-2} \|G_q^{(k)}(\lambda)\|_2^2 \} \leq \\ &\sum_{q=1}^{\infty} \text{ent} \{ c'_9 \delta^{-2} q^{-(\alpha-1-\varepsilon)/d} \lambda^{-(1-\varepsilon)} \} \leq c_9 \lambda^{-d(1-\varepsilon)/(\alpha-1-\varepsilon)} \end{aligned}$$

where the constant $c_9 = c_9(\delta, \varepsilon)$ is independent of λ , and ε can be chosen as in (4.25). Since $\alpha > 2$, the estimate $d(1 - \varepsilon)/(\alpha - 1 - \varepsilon) < d/(\alpha - 1)$ is valid, and it allows us to conclude that (4.32) holds.

Finally, the estimate (4.33) follows directly from (4.37) - (4.38) with $k = 2$, and (4.30).

Combining (4.21) - (4.24) with (4.31) - (4.33), we get the following proposition.

Proposition 4.2 *Under the hypotheses of Theorem 4.5 we have*

$$\mathbf{N}_1^-(\lambda) \leq n(1 - \delta; \mathcal{M}_1^{(k)}(\lambda)) + o(\nu_{3,k}(\lambda)), \lambda \downarrow 0, \forall \delta \in (0, 1), k = 1, 2, \quad (4.39)$$

$$\mathbf{N}_1^-(\lambda) \geq n(1 + \delta; \mathcal{M}_1^{(k)}(\lambda)) + o(\nu_{3,k}(\lambda)), \lambda \downarrow 0, \forall \delta > 0, k = 1, 2. \quad (4.40)$$

4.2.5. Set $\tilde{\mathbf{M}}^{(k)} = r_1^{(k)}(\lambda)^{-1} \mathcal{M}_1^{(k)}(\lambda)$, $k = 1, 2$. Note that the operator $\tilde{\mathbf{M}}^{(k)}$ is independent of λ .

Further, define the operator $\mathbf{M} \equiv \mathbf{M}^{(k)}$, $k = 1, 2$, as a Ψ DO with antiwick symbol $\mathbf{w}(y, \eta) := - \int_{\mathbf{R}^2} V_{\mathbf{b}}(-\eta, y, z) dz$. Obviously, we have $\mathbf{w} \in \mathcal{D}_{\alpha-k, \infty}^{++}$, $k = 1, 2$. Therefore, the operator inequalities

$$\mathbf{M}^{(k)} \leq c_{11}^+ h_1^{-(\alpha-k)/2}, k = 1, 2, \quad (4.41)$$

$$\mathbf{M}^{(k)} \geq c_{11}^- h_1^{-(\alpha-k)/2}, k = 1, 2, \quad (4.42)$$

hold with $0 < c_{11}^- < c_{11}^+ < \infty$ (see the proof of Lemma 4.2.) In particular, (4.42) implies the invertibility of \mathbf{M} , and, therefore, of $\mathbf{M}^{1/2}$.

Lemma 4.7 *The nonzero eigenvalues of the operators $\tilde{\mathbf{M}}^{(k)}$ and $\mathbf{M}^{(k)}$, $k = 1, 2$, coincide.*

Proof. Denote by \mathcal{L}_1 the subspace of $L^2(\mathbf{R}^{d+k})$ consisting of functions in the form $w(y, z) = (h_4^{1/2}(z)u)(y)$ where $u \in L^2(\mathbf{R}^d)$ (see Lemma 4.5 b.) Set $\mathcal{L}_2 = L^2(\mathbf{R}^{d+k}) \ominus \mathcal{L}_1$. Note that $\mathcal{L}_2 \subset \text{Ker } \tilde{\mathbf{M}}$. Write the arbitrary function $w \in L^2(\mathbf{R}^{d+k})$ as

$$w(y, z) = (h_4^{1/2}(z)u)(y) + v(y, z)$$

where $u \in L^2(\mathbf{R}^d)$ and $v \in \mathcal{L}_2$. Then we have

$$\|w\|_{L^2(\mathbf{R}^{d+k})}^2 = \|\mathbf{M}^{1/2}u\|_{L^2(\mathbf{R}^d)}^2 + \|v\|_{L^2(\mathbf{R}^{d+k})}^2$$

and

$$(\tilde{\mathbf{M}}w, w)_{L^2(\mathbf{R}^{d+k})} = \|\mathbf{M}u\|_{L^2(\mathbf{R}^d)}^2.$$

Since the operator $\mathbf{M}^{1/2}$ is invertible, we find that $\tilde{\mathbf{M}}$ is unitarily equivalent to $\mathbf{M} \oplus \mathbf{K}$ where \mathbf{K} is the identical-zero operator in \mathcal{L}_2 .

Corollary 4.1 *For each $\mu > 0$ we have*

$$n(\mu; \mathcal{M}_1^{(k)}(\lambda)) = n(r_1^{(k)}(\lambda)^{-1}\mu; \mathbf{M}^{(k)}), \lambda > 0, k = 1, 2. \quad (4.43)$$

Applying the general results on the eigenvalue asymptotics for Ψ DO of negative order in [Dau.Rob], we obtain the asymptotic estimates

$$\lim_{\delta \downarrow 0} \limsup_{\lambda \downarrow 0} \pm \nu_{3,k}(\lambda)^{-1} n((1 \mp \delta)r_1^{(k)}(\lambda)^{-1}\mu; \mathbf{M}^{(k)}) = \pm 1, \lambda > 0, k = 1, 2. \quad (4.44)$$

Putting together (4.39), (4.40), (4.43) and (4.44), we come to (4.14) or, respectively, to (4.15).

4.2.6. In this subsection we complete the proof of Theorem 4.6. By analogy with (4.34) - (4.35), we find that Lemma 4.2 b) entails the operator inequality $3\mathcal{H}_{2,2} \leq 2c_{12}\tilde{\mathcal{H}}^{-1}$ with some positive constant c_{12} . Hence, we obtain

$$N(-\lambda; \mathcal{H}_2(1/2)) \leq N(0; \mathcal{H}_{2,1} - c_{12}\tilde{\mathcal{H}}^{-1}). \quad (4.45)$$

On $W_2^2(\mathbf{R}_z^k)$ define the operators $\mathbf{h}_q = -\Delta - c_{12}\chi_q^{-\alpha/2}$, $q \geq 1$. Then we have

$$N(0; \mathcal{H}_{2,1} - c_{12}\tilde{\mathcal{H}}^{-1}) = \sum_{q=1}^{\infty} \kappa_q N(0; \mathbf{h}_q). \quad (4.46)$$

Since $k \geq 3$, we can use the Rozenbljum estimate of the negative-eigenvalues total multiplicity for the operators \mathbf{h}_q (see Proposition 2.3 with $A \equiv 0$.) Namely, we get the estimate

$$N(0; \mathbf{h}_q) \leq c'_{13} \int_{\mathbf{R}^k} \chi_q(z)^{-\alpha k/4} dz = c_{13} \Lambda_q^{-k(\alpha-2)/4} \quad (4.47)$$

where c'_{13} and c_{13} are independent of q . Since $N(0; \mathbf{h}_q)$ is integer-valued, we can substitute the estimating quantity in (4.47) for its integer part. Finally, since $\alpha > 2$ and $\Lambda_q \rightarrow \infty$ as $q \rightarrow \infty$, we find that (4.21) and (4.45) - (4.47) yield (4.16).

4.2.7. In this paragraph we consider potentials satisfying the estimates $-V(X) \asymp |X|^{-2}$ as $|X| \rightarrow \infty$. This is the border-line case between the potentials treated in Theorem 4.4 and the ones considered in Theorems 4.5 - 4.6. However, in this case we deal with a class of potentials V which is essentially narrower than \mathcal{D}_{-2}^+ . Namely, we assume that V obeys the asymptotics

$$- \lim_{|X| \rightarrow \infty} |X|^2 V(X) = g, \quad (4.48)$$

g being a positive constant.

Introduce the auxiliary operator

$$\mathbf{h}^{as} \equiv \mathbf{h}^{as}(g) := - \sum_{l=1}^k \frac{\partial^2}{\partial z_l^2} - \frac{g}{1 + |z|^2}, \quad z \in \mathbf{R}^k \equiv \text{Ker } B.$$

The negative spectrum of \mathbf{h}^{as} is purely discrete. Denote by $\{-\gamma_l\}_{l \geq 1} \equiv \{-\gamma_l(g)\}_{l \geq 1}$ the nondecreasing sequence of the negative eigenvalues of $\mathbf{h}^{as}(g)$.

The following lemma summarizes some properties of $\{\gamma_l\}$ which are implied by the results in [Bir 1] and [Ki.Sim].

Lemma 4.8 (i) *If $k = 1, 2$, and $g > 0$, then $\mathbf{h}^{as}(g)$ has at least one negative eigenvalue, while in the case $k \geq 3$ the operator $\mathbf{h}^{as}(g)$ is nonnegative if and only if $g \leq (k - 2)^2/4$.*

(ii) *The estimates*

$$\gamma_l \leq c_{k,1} \exp(-c_{k,2}l), \quad k \geq 1, \quad (4.49)$$

are valid for some constants $c_{k,j}$, $j = 1, 2$, $k \geq 1$, which are independent of l .

(iii) *For each $p > 0$ the sum $\sum_{l \geq 1} \gamma_l(g)^p$ is finite and continuous with respect to $g \geq 0$.*

Theorem 4.7 [Rai 3, Theorem 2.4] *Suppose that the potential $V \in L^\infty(\mathbf{R}^m)$ satisfies (4.48).*

(i) *For each $k \geq 1$ we have*

$$\lim_{\lambda \downarrow 0} \lambda^d \mathbf{N}_1^-(\lambda) = (2^d d!)^{-1} b_1 \dots b_d \sum_{l \geq 1} \gamma_l(g)^d. \quad (4.50)$$

(ii) *Moreover, if $k \geq 3$ and $g < (k - 2)^2/4$ (and, hence, $\mathbf{h}^{as}(g) \geq 0$) we have*

$$\mathbf{N}_1^-(\lambda) = O(1), \quad \lambda \downarrow 0. \quad (4.51)$$

Sketch of the proof. For $\mu > \sum_{j=1}^d b_j^{-2}$ introduce the ‘‘asymptotic’’ potential

$$V^{as} \equiv V^{as}(x, y, z) = -g \int_0^\infty \exp \left\{ -(\mu + |z|^2)t + \sum_{j=1}^d [b_j^{-1} - (x_j^2 + y_j^2)/b_j^{-1} (\coth b_j^{-1} t - 1)] \right\} dt.$$

Applying the Laplace method for the calculation of the asymptotics of integrals depending on a large parameter, we easily verify the relation

$$V(X) = V^{as}(X) + O(|X|^{-2-\delta}) \text{ as } |X| \rightarrow \infty, \delta > 0. \quad (4.52)$$

Employing Theorems 4.5 - 4.6 together with (4.52), and using a standard variational argument (see [Rai 3, Proposition 3.1]), we easily find that the estimates

$$\pm \mathbf{N}_1^-(\lambda) \leq \pm N(\Lambda_1 - \lambda; H(A, (1 \pm \varepsilon)V^{as})) + o(\lambda^{-d}), \lambda \downarrow 0, k = 1, 2, \quad (4.53)$$

$$\pm \mathbf{N}_1^-(\lambda) \leq \pm N(\Lambda_1 - \lambda; H(A, (1 \pm \varepsilon)V^{as})) + o(1), \lambda \downarrow 0, k \geq 3, \quad (4.54)$$

hold for each $\varepsilon \in (0, 1)$.

Next, by analogy with with the operator $\mathcal{H}_{1,3}$ introduce the operator $\mathcal{H}_{1,3}^{as}$ substituting V for V^{as} . For $\varepsilon \in (-1, 1)$ set

$$\mathcal{H}_1^{as}(\varepsilon) := \mathcal{H}_{1,1} + \mathcal{H}_{1,2} + (1 + \varepsilon)\mathcal{H}_{1,3}^{as}$$

(cf. (2.38).) Then the operators $H(A, (1 + \varepsilon)V^{as})$ and $\mathcal{H}_1^{as}(\varepsilon)$ are unitary equivalent for each $\varepsilon \in (-1, 1)$ (cf. Lemma 2.4.) Hence, we have

$$N(\Lambda_1 - \lambda; H(A, (1 + \varepsilon)V^{as})) = N(\Lambda_1 - \lambda; \mathcal{H}_1^{as}(\varepsilon)), \forall \varepsilon \in (-1, 1). \quad (4.55)$$

Further, by analogy with the operator $\mathcal{H}_2(\varepsilon)$ introduce the operator $\mathcal{H}_2^{as}(\varepsilon)$ replacing again V by V^{as} . Similarly to (4.21) - (4.22) we get the estimates

$$\pm N(\Lambda_1 - \lambda; \mathcal{H}_1^{as}(\pm\varepsilon)) \leq$$

$$\pm N(-\lambda; \mathcal{H}_2^{as}(\pm\varepsilon')) + O(1), \lambda \downarrow 0, \forall \varepsilon \in (0, 1), \forall \varepsilon' \in (\varepsilon, 1). \quad (4.56)$$

Note that the operator $\mathcal{H}_{2,2}$ coincides with the operator $\int_{\mathbf{R}^k}^{\oplus} (\tilde{\mathbf{h}}_1(\mathbf{b}) + |z|^2 + \mu)^{-1} dz$ where the selfadjoint operator

$$\tilde{\mathbf{h}}_1(\mathbf{b}) : \sum_{j=1}^d b_j^{-1} (-\partial_j^2 + x_j^2)$$

is defined originally on $C_0^\infty(\mathbf{R}^d)$, and then is closed in $L^2(\mathbf{R}^d)$. Denote by $\{\tilde{\Lambda}_q(\mathbf{b})\}_{q \geq 1}$ the *nondecreasing* sequence of the eigenvalues of $\tilde{\mathbf{h}}(\mathbf{b})$ *counted with the multiplicities* (cf (2.28).)

The standard Weylian asymptotic relation

$$N(\lambda; \tilde{\mathbf{h}}_1(\mathbf{b})) = (2\pi)^{-d} \left\{ (x, \xi) \in T^*\mathbf{R}^d : \sum_{j=1}^d b_j^{-1} (x_j^2 + \xi_j^2) < \lambda \right\} (1 + o(1)) =$$

$$\frac{b_1 \cdots b_d}{2^d d!} \lambda^d (1 + o(1)), \lambda \rightarrow \infty,$$

(see e.g. [Shu 1, Theorem 30.1]), implies

$$\tilde{\Lambda}_q = \left(2^d d! / b_1 \dots b_d\right)^{1/d} q^{1/d} (1 + o(1)), q \rightarrow \infty, \quad (4.57)$$

(see [Shu 1, Proposition 13.1].)

On $W_2^2(\mathbf{R}^k)$ define the operators

$$\mathbf{h}_q^{as}(\varepsilon) := - \sum_{l=1}^k \frac{\partial^2}{\partial z_l^2} - \frac{(1 + \varepsilon)g}{\mu + \tilde{\Lambda}_q + |z|^2}, \varepsilon \in (-1, 1), q \geq 1.$$

Obviously we have

$$\begin{aligned} N(-\lambda; \mathcal{H}_2(\varepsilon)) &= \sum_{q \geq 1} N(-\lambda; \mathbf{h}_q^{as}(\varepsilon)) = \sum_{q \geq 1} N(-(\mu + \tilde{\Lambda}_q)\lambda; \mathbf{h}^{as}((1 + \varepsilon)g)) = \\ &\# \left\{ q \geq 1, l \geq 1 : \lambda^{-1} \gamma_l((1 + \varepsilon)g) > \mu + \tilde{\Lambda}_q \right\}, \forall \varepsilon \in (-1, 1). \end{aligned} \quad (4.58)$$

Using (4.49), we find that for each positive inetger \mathcal{Q} we have

$$\# \left\{ q = 1, \dots, \mathcal{Q}, l \geq 1 : \lambda^{-1} \gamma_l((1 + \varepsilon)g) > \mu + \tilde{\Lambda}_q \right\} = O(|\log \lambda|) = o(\lambda^{-d}), \lambda \downarrow 0.$$

Hence we can replace in the rightmost quantity $\mu + \tilde{\Lambda}_q$ in (4.58) by its asymptotic expression as $q \rightarrow \infty$ given in (4.57). More precisely, we obtain the estimates

$$\begin{aligned} &\pm \# \left\{ q \geq 1, l \geq 1 : \lambda^{-1} \gamma_l((1 + \varepsilon)g) > \mu + \tilde{\Lambda}_q \right\} \leq \\ &\pm \# \left\{ q \geq 1, l \geq 1 : \lambda^{-1} \gamma_l((1 + \varepsilon)g) > (1 \mp \delta)(2^d d! / b_1 \dots b_d)^{1/d} q^{1/d} \right\} + o(\lambda^{-d}) \leq \\ &\pm (1 \mp \delta)^{-d} \lambda^{-d} (2^d d!)^{-1} b_1 \dots b_d \sum_{l \geq 1} \gamma_l((1 + \varepsilon)g)^d + o(\lambda^{-d}), \lambda \downarrow 0, \forall \delta \in (0, 1), \forall \varepsilon \in (-1, 1). \end{aligned} \quad (4.59)$$

The combination of (4.53) or (4.54) with (4.55), (4.56), (4.58) and (4.59) yields

$$\begin{aligned} &\limsup_{\lambda \downarrow 0} \pm \lambda^d \mathbf{N}_1^-(\lambda) \leq \\ &\pm (1 \mp \delta)^{-d} (2^d d!)^{-1} b_1 \dots b_d \sum_{l \geq 1} \gamma_l((1 \pm \varepsilon)g)^d, \lambda \downarrow 0, \forall \delta \in (0, 1), \forall \varepsilon \in (0, 1). \end{aligned}$$

Letting $\delta \downarrow 0$ and $\varepsilon \downarrow 0$, and taking into account Lemma 4.8 (iii), we come to (4.50).

Now, assume that the hypotheses of Theorem 4.7 (ii) hold. Note that $g < (k - 2)^2 / 4$ entails $(1 + \varepsilon)g < (k - 2)^2 / 4$ for $\varepsilon > 0$ small enough. Hence, by Lemma 4.8 (i), we have $\mathbf{h}^{as}((1 + \varepsilon)g) \geq 0$. Putting together (4.54), (4.55), (4.56) and (4.58), we come to (4.51).

4.3 Eigenvalue asymptotics for Hamiltonians $H(A, V)$ with constant magnetic field B and electric potential which decays at infinity. The case $\text{Ker } B = \{0\}$

4.3.1. In this subsection we consider the case $k = 0$.

Fix the Landau level $\Lambda_q, q \geq 1$, (resp., $q > 1$), choose any $\mu \in (\Lambda_q, \Lambda_{q+1})$ (resp. $\mu \in (\Lambda_{q-1}, \Lambda_q)$), and for $\lambda > 0$ small enough set $\mathbf{N}_q^+(\lambda) = \mathcal{N}(\Lambda_q + \lambda, \mu | H(A, V))$, $q \geq 1$, (resp., $\mathbf{N}_q^-(\lambda) = \mathcal{N}(\mu, \Lambda_q - \lambda | H(A, V))$, $q > 1$); the function $\mathbf{N}_1^-(\lambda)$ is defined as in (4.12).

Theorem 4.8 [Rai 2, Theorem 2.6] *Let $k = 0$. Suppose that (2.26) holds. Assume*

$$V \in \mathcal{D}_{-\alpha, \infty}, \alpha > 0. \quad (4.60)$$

For $\lambda > 0$ put

$$\nu_4^\pm(\lambda) = (2\pi)^{-d} b_1 \dots b_d \text{vol} \left\{ (x, y) \in \mathbf{R}^{2d} \equiv \mathbf{R}^m : \pm V(x, y) > \lambda \right\}.$$

Suppose that $\nu_4^+(\lambda)$ and $\nu_4^-(\lambda)$ satisfy the condition \mathcal{T} . Moreover, assume that the estimate

$$\nu_4^\pm(\lambda) \geq c_0 \lambda^{-2d/\alpha}, \lambda \downarrow 0. \quad (4.61)$$

holds for some constant $c_0 > 0$. Then the asymptotic relations

$$\mathbf{N}_q^+(\lambda) = \kappa_q \nu_4^+(\lambda) (1 + o(1)), \lambda \downarrow 0, \quad (4.62)$$

$$\mathbf{N}_q^-(\lambda) = \kappa_q \nu_4^-(\lambda) (1 + o(1)), \lambda \downarrow 0, \quad (4.63)$$

hold for each Landau level $\Lambda_q, q \geq 1$.

The proof of the theorem is contained in the next three paragraphs.

Remark. The inclusion (4.60) and the inequality (4.61) entail

$$\nu_4^\pm(\lambda) \asymp \lambda^{-2d/\alpha}, \lambda \downarrow 0.$$

4.3.2. We shall verify only the relation (4.62); the validity of (4.63) can be demonstrated in a completely analogous manner.

Until the end of the section we keep the Landau level $\Lambda_q, q \geq 1$, fixed. For brevity's sake set

$$\Lambda \equiv \Lambda_q, \kappa \equiv \kappa_q, f_j \equiv f_{q,j}, j = 1, \dots, \kappa,$$

(see (2.28) - (2.30).) Define the orthogonal projection $P : L^2(\mathbf{R}^m) \rightarrow L^2(\mathbf{R}^m)$ by

$$(Pu)(x, y) = \sum_{j=1}^{\kappa} f_j(x) \int_{\mathbf{R}^d} f_j(x') u(x', y) dx', u \in L^2(\mathbf{R}^m).$$

Put $Q = \text{Id} - P$.

Lemma 4.9 [Rai 2, Lemma 5.1] *The operator $\mathcal{L} := P\mathcal{H}_{1,3}$ is compact.*

Proof. Obviously we have $\mathcal{L} = \Lambda P\mathcal{H}_{1,1}^{-1}\mathcal{H}_{1,3}$. Hence the assertion the lemma follows directly from the fact that the multiplier by V is Δ -compact, combined with Lemma 2.2.

4.3.3. Let

$$F \in \mathcal{D}_{-\beta, \infty}, \beta > 0. \quad (4.64)$$

For $\lambda > 0$ set $\varphi^{\pm}(\lambda) := (2\pi)^{-d}\Phi_{\pm F}(\lambda)$.

Define the operator $\mathcal{F} = \mathcal{F}(F)$ as a Ψ DO with matrix-valued Weyl symbol $\mathbf{F}(y, \eta) = \{F_{jl}(y, \eta)\}_{j,l=1}^{\kappa}$ where

$$F_{jl}(y, \eta) =$$

$$\frac{1}{(2\pi)^d} \int_{\mathbf{R}^{3d}} f_j(x) f_l(x') \exp\{i\xi(x - x')\} F\left(\frac{1}{2}(x + x') - \eta, y - \xi\right) dx dx' d\xi, j, l = 1, \dots, \kappa.$$

Obviously, the operator \mathcal{F} is selfadjoint and compact on $L^2(\mathbf{R}^d)^{\kappa}$.

Lemma 4.10 [Rai 2, Lemma 5.2] *Let (4.64) hold. Assume that the function $\varphi^+(\lambda)$ satisfies the condition \mathcal{T} and, moreover,*

$$\liminf_{\lambda \downarrow 0} \lambda^{2d/\beta} \varphi^+(\lambda) > 0.$$

Suppose that φ^- satisfies the condition \mathcal{T} . Then we have

$$n(\lambda; \mathcal{F}) = \kappa \varphi^+(\lambda) (1 + o(1)), \lambda \downarrow 0. \quad (4.65)$$

Proof. Straightforward calculations based on the first-order Taylor expansion of F yield

$$F_{jl}(y, \eta) = F_{jl}^{(1)}(y, \eta) + F_{jl}^{(2)}(y, \eta)$$

where

$$F_{jl}^{(1)}(y, \eta) = \delta_{jl}F(-\eta, y),$$

$$F_{jl}^{(2)}(y, \eta) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^{3d}} \mathbf{P}_{jl}(x, \xi) \exp\{-(|x|^2 + |\xi|^2)\} \\ \int_0^1 \{x \cdot \nabla_x F(\tau x - \eta, y - \tau\xi) - \xi \cdot \nabla_y F(\tau x - \eta, y - \tau\xi)\} d\tau dx d\xi,$$

and $\mathbf{P}_{jl}(x, \xi)$ are polynomials, $j, l = 1, \dots, \kappa$. Put $\mathbf{F}^{(i)} = \{F_{jl}^{(i)}\}_{j,l=1}^{\kappa}$ and define \mathcal{F}_i as the Ψ DOs with matrix-valued Weyl symbols $\mathbf{F}^{(i)}$, $i = 1, 2$. Applying the Weyl - Ky Fan inequalities, we obtain the estimates

$$\pm n(\lambda; \mathcal{F}) \leq \pm n((1 \mp \tau)\lambda; \mathcal{F}_1) + n(\tau\lambda; \pm \mathcal{F}_2) \quad (4.66)$$

which are valid for each $\tau \in (0, 1)$. Obviously, we have $F_{jl}^{(2)} \in \mathcal{D}_{-1-\beta, \infty}$, $j, l = 1, \dots, \kappa$. Hence, the operator inequalities

$$\pm \mathcal{F}_2 \leq c_3 \left(\sum_{j=1}^{\kappa} \oplus h_1 \right)^{-(1+\beta)/2}$$

are valid for some $0 < c_3 < \infty$ (see e.g. [Shu 1, Theorem 25.2].) Consequently, we have

$$n(\tau\lambda; \pm \mathcal{F}_2) = O(\lambda^{-2d/(\beta+1)}) = o(\varphi^+(\lambda)), \lambda \downarrow 0, \forall \tau \in (0, 1). \quad (4.67)$$

Further, the general results in [Dau.Rob] entail

$$n((1 \mp \tau)\lambda; \mathcal{F}_1) = \kappa\varphi^+((1 \mp \tau)\lambda)(1 + o(1)), \lambda \downarrow 0. \quad (4.68)$$

Finally, the obvious relation

$$\lim_{\tau \downarrow 0} \limsup_{\lambda \downarrow 0} \pm \varphi^+((1 \mp \tau)\lambda) / \varphi^+(\lambda) = \pm 1 \quad (4.69)$$

follows from the fact that the function $\varphi^+(\lambda)$ satisfies the condition \mathcal{T} , and is equivalent to the fact that this function satisfies the condition \mathcal{T}_0 .

Putting together (4.66)–(4.69), we come to (4.65).

4.3.4. In this subsection we conclude the demonstration of (4.62). To do this, we need the following abstract lemma.

Lemma 4.11 [Rai 2, Lemma 5.4] *Let $T_i, i = 0, 1$, be selfadjoint operators in Hilbert space. Assume that $T_1 \in \mathbf{S}_\infty$. Set $T = T_0 + T_1$. Then the estimates*

$$\mathcal{N}(\mu_1, \mu_2 | T_0) \leq \mathcal{N}(\mu_1 - \tau_1, \mu_2 + \tau_2 | T) + n(\tau_1; -T_1) + n(\tau_2; T_1)$$

hold for each interval $(\mu_1, \mu_2) \subset \mathbf{R}$ and every $\tau_i \geq 0$, $i = 1, 2$.

Set $\mathbf{L} = i(\mathcal{L} - \mathcal{L}^*)$. By Lemma 4.9, the operator $\mathbf{L} = \mathbf{L}^*$ is compact in $L^2(\mathbf{R}^m)$. Denote by \mathbf{L}_+ and \mathbf{L}_- respectively the positive and the negative part of \mathbf{L} , so that $\mathbf{L} = \mathbf{L}_+ - \mathbf{L}_-$ and $|\mathbf{L}| = \mathbf{L}_+ + \mathbf{L}_-$.

Further, for $\varepsilon > 0$ denote by $\mathcal{Y}^\pm(\varepsilon)$ the operator generated in $L^2(\mathbf{R}^m)$ by the quadratic form

$$\int_{\mathbf{R}^{2d}} \left\{ |i\varepsilon^{-1}\mathbf{L}_+^{1/2}Qu \pm \varepsilon\mathbf{L}_+^{1/2}Pu|^2 + |i\varepsilon^{-1}\mathbf{L}_-^{1/2}Qu \mp \varepsilon\mathbf{L}_-^{1/2}Pu|^2 \right\} dx dy.$$

The operators $\mathcal{Y}^\pm(\varepsilon)$, $\varepsilon > 0$, are compact and nonnegative. Besides, we have

$$\mathcal{H}_1 = P\mathcal{H}_1P + Q(\mathcal{H}_1 \pm \varepsilon^{-2}|\mathbf{L}|)Q \pm \varepsilon^2P|\mathbf{L}|P \mp \mathcal{Y}^\pm(\varepsilon), \varepsilon > 0.$$

Applying Lemma 4.11, we find that the estimates

$$\begin{aligned} \pm \mathcal{N}(\Lambda + \lambda, \mu | \mathcal{H}_1) &\leq \pm \mathcal{N}(\Lambda + (1 \mp \varepsilon)\lambda, \mu \pm \varepsilon | P\mathcal{H}_1P + Q(\mathcal{H}_1 \pm \varepsilon^{-2}|\mathbf{L}|)Q) + \\ &\quad n(\lambda\varepsilon^{-1}; P|\mathbf{L}|P) + n(\varepsilon; \mathcal{Y}^\pm(\varepsilon)) \end{aligned} \quad (4.70)$$

hold for any $\mu \in (\Lambda_q, \Lambda_{q+1})$, and sufficiently small $\varepsilon > 0$.

Let \mathcal{Z}_1 (resp. $\mathcal{Z}^\pm \equiv \mathcal{Z}_2^\pm(\varepsilon)$) be the operator $P\mathcal{H}_1P$ (resp. $Q(\mathcal{H}_1 \pm \varepsilon^{-2}|\mathbf{L}|)Q$) with domain $PD(\mathcal{H}_1)$ (resp. $QD(\mathcal{H}_1)$). Then we have

$$\begin{aligned} \mathcal{N}(\Lambda + (1 \mp \varepsilon)\lambda, \mu \pm \varepsilon | P\mathcal{H}_1P + Q(\mathcal{H}_1 \pm \varepsilon^{-2}|\mathbf{L}|)Q) = \\ \mathcal{N}(\Lambda + (1 \mp \varepsilon)\lambda, \mu \pm \varepsilon | \mathcal{Z}_1) + \mathcal{N}(\Lambda + (1 \mp \varepsilon)\lambda, \mu \pm \varepsilon | \mathcal{Z}_2^\pm(\varepsilon)). \end{aligned} \quad (4.71)$$

Since the operator \mathcal{Z}_1 is unitarily equivalent to $\mathcal{F}(-V_{\mathbf{b}}) + \Lambda \text{Id}$, we have

$$\begin{aligned} \mathcal{N}(\Lambda + (1 \mp \varepsilon)\lambda, \mu \pm \varepsilon | \mathcal{Z}_1) &= n((1 \mp \varepsilon)\lambda; \mathcal{F}(-V_{\mathbf{b}})) - \\ &\quad \lim_{\tau \downarrow 0} n(\mu \pm \varepsilon - \Lambda - \tau; \mathcal{F}(-V_{\mathbf{b}})). \end{aligned} \quad (4.72)$$

Applying Lemma 4.10, we easily get

$$\lim_{\varepsilon \downarrow 0} \limsup_{\lambda \downarrow 0} \pm \nu_4^+(\lambda)^{-1} n((1 \mp \varepsilon)\lambda; \mathcal{F}(-V_{\mathbf{b}})) = \pm \kappa. \quad (4.73)$$

Moreover, since $\mu - \Lambda > 0$ and the operator $\mathcal{F}(-V_{\mathbf{b}})$ is compact, the second term at the right-hand of (4.72) (which is independent of λ) is finite for $\varepsilon > 0$ small enough. Further, it is easy to check that we have $\sigma_{ess}(\mathcal{Z}_2^\pm(\varepsilon)) \cap [\Lambda, \mu \pm \varepsilon] = \emptyset$ for $\mu \in (\Lambda_q, \Lambda_{q+1})$ and $\varepsilon > 0$ small enough, so that the second term at the right-hand side of (4.71) remains uniformly bounded as $\lambda \downarrow 0$.

Next, employing some elementary properties of the singular numbers of compact operators, we get

$$n(\lambda\varepsilon^{-1}; P|\mathbf{L}|P) \leq 2n(\lambda^2\varepsilon^{-2}/4; \mathcal{F}(V_{\mathbf{b}}^2)), \forall \varepsilon > 0. \quad (4.74)$$

Taking into account Lemma 4.10, we obtain the estimate

$$\lim_{\varepsilon \downarrow 0} \limsup_{\lambda \downarrow 0} \nu_4^+(\lambda)^{-1} n(\lambda^2\varepsilon^{-2}/4; \mathcal{F}(V_{\mathbf{b}}^2)) = 0. \quad (4.75)$$

Finally, since the operators $\mathcal{Y}^\pm(\varepsilon)$, $\varepsilon > 0$, are compact, the last term at the right-hand side of (4.70) (which is independent of λ) is finite for each $\varepsilon > 0$.

Now, (4.62) follows directly from (4.70) - (4.75).

4.4 Asymptotic estimates of $N(\lambda; H(A, V))$ for general potentials (A, V)

4.4.1. In this paragraph we investigate the behaviour of the quantity $N(\lambda; H(A, V))$ for large λ in the case where the resolvent of $H(A, V)$ is compact. More precisely, we assume that the hypotheses of Theorem 3.4 or of Theorem 3.5 hold and the relations (3.12) or, respectively, (3.16) are fulfilled.

Moreover, we suppose that the magnetic field $B(X)$ is of C^{r+2} -class and there exists a constant C_0 such that we have

$$\sum_{|\beta|=r+2} |D^\beta B(X)| \leq C_0 \phi(X)$$

with $\phi(X) = m(X)$ if the hypotheses of Theorem 3.4 hold, or with $\phi(X) = M(X)$ if the hypotheses of Theorem 3.5 hold.

Under the hypotheses of Theorem 3.4 set

$$\mathcal{W}(X) = \sum_{|\alpha| \leq r+1} |D^\alpha V(X)|^{2/(2+|\alpha|)} + \sum_{|\beta| \leq r} |D^\beta B(X)|^{2/(2+|\beta|)},$$

and under the hypotheses of Theorem 3.5 set

$$\mathcal{W}(X) = V(X) + \sum_{l=1}^{r+1} \left\{ \sum_{|\beta|=l-1} |D^\beta B(X)| + \sum_{j=1}^p \sum_{|\alpha|=l} |D^\alpha V_j(X)| \right\}^{2/(l+1)}.$$

Theorem 4.9 (cf. [Moh.Nou, Theorems 1.2 -1.3]) *Under the preceding hypotheses there exist numbers $\varepsilon \in (0, 1]$ and $C \geq 1$ such that we have*

$$C^{-1} \int_{\mathbf{R}^m} (\varepsilon\lambda - \mathcal{W}(X))_+^{m/2} dX \leq N(\lambda; H(A, V)) \leq C \int_{\mathbf{R}^m} (\varepsilon^{-1}\lambda - \mathcal{W}(X))_+^{m/2} dX$$

for sufficiently large λ .

Sketch of the proof. The proof is performed in two steps. First, we show that there exists a constant $C > 0$ for which the estimate

$$\|\mathcal{W}^{1/2}u\|^2 \leq C \left((H(A, V)u, u) + \|u\|^2 \right)$$

is valid for each $u \in C_0^\infty(\mathbf{R}^m)$.

Further, we make use of this estimate, apply an appropriate partition of unity and utilize a sophisticated version of the minimax principle in order to compare $N(\lambda; H(A, V))$ with $N(\varepsilon\lambda; H(0, C\mathcal{W}))$ or, respectively, with $N(\varepsilon^{-1}\lambda; H(0, C^{-1}\mathcal{W}))$; this is the method of Rayleigh-Ritz.

Theorem 4.9 allows us to estimate the growth-rate of $N(\lambda; H(A, V))$ for degenerate potentials $V(X)$ (i.e. potentials which do not tend to infinity in all directions as $|X| \rightarrow \infty$) more general than the ones treated in [Rob], [Sim 3], [Gur], [Fef] and [So]. In the case of magnetic bottles of the third kind ¹ with degenerate magnetic field we establish nonclassical growth-rate as in the following example.

Let $m = 3$ and

$$A_1(X) = X_1X_2^2(\varrho(X) + X_3^2/|X|), A_2(X) = -X_1^2X_2(\varrho(X) + X_3^2/|X|), A_3(X) = 0,$$

where $\varrho(X) = \int_0^{X_3} \tau^3(X_1^2 + X_2^2 + \tau^2)^{-1/2} d\tau$. Then $|B(X)|$ behaves at infinity as $|X_1X_2X_3|$ and, hence, we have

$$N(\lambda; H(A, 0)) \asymp \lambda^{5/2}(\log \lambda)^2, \lambda \rightarrow \infty.$$

Some of the ideas employed in the proof of Theorem 4.9 have been developed in [Lev.Moh.Nou] and [Moh.Lev.Nou] for the case where the operator $H(A, 0)$ is replaced by the Laplacian acting in $L^2(M)$, M being a nilpotent Lie group.

4.4.2. In this paragraph we discuss the case where the resolvent of the operator $H(A, V)$ is not compact. In other words we suppose

$$\mathcal{E} \equiv \mathcal{E}(A, V) := \inf \sigma_{ess}(H(A, V)) < \infty.$$

The first problem is to know whether the quantity $N(\mathcal{E}; H(A, V))$ is finite or not. When the magnetic field $B(X)$ can be considered as a perturbation of a constant field, we can formulate the following theorem.

Theorem 4.10 [Moh 1] *Let $-V \in \mathcal{D}_{-\sigma, 1}^+$, $\sigma > 0$, and $A \in C^3(\mathbf{R}^m; \mathbf{R}^m)$. Suppose that there exists a constant magnetic field B_0 such that the estimates*

¹Here we use the physical terminology introduced in [Av.Her.Sim 1, p.849]. The Hamiltonian $H(A, 0)$ is called a magnetic bottle of the first (resp., second) [resp., third] kind if and only if it has at least one eigenvalue (resp., $\sigma(H(A, 0))$ is purely point) [resp., $\sigma(H(A, 0))$ is purely discrete].

$$\sum_{|\alpha|=1}^3 |D^\alpha(A(X) - \frac{1}{2}B_0 \cdot X)| \langle X \rangle^{\sigma-\eta+|\alpha|} \leq C_0$$

hold for some constants $\eta > 1$ and $C_0 > 1$.

(i) If $\sigma < 2$, or if the rank of B_0 equals m , then the operator $H(A, V)$ has infinitely many eigenvalues smaller than \mathcal{E} .

(ii) If $\sigma < 2$, then the asymptotic estimates

$$C_1^{-1} \int_{\mathbf{R}^m} \left(\mathcal{E} - \varepsilon^{-1}\lambda - \text{Tr}^+ B(X) - V(X) \right)_+^{m/2} dX \leq N(\mathcal{E} - \lambda; H(A, V)) \leq$$

$$C_1 \int_{\mathbf{R}^m} \left(\mathcal{E} - \varepsilon\lambda - \text{Tr}^+ B(X) - V(X) \right)_+^{m/2} dX$$

hold as $\lambda \downarrow 0$ with some constant $\varepsilon \in (0, 1]$ and $C_1 \geq 1$.

For the proof of the theorem we refer to the original paper [Moh 1].

Remark. The hypotheses of Theorem 4.10 imply $\mathcal{E} = \text{Tr}^+ B_0 \equiv \Lambda_1$ (cf. Corollary 3.3.)

Remark. If $\eta < \sigma/2$ we can arrange that $|A(X)|^2 = o(|V(X)|)$ as $|X| \rightarrow \infty$. In this case Theorem 4.10 can be obtained without any difficulty if one employs the methods developed in Subsect. 4.2. Analogously, we can establish in a similar way that $H(A, V)$ has infinitely many eigenvalues on $(-\infty, \mathcal{E})$, in the cases where the rank of B_0 equals $m - 1$ or $m - 2$, even if $\sigma \geq 2$.

Example. Let $m = 2$ and

$$A_1(X) = X_2, A_2(X) = \langle X \rangle^{-2(\sigma-\varepsilon)}, V(X) = -\langle X \rangle^{-2\sigma}$$

with $\varepsilon \in (0, 1/2)$ and $\sigma > 0$. Then the operator $H(A, V)$ has infinitely many eigenvalues lying on the interval $(-\infty, 1)$.

5 Spectral asymptotics for the operator $H_{h,\mu,g}$ corresponding to the characteristic behaviour of the parameters h , μ and g

In this section we investigate the asymptotic behaviour of the spectrum of the operator $H_{h,\mu,g}$ as $g \rightarrow \infty$ (strong-electric-field approximation - see Subsect. 5.1), or as $g \downarrow 0$

(weak-electric-field approximation - see Subsect. 5.2), or as $\mu \rightarrow \infty$ (strong-magnetic-field approximation - see Subsect. 5.3), or as $\mu \downarrow 0$ (weak-magnetic-field approximation - see Subsect. 5.4), or as $h \downarrow 0$ (semiclassical approximation - see Subsect. 5.5). In order to use readily the necessary results of Sections 2 – 4 where we assumed $h = \mu = g = 1$, note the obvious identity

$$H_{h,\mu,g}(A, V) = h^2 H_{1,1,1}(h^{-1}\mu A, h^{-2}gV).$$

5.1 Strong-electric-field approximation

5.1.1. In this paragraph we consider quite arbitrary A and electric potentials V which decay rapidly at infinity in a certain sense. It will be shown that in this case Conjecture 1.2 concerning the asymptotics as $g \rightarrow \infty$, is correct.

Theorem 5.1 [Rai 4, Theorem 1.1] *Let $m \geq 3$. Assume $A \in L_{loc}^m(\mathbf{R}^m)^m$ and $V \in L^{m/2}(\mathbf{R}^m)$. Fix $\lambda < \mathcal{E} \equiv \inf \sigma_{ess}(H(A, 0))$. Then we have*

$$\lim_{g \rightarrow \infty} g^{-m/2} N(\lambda; H_{1,1,g}) = \omega_m \int_{\mathbf{R}^m} V(X)_-^{m/2} dX / (2\pi)^m. \quad (5.1)$$

Remark. If $m = 2$, analogues of Theorem 5.1 are valid under more complicated assumptions. For example, the asymptotics (5.1) hold if $A \in L_{loc}^p(\mathbf{R}^2)^2$, $p \geq 2$, $V \in L^q(\mathbf{R}^2)$, $q > 1$, and $\text{supp } V$ is compact.

Proof of Theorem 5.1. The asymptotics (5.1) will follow from the estimates

$$\limsup_{g \rightarrow \infty} g^{-m/2} N(\lambda; H_{1,1,g}) \leq \omega_m \int_{\mathbf{R}^m} V(X)_-^{m/2} dX / (2\pi)^m, \quad (5.2)$$

$$\liminf_{g \rightarrow \infty} g^{m/2} N(\lambda; H_{1,1,g}) \geq \omega_m \int_{\mathbf{R}^m} V(X)_-^{m/2} dX / (2\pi)^m. \quad (5.3)$$

First, we prove (5.2). Fix $\varepsilon > 0$, and write V as the sum $V = V_1 + V_2$ where $V_1 \in C_0^\infty(\mathbf{R}^m)$, and V_2 satisfies the estimate

$$\int_{\mathbf{R}^m} |V_2(X)|^{m/2} dX \leq \varepsilon. \quad (5.4)$$

The minimax principle yields

$$N(\lambda; H_{1,1,g}) \leq N(0; H(A, (1-\tau)^{-1}(gV_1 - \lambda))) + N(0; H(A, \tau^{-1}gV_2)), \forall \tau \in (0, 1). \quad (5.5)$$

By Proposition 2.3 and (5.4), we have

$$N(0; \tau^{-1}gV_2) \leq c_m \varepsilon \tau^{-m/2} g^{m/2} \quad (5.6)$$

where c_m is the same constant as in (2.15). Let $\mathbf{B}_l, l = 0, 1$, be open balls in \mathbf{R}^m such that $\text{supp } V_1 \subset \mathbf{B}_0$, and $\bar{\mathbf{B}}_0 \subset \mathbf{B}_1$. Set $\mathbf{B}_2 = \mathbf{R}^m \setminus \bar{\mathbf{B}}_0$. Let $\{\phi_l\}_{l=1,2}$ be a partition of unity satisfying:

- (i) $\phi_l \in C^\infty(\mathbf{R}^m), l = 1, 2$;
- (ii) $\text{supp } \phi_l \subset \mathbf{B}_l, l = 1, 2$;
- (iii) $\sum_{l=1,2} \phi_l^2(X) = 1, \forall X \in \mathbf{R}^m$.

Then we have

$$(H(A, (1 - \tau)^{-1}(gV_1 - \lambda))u, u) = \sum_{l=1,2} \left\{ (H(A, (1 - \tau)^{-1}(gV_1 - \lambda))\phi_l u, \phi_l u) - (V_3 \phi_l u, \phi_l u) \right\}, \forall u \in C_0^\infty(\mathbf{R}^m),$$

where $V_3 := \sum_{l=1,2} |\nabla \phi_l|^2$. Hence the minimax principle entails

$$\begin{aligned} N(0; H(A, (1 - \tau)^{-1}(gV_1 - \lambda))) &\leq N(0; H_{B_1}^D(A, (1 - \tau)^{-1}(gV_1 - \lambda) - V_3)) + \\ &N(0; H_{B_2}^D(A, -(1 - \tau)^{-1}\lambda - V_3)), \forall \tau \in (0, 1). \end{aligned} \quad (5.7)$$

The elementary inequality

$$|i\nabla u + Au|^2 \geq (1 - \tau)|\nabla u|^2 - \tau^{-1}|A|^2|u|^2, \forall \tau \in (0, 1),$$

combined with the minimax principle yields

$$\begin{aligned} N(0; H_{B_1}^D(A, (1 - \tau)^{-1}(gV_1 - \lambda) - V_3)) &\leq N(0; H_{B_1}^D(0, (1 - \tau)^{-3}gV_1)) + \\ N(0; H_{B_1}^D(A, -\tau^{-1}(1 - \tau)^{-1}(\tau^{-1}|A|^2 + (1 - \tau)^{-1}\lambda + V_3))) &, \forall \tau \in (0, 1). \end{aligned} \quad (5.8)$$

Applying the standard Weylian asymptotic formula, we get

$$\lim_{g \rightarrow \infty} g^{-m/2} N_{B_1}^D(0, (1 - \tau)^{-3}gV_1) \leq \omega_m (1 - \tau)^{-3m/2} \int_{\mathbf{R}^m} V_1(X)_-^{m/2} dX / (2\pi)^m \quad (5.9)$$

(see e.g. [Re.Sim 4, Theorem XIII.79].)

Since the function $\tau^{-1}(1 - \tau)^{-1}(\tau^{-1}|A|^2 + (1 - \tau)^{-1}\lambda + V_3)$ is in $L^{m/2}(\mathbf{B}_1)$ for each $\tau \in (0, 1)$, Proposition 2.3 implies the boundedness of the second term at the right-hand side of (5.8). Moreover, this term is independent of g . Further, using the minimax principle, we get

$$N(0; H_{B_2}^D(A, -(1 - \tau)^{-1}\lambda - V_3))$$

$$\leq N((1 - \tau)^{-2}\lambda; H(A, 0)) + N(0; H(A, -\tau^{-1}V_3)), \forall \tau \in (0, 1). \quad (5.10)$$

Since $\lambda < \mathcal{E}$, we may choose $\tau > 0$ small enough so that $(1 - \tau)^{-2}\lambda < \mathcal{E}$. Hence, the first term at the right-hand side of (5.10) is finite. Proposition 2.3 implies that the second term at the right-hand side is finite as well. Combining (5.5) – (5.10), we come to (5.2).

Finally, we just outline the demonstration of (5.3). The minimax principle entails the validity of the inequality

$$N(\lambda; H_{1,1,g}) \geq N(0; H(A, (1 + \tau)^{-1}(gV_1 - \lambda))) - N(0; H(A, -\tau^{-1}gV_2)), \forall \tau > 0,$$

which is the analogue of (5.5). Further, the derivation of (5.3) is quite similar (and simpler) to the derivation of (5.2).

Our next theorem can be regarded as a generalization of Theorem 5.1 for the case where the spectral parameter $\lambda \in \mathbf{R}$ satisfies $\lambda \in \rho(H(A, 0))$, but not necessarily $\lambda < \mathcal{E}$. In this case, however, we must choose a well-defined analogue of the counting function $N(\lambda; H_{1,1,g})$.

Let $\lambda = \bar{\lambda} \in \rho(H(A, 0))$. Assume that $V \geq 0$, and the multiplier by V is $-\Delta$ -form-compact. Denote by $\tilde{\mathcal{N}}_g^\pm(\lambda)$ the number of the eigenvalues of the operator $H(A, \mp tV)$ crossing λ as the parameter t grows from 0 to the value $g > 0$.

Note that we have

$$\tilde{\mathcal{N}}_g^\pm(\lambda) = n(g^{-1}; \pm V^{1/2}(H(A, 0) - \lambda)^{-1}V^{1/2}). \quad (5.11)$$

Moreover, the equality

$$\tilde{\mathcal{N}}_g^+(\lambda) = N(\lambda; H(A, -gV)) \quad (5.12)$$

holds if $\lambda < 0$ (see [Bir 2].)

Theorem 5.2 (cf. [Bir.Rai, Theorem 1.1]) *Let $m \geq 3$, $A \in L_{loc}^m(\mathbf{R}^m)^m$, $V \geq 0$, $V \in L^{m/2}(\mathbf{R}^m)$. Assume $\lambda = \bar{\lambda} \in \rho(H(A, 0))$. Then we have*

$$\lim_{g \rightarrow \infty} g^{-m/2} \tilde{\mathcal{N}}_g^+(\lambda) = \omega_m \int_{\mathbf{R}^m} V(X)^{m/2} dX / (2\pi)^m, \quad (5.13)$$

$$\tilde{\mathcal{N}}_g^-(\lambda) = o(g^{m/2}), g \rightarrow \infty. \quad (5.14)$$

Remark. Assume that multiplier by the real-valued function F is $-\Delta$ -form-found with zero form bound, i.e. the estimate

$$\int_{\mathbf{R}^m} |F||u|^2 dX \leq \varepsilon \int_{\mathbf{R}^m} |\nabla u|^2 dX + c(\varepsilon) \int_{\mathbf{R}^m} |u|^2 dX$$

holds for each $\varepsilon > 0$ and every $u \in C_0^\infty(\mathbf{R}^m)$. Then Theorem 5.2 remains valid if we replace the unperturbed operator $H(A, 0)$ by $H(A, F)$.

Sketch of the proof of Theorem 5.2. Write the weighted resolvent identity

$$V^{1/2}(H(A, 0) - \lambda)^{-1}V^{1/2} =$$

$$V^{1/2}(H(A, 0) + 1)^{-1}V^{1/2} - (\lambda + 1)V^{1/2}(H(A, 0) - \lambda)^{-1}(H(A, 0) + 1)^{-1}V^{1/2} \quad (5.15)$$

with $\lambda = \bar{\lambda} \in \rho(H(A, 0))$. Bearing in mind (5.1) and (5.11), we find that it suffices to verify the asymptotic estimates

$$n(s; (H(A, 0) + 1)^{-1/2}V(H(A, 0) + 1)^{-1/2}) = O(s^{-m/2}), s \downarrow 0, \quad (5.16)$$

$$n(s; (H(A, 0) + 1)^{-1}V(H(A, 0) + 1)^{-1}) = o(s^{-m/2}), s \downarrow 0, \quad (5.17)$$

in order to conclude the validity of (5.13) (see [Bir 2, Theorem 1.2].) The Birman-Schwinger principle entails

$$n(s; (H(A, 0) + 1)^{-1/2}V(H(A, 0) + 1)^{-1/2}) = N(-1; H(A, -s^{-1}V)) \leq$$

$$N(0; H(A, -s^{-1}V)), \forall s > 0,$$

and, hence, the estimate (5.16) follows directly from Proposition 2.3. Further, fix an arbitrary $\varepsilon > 0$, and write $V = V_1 + V_2$ where $V_1 \in C_0^\infty(\mathbf{R}^m)$ and $\int_{\mathbf{R}^m} |V_2(X)|^{m/2} dX < \varepsilon$. Then we have

$$n(s; (H(A, 0) + 1)^{-1}V(H(A, 0) + 1)^{-1}) \leq n(s/2; (H(A, 0) + 1)^{-1}V_1(H(A, 0) + 1)^{-1}) +$$

$$n(s/2; (H(A, 0) + 1)^{-1}V_2(H(A, 0) + 1)^{-1}), \forall s > 0. \quad (5.18)$$

Proposition 2.3 combined with the minimax principle implies

$$n(s; (H(A, 0) + 1)^{-1}V_2(H(A, 0) + 1)^{-1}) \leq$$

$$n(s; (H(A, 0) + 1)^{-1/2}V_2(H(A, 0) + 1)^{-1/2}) \leq c_m \varepsilon s^{-m/2}, \forall s > 0. \quad (5.19)$$

Pick a function $\zeta \in C_0^\infty(\mathbf{R}^m)$ such that $0 \leq \zeta(X) \leq 1$ for each $X \in \mathbf{R}^m$ and $\zeta(X) = 1$ for each $X \in \mathbf{B}$ where \mathbf{B} is an open ball containing $\text{supp } V_1$. Since the commutator

$$[(H(A, 0) + 1), \zeta] = -2i \sum_{j=1}^m \partial_j \zeta \cdot L_j(A) - \Delta \zeta$$

is relatively compact with respect to the operator $H(A, 0) + 1$, we get

$$n(s; (H(A, 0) + 1)^{-1} V_1 (H(A, 0) + 1)^{-1}) \leq$$

$$n(s\tau; (H_{\mathbf{B}}^D(A, 0) + 1)^{-1} V_{1|\mathbf{B}} (H_{\mathbf{B}}^D(A, 0) + 1)^{-1}) + O(1), s \downarrow 0, \quad (5.20)$$

for some $\tau \in (0, 1]$ independent of s (see [Bir.Sol 1, Ch. 4].) Since V_1 is bounded, we have

$$n(s; (H_{\mathbf{B}}^D(A, 0) + 1)^{-1} V_{1|\mathbf{B}} (H_{\mathbf{B}}^D(A, 0) + 1)^{-1}) \leq n(s\tau; (H_{\mathbf{B}}^D(A, 0) + 1)^{-2}), \forall s > 0, \quad (5.21)$$

for some $\tau \in (0, 1)$ which is independent of s . Finally, since the operator

$$H_{\mathbf{B}}^D(A, 0) + 1 - (-\Delta_{\mathbf{B}}^D + 1) = 2iA \cdot \nabla + |A|^2$$

is relatively compact with respect to $H_{\mathbf{B}}^D(A, 0) + 1$, we get

$$n(s; (H_{\mathbf{B}}^D(A, 0) + 1)^{-2}) \leq n(s\tau; (-\Delta_{\mathbf{B}}^D(A, 0) + 1)^{-2}) + O(1), s \downarrow 0, \quad (5.22)$$

with some $\tau \in (0, 1)$ independent of s (see [Bir.Sol 1, Chapter 4].) The standard Weylian asymptotic formula

$$N(\lambda; -\Delta_{\mathbf{B}}^D) = O(\lambda^{m/2}), \lambda \rightarrow \infty,$$

implies

$$n(s; (-\Delta_{\mathbf{B}}^D(A, 0) + 1)^{-2}) = O(s^{-m}), s \downarrow 0. \quad (5.23)$$

Combining (5.18) – (5.23), we come to (5.17), and, whence to (5.13). In order to verify (5.14), we must take into account the estimates (5.11) and (5.15) – (5.17) together with the elementary fact that the operator $V^{1/2}(H(A, 0) + 1)^{-1}V^{1/2}$ is nonnegative, so that we have

$$n(s; -V^{1/2}(H(A, 0) + 1)^{-1}V^{1/2}) = 0, \forall s > 0.$$

5.1.2. In this paragraph we deal with magnetic potentials A generating constant magnetic fields B , and electric potentials V which decay slowly at infinity.

Throughout the paragraph we assume that $B = \text{curl } A$ satisfies (2.26), and use the notations $2d = \text{rank } B$ and $k = \dim \text{Ker } B$. Recall that, by Corollary 3.3, we have

$$\mathcal{E} \equiv \inf \sigma_{\text{ess}}(H(A, 0)) = \Lambda_1.$$

Assume $-V \in \mathcal{D}_{-\alpha}^+$, $\alpha > 0$. For $\lambda < \Lambda_1$ set

$$\nu_5(g) \equiv \nu_5(g; \lambda) := \int_{\mathbf{R}^m} \Theta(-gV(X) + \lambda; B) dX.$$

Note that if $-V \in \mathcal{D}_{-\alpha}^+$, $\alpha > 0$, then the asymptotic estimates

$$\nu_5(g; \lambda) \asymp g^{m/\alpha}, \alpha \in (0, 2), \quad (5.24)$$

$$\nu_5(g; \lambda) \asymp g^{m/2} \log g, \alpha = 2, \quad (5.25)$$

$$\nu_5(g; \lambda) = \omega_m g^{m/2} \int_{\mathbf{R}^m} V(X)^{m/2} dX / (2\pi)^m (1 + o(1)), \alpha > 2, \quad (5.26)$$

hold as $g \rightarrow \infty$ for any fixed $\lambda < \Lambda_1$. Moreover, if V obeys the asymptotics

$$-\lim_{|X| \rightarrow \infty} |X|^\alpha V(X) = v(\hat{X}), \hat{X} := X/|X|, \quad (5.27)$$

with $\alpha \in (0, 2]$ and a positive function $v \in C(\mathbf{S}^{m-1})$, then we have

$$\begin{aligned} \lim_{g \rightarrow \infty} g^{-m/\alpha} \nu_5(g; \lambda) &= \frac{b_1 \dots b_d}{2^{k+d} \pi^{m/2}} \frac{\Gamma(m/\alpha - k/2)}{\alpha \Gamma(1 + m/\alpha)} \\ &\sum_{q=1}^{\infty} \kappa_q (\Lambda_q - \lambda)^{k/2 - m/\alpha} \int_{\mathbf{S}^{m-1}} v(\omega)^{m/\alpha} dS(\omega), \alpha \in (0, 2), \end{aligned} \quad (5.28)$$

$$\lim_{g \rightarrow \infty} (g^{m/2} \log g)^{-1} \nu_5(g; \lambda) = \int_{\mathbf{S}^{m-1}} v(\omega)^{m/2} dS(\omega) / 2(4\pi)^{m/2} \Gamma(1 + m/2), \alpha = 2. \quad (5.29)$$

Lemma 5.1 [Rai 6, Lemma 2.3] *Assume that (2.26) holds, and $-V \in \mathcal{D}_{-\alpha,1}^+$, $\alpha \in (0, 2]$. If $k = 0$ and $\alpha \neq 2$ suppose in addition that the function $\Psi_{-V}(s)$ satisfies the condition \mathcal{T}_0 as $s \downarrow 0$. Then we have*

$$\lim_{\delta \downarrow 0} \limsup_{g \rightarrow \infty} \nu_5((1 + \delta)g; \lambda) / \nu_5(g; \lambda) = 1, \forall \lambda < \Lambda_1,$$

or in other words, for each fixed $\lambda < \Lambda_1$ the function $\nu_5(g; \lambda)$ satisfies the condition \mathcal{T}_0 as $g \rightarrow \infty$.

Remark. If $-V \in \mathcal{D}_{-\alpha,1}^+$ with $\alpha > 2$, then the estimate (5.26) is valid, so that the function $\nu_5(g; \lambda)$ again satisfies the condition \mathcal{T}_0 as $g \rightarrow \infty$ for trivial reasons.

Theorem 5.3 [Rai 4, Theorem 2.1] *Let $m \geq 2$ and (2.26) hold. Assume $-V \in \mathcal{D}_{-\alpha,1}^+$ with $\alpha \in (0, 2]$. If $k = 0$ and $\alpha \neq 2$, assume in addition that the function $\Psi_{-V}(s)$ satisfies the condition \mathcal{T}_0 . Then we have*

$$N(\lambda; H_{1,1,g}) = \nu_5(g; \lambda)(1 + o(1)), g \rightarrow \infty, \quad (5.30)$$

for each $\lambda < \Lambda_1$.

Remark. If $-V \in \mathcal{D}_{-\alpha}^+$ with $\alpha > 2$, then $V \in L^2(\mathbf{R}^m)$. Hence, in this case Theorem 5.1 is valid (provided that $m \geq 3$.)

Proof of Theorem 5.3. Modifying in a straightforward manner the argument in [Roz, Lemma 4], we can easily verify that for each sufficiently small $\delta > 0$ there exist a disjoint covering of \mathbf{R}^m by open cubes $q_l \equiv q_l(r_l; X_l)$, $l \geq 1$, with centres at the points X_l and side-lengths equal to r_l satisfying

$$C_0^{-1}\delta(1 + |X_l|) \leq r_l \leq C_0\delta(1 + |X_l|),$$

with a positive constant C_0 independent of l and δ . Let $\{\varphi_l\}_{l=1}^\infty$ be a partition of unity subject to this covering possessing the following properties:

- (i) $\varphi_l \in C_0^\infty(\mathbf{R}^m)$, $\forall l \geq 1$;
- (ii) $\text{supp } \varphi_l \subset \tilde{q}_{l,\delta} := q_l((1 + \delta)r_l; X_l)$, $\forall l \geq 1$;
- (iii) $0 \leq \varphi_l(X) \leq 1$, $l \geq 1$, $\sum_{l=1}^\infty \varphi_l^2(X) = 1$, $\forall X \in \mathbf{R}^m$;
- (iv) the estimates $|D^\gamma \varphi_l(X)| \leq C_\gamma(\delta r_l)^{-|\gamma|}$, $\forall \gamma \in \mathbf{N}^d$, hold with some constants C_γ independent of l and δ .

The quantity $\#\{j : \text{supp } \varphi_j \cap \text{supp } \varphi_l \neq \emptyset\}$ is uniformly bounded with respect to l and sufficiently small δ . Moreover, the estimates

$$C_2^{-1} \leq (1 + |X_j|)/(1 + |X_l|) \leq C_2$$

are valid with a quantity C_2 independent of the integers l, j satisfying $\text{supp } \varphi_j \cap \text{supp } \varphi_l \neq \emptyset$.

Applying the relations

$$\begin{aligned} (H_{1,1,g}u, u) &= \sum_{l=1}^\infty \left\{ (H_{1,1,g}\varphi_l u, \varphi_l u) - (|\nabla \varphi_l|^2 u, u) \right\} \geq \\ &\sum_{l=1}^\infty \left\{ (H_{1,1,g}\varphi_l u, \varphi_l u) - C_3^{-1}(\delta r_l)^{-1} \|\varphi_l u\|^2 \right\}, \forall u \in C_0^\infty(\mathbf{R}^m), \end{aligned}$$

where C_3 is independent of l and sufficiently small $\delta > 0$, we get

$$N(\lambda; H_{1,1,g}) \leq \sum_{l=1}^\infty N(\lambda + C_3(\delta r_l)^{-2}; H_{\tilde{q}_{l,\delta}}^D(A, gV)). \quad (5.31)$$

Put

$$V_l^+ := \sup_{X \in \tilde{q}_l, \delta} (-V(X)).$$

Using Theorem 2.7, we obtain

$$\begin{aligned} N(\lambda + C_3(\delta r_l)^{-2}; H_{\tilde{q}_l, \delta}^D(A, gV)) &\leq \\ (1 + \delta)^m \text{vol } q_l \Theta(gV_l^+ + C_3(\delta r_l)^{-2} + \lambda; B), \forall l \geq 1. \end{aligned} \quad (5.32)$$

The inclusion $-V \in \mathcal{D}_{-\alpha, 1}^+$ with $\alpha \in (0, 2]$ implies that for a given $\delta > 0$, and sufficiently large g , we have

$$gV_l^+ + C_3(\delta r_l)^{-2} \leq -(1 + C_4\delta)gV(X), \forall X \in q_l, \forall l \geq 1, \quad (5.33)$$

where C_4 is independent of δ , l and X . The combination of (5.31) – (5.33) yields

$$N(\lambda; H_{1,1,g}) \leq (1 + \delta)^m \int_{\mathbf{R}^m} \Theta(-(1 + C_4\delta)gV(X) + \lambda; B) dX, \quad (5.34)$$

for sufficiently small $\delta > 0$. Applying Lemma 5.1, we get

$$\lim_{\delta \downarrow 0} \limsup_{g \rightarrow \infty} (1 + \delta)^m \int_{\mathbf{R}^m} \Theta(-(1 + C_4\delta)gV(X) + \lambda; B) dX / \nu_5(g; \lambda) = 1.$$

Hence, the estimate (5.34) implies

$$\limsup_{g \rightarrow \infty} N(\lambda; H_{1,1,g}) / \nu_5(g; \lambda) \leq 1. \quad (5.35)$$

Further, by the minimax principle we have

$$N(\lambda; H_{1,1,g}) \geq \sum_{l=1}^{\infty} N(\lambda; H_{q_l}^D(A, gV)). \quad (5.36)$$

Arguing as in the derivation of (5.34), we get

$$N(\lambda; H_{1,1,g}) \geq (1 - \delta)^m \int_{\mathbf{R}^m} \Theta(-(1 + C'_4\delta)gV(X) + \lambda; B) dX \quad (5.37)$$

where C'_4 is independent of δ and g . Now (5.37) entails

$$\liminf_{g \rightarrow \infty} N(\lambda; H_{1,1,g}) / \nu_5(g; \lambda) \geq 1. \quad (5.38)$$

Putting together (5.35) and (5.38), we come to (5.30).

At the end of this paragraph we shall establish an analogue of Theorem 5.2 for the case where (2.26) holds, and V decays slowly at infinity. Note that if $\dim \text{Ker } B > 0$,

then $\lambda = \bar{\lambda} \in \rho(H(A, 0))$ implies $\lambda < \Lambda_1 \equiv \inf \sigma_{ess}(H(A, 0))$ and therefore this case is exhausted in Theorem 5.3. However, if $k = 0$, then the relation $\lambda = \bar{\lambda} \in \rho(H(A, 0))$ does not imply necessarily $\lambda < \Lambda_1$, since in this case λ can be situated between two subsequent distinct Landau levels Λ_q and Λ_{q+1} , $q \geq 1$. We shall deal exactly with this case in the remaining theorem of this paragraph. To this end, we assume $-V \in \mathcal{D}_{-\alpha, 1}^+$, $\alpha \in (0, 2]$, fix $\lambda = \bar{\lambda} \in \rho(H(A, 0))$ and introduce the quantities

$$\tilde{\nu}_5^+(g; \lambda) = \frac{b_1 \dots b_d}{(2\pi)^{m/2}} \sum_{q \geq 1: \Lambda_q > \lambda} \kappa_q \Psi_V(g^{-1}(\Lambda_q - \lambda)),$$

and

$$\tilde{\nu}_5^-(g; \lambda) = \frac{b_1 \dots b_d}{(2\pi)^{m/2}} \sum_{q \geq 1: \Lambda_q < \lambda} \kappa_q \Psi_V(g^{-1}(\lambda - \Lambda_q)).$$

Note that we have $\tilde{\nu}_5^+(g; \lambda) = \nu_5(g; \lambda)$ if $\lambda < \Lambda_1$. Moreover, the relations (5.24) – (5.26) remain valid if we replace in them $\nu_5(g; \lambda)$ by $\tilde{\nu}_5^+(g; \lambda)$.

If $\lambda > \Lambda_1$, we have

$$\tilde{\nu}_5^-(g; \lambda) \asymp g^{m/\alpha}, \quad \forall \alpha > 0, g \rightarrow \infty. \quad (5.39)$$

Besides, if the potential V obeys the asymptotics (5.27) with $\alpha \in (0, 2)$, then similarly to (5.28), we have

$$\begin{aligned} & \lim_{g \rightarrow \infty} g^{-m/\alpha} \tilde{\nu}_5^+(g; \lambda) = \\ & \frac{b_1 \dots b_d}{m(2\pi)^{m/2}} \sum_{q \geq 1: \Lambda_q > \lambda} \kappa_q (\Lambda_q - \lambda)^{-m/\alpha} \int_{\mathbf{S}^{m-1}} v(\omega)^{m/\alpha} dS(\omega), \quad \alpha \in (0, 2). \end{aligned}$$

If (5.27) holds with $\alpha = 2$, then the asymptotic relation (5.29) remains valid if we replace in it $\nu_5(g; \lambda)$ by $\tilde{\nu}_5^+(g; \lambda)$. Moreover, if $\lambda > \Lambda_1$, we have

$$\lim_{g \rightarrow \infty} g^{-m/\alpha} \tilde{\nu}_5^-(g; \lambda) = \frac{b_1 \dots b_d}{m(2\pi)^{m/2}} \sum_{q \geq 1: \Lambda_q < \lambda} \kappa_q (\lambda - \Lambda_q)^{-m/\alpha} \int_{\mathbf{S}^{m-1}} v(\omega)^{m/\alpha} dS(\omega), \quad \forall \alpha > 0.$$

Theorem 5.4 [Rai 5, Theorem 1.1 - 1.3] *Let (2.26) hold and $\dim \text{Ker } B = 0$. Assume $V \in \mathcal{D}_{-\alpha, \infty}^{++}$. Fix $\lambda = \bar{\lambda} \in \rho(H(A, 0))$. Then we have*

(i) *Assume $\alpha \in (0, 2]$. Then we have*

$$\tilde{\mathcal{N}}_g^+(\lambda) = \tilde{\nu}_5^+(g; \lambda)(1 + o(1)), \quad g \rightarrow \infty. \quad (5.40)$$

(ii) *Assume $\alpha > 0$ and $\lambda > \Lambda_1$. Then we have*

$$\tilde{\mathcal{N}}_g^-(\lambda) = \tilde{\nu}_5^-(g; \lambda)(1 + o(1)), \quad g \rightarrow \infty. \quad (5.41)$$

Remark. If $\lambda < \Lambda_1$, then $\tilde{\nu}_5^-(g; \lambda) \equiv 0$ for each nonnegative g .

Sketch of the proof of Theorem 5.4. For definiteness we outline the demonstration of (5.41). Taking into account (5.11), and the unitary equivalence of the operators $H(A, gV)$ and $\mathcal{H}_{1,1} + g\mathcal{H}_{1,3}$ (see Lemma 2.4), we get

$$\tilde{\mathcal{N}}_g^-(\lambda) = n(g^{-1}; -\mathcal{H}_{1,3}^{1/2}(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{H}_{1,3}^{1/2}).$$

Fix some $\Lambda > \lambda$ and denote by \mathcal{P}_- , \mathcal{P}_+ , \mathcal{P}_∞ the spectral projections of the operator $\mathcal{H}_{1,1}$ corresponding respectively to the intervals $(-\infty, \lambda)$ (or, equivalently, to $[\Lambda_1, \lambda)$), (λ, Λ) and $[\Lambda, \infty)$. Obviously, the projections \mathcal{P}_- , \mathcal{P}_+ and \mathcal{P}_∞ are pairwise orthogonal, and we have $\mathcal{P}_- + \mathcal{P}_+ + \mathcal{P}_\infty = \text{Id}$. Replacing $(\mathcal{H}_{1,1} - \lambda)^{-1}$ by its negative part we get

$$n(s; -\mathcal{H}_{1,3}^{1/2}(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{H}_{1,3}^{1/2}) \leq n(s; -\mathcal{H}_{1,3}^{1/2}\mathcal{P}_-(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{P}_-\mathcal{H}_{1,3}^{1/2}), \forall s > 0. \quad (5.42)$$

On the other hand, restricting the operator $\mathcal{H}_{1,3}^{1/2}(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{H}_{1,3}^{1/2}$ on the range of \mathcal{P}_- , we get

$$\begin{aligned} n(s; -\mathcal{H}_{1,3}^{1/2}(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{H}_{1,3}^{1/2}) &\geq \\ n(s; -\mathcal{P}_-\mathcal{H}_{1,3}^{1/2}(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{H}_{1,3}^{1/2}\mathcal{P}_-) &\geq \\ n((1 - 2\tau)s; -\mathcal{P}_-\mathcal{H}_{1,3}^{1/2}\mathcal{P}_-(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{P}_-\mathcal{H}_{1,3}^{1/2}\mathcal{P}_-) - \\ n(\tau s; -\mathcal{P}_-\mathcal{H}_{1,3}^{1/2}\mathcal{P}_+(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{P}_+\mathcal{H}_{1,3}^{1/2}\mathcal{P}_-) - \\ n(\tau s; -\mathcal{P}_-\mathcal{H}_{1,3}^{1/2}\mathcal{P}_\infty(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{P}_\infty\mathcal{H}_{1,3}^{1/2}\mathcal{P}_-) &\geq 0, \forall \tau \in (0, 1/2). \end{aligned} \quad (5.43)$$

Applying arguments from the theory of Ψ DO of negative order with Weyl symbols, similar to the ones used in the proof of Lemma 4.10, we get the following estimates

$$n(s; -\mathcal{P}_-\mathcal{H}_{1,3}^{1/2}(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{H}_{1,3}^{1/2}\mathcal{P}_-) = \tilde{\nu}_5^-(s^{-1}; \lambda)(1 + o(1)), s \downarrow 0,$$

$$n(s; -\mathcal{P}_-\mathcal{H}_{1,3}^{1/2}\mathcal{P}_-(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{P}_-\mathcal{H}_{1,3}^{1/2}\mathcal{P}_-) = \tilde{\nu}_5^-(s^{-1}; \lambda)(1 + o(1)), s \downarrow 0,$$

(see [Rai 5, Lemma 2.2]),

$$n(s; -\mathcal{P}_-\mathcal{H}_{1,3}^{1/2}\mathcal{P}_+(\mathcal{H}_{1,1} - \lambda)^{-1}\mathcal{P}_+\mathcal{H}_{1,3}^{1/2}\mathcal{P}_-) = O(s^{-d/(\alpha+2)}) = o(\tilde{\nu}_5^-(s^{-1}; \lambda)), s \downarrow 0,$$

(see [Rai 5, Lemma 2.4]), and

$$\lim_{\Lambda \rightarrow \infty} \limsup_{s \downarrow 0} s^{d/\alpha} n(s; -\mathcal{P}_- \mathcal{H}_{1,3}^{1/2} \mathcal{P}_\infty (\mathcal{H}_{1,1} - \lambda)^{-1} \mathcal{P}_\infty \mathcal{H}_{1,3}^{1/2} \mathcal{P}_-) = 0,$$

(see [Rai 5, Lemma 2.6].)

Taking into account the estimate (5.39), and the fact that the function $\tilde{v}_5^-(g; \lambda)$ satisfies the property \mathcal{T}_0 as $g \rightarrow \infty$, we find that (5.42) and (5.43) entail (5.41).

5.2 Weak electric field approximation

In this subsection we consider the case $m = 2$ and concentrate on constant magnetic field and axisymmetric electric potentials. More precisely, if (2.26) holds and (r, φ, z) are cylindrical coordinates in \mathbf{R}^3 such that $z \in \text{Ker } B$, then we suppose that V is independent on φ . Assume that V satisfies the estimates

$$V \leq 0, V \not\equiv 0, \quad (5.44)$$

$$|V(r, z)| \leq c(r^2 + z^2)^{-1-\varepsilon_1}, \varepsilon_1 > 0, \text{ for } r^2 + z^2 \geq 1, \quad (5.45)$$

$$|V(r, z)| \leq c(r^2 + z^2)^{-1+\varepsilon_2}, \varepsilon_2 > 0, \text{ for } r^2 + z^2 \leq 1. \quad (5.46)$$

Without any loss of generality, we assume $b_1 = 2$. Then the operator $H_{1,1,g}(A, V)$ can be written in the coordinates (r, φ, z) as

$$H_{1,1,g}(A, V) = -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{\partial^2}{\partial z^2} - \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} - 2i \frac{\partial}{\partial \varphi} + r^2 + gV(r, z).$$

Since V is independent of φ , the operator $H_{1,1,g}(A, V)$ is unitarily equivalent to the countable orthogonal sum $\sum_{l \in \mathbf{Z}} \oplus H_g^{(l)}$ where $H_g^{(l)}$ is the restriction of $H_{1,1,g}(A, V)$ onto the subspace $\{u \in D(H_{1,1,g}) : u(r, \varphi, z) = e^{il\varphi} w(r, z)\}$ corresponding to a fixed magnetic wavenumber $l \in \mathbf{Z}$. Namely,

$$H_g^{(l)} = -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{\partial^2}{\partial z^2} + \frac{l^2}{r^2} + 2l + r^2 + gV(r, z), l \in \mathbf{Z}.$$

The lower bound of the essential spectrum of $H_g^{(l)}$ coincides with the the minimum eigenvalue $2(l + |l| + 1)$ of the operator

$$h^{(l)} = -\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + \frac{l^2}{r^2} + 2l + r^2, l \in \mathbf{Z},$$

which is selfadjoint in $L^2(\mathbf{R}_+; r dr)$. Moreover, the normalized eigenfunction of $h^{(l)}$ corresponding to $2(l + |l| + 1)$ equals $f_l(r) := (2/|l|!)^{1/2} \exp(-r^2/2)r^{|l|}$. Set

$$V_l = \int_{\mathbf{R}} dz \int_0^\infty V(r, z) f_l^2(r) r dr.$$

Introduce the quantity $G \equiv G_l := g(|l|+1)^{-\varepsilon}$, $l \in \mathbf{Z}$, where $\varepsilon = \min(\varepsilon_1, 1)$ (see (5.45).)

Theorem 5.5 [Sol, Theorem 3.3] *Let (2.26) and (5.44) – (5.46) hold. Fix $l \in \mathbf{Z}$. Denote by $\lambda_l(g)$ the minimum eigenvalue of the operator $H_g^{(l)}$. Then for sufficiently small g we have*

$$\lambda_l(g) = 2(l + |l| + 1) - (gV_l/2)^2(1 + R) \quad (5.47)$$

where the remainder R satisfies the inequalities

$$-c_1 G \leq R \leq c_2 G(|l| + 1) - 1$$

and the constants $c_j > 0$, $j = 1, 2$, are independent of g and l .

For the proof we refer the reader to the original work [Sol].

Remark. The result of [Av.Her.Sim 1, Theorem 5.4] is quite similar to (5.47) but, however, does not contain any information about the dependence of the remainder R on the integer l .

It is noticed in [Sol, p.275] that if (5.45) is valid for some $\varepsilon_1 \in (-1/2, 0]$, then the following analogue of (5.47)

$$\lambda_l(g) = 2(l + |l| + 1) - (gV_l/2)^2 + \tilde{R}(g)$$

holds with $\tilde{R}(g) = O(g^{2+\delta})$, $\delta \in (0, 1 + 2\varepsilon_1)$, for each fixed $l \in \mathbf{Z}$.

Note that if (5.45) holds with $\varepsilon_1 > 0$, then the operator $H_g^{(l)}$ has finitely many isolated eigenvalues situated below the essential-spectrum lower bound, while in the case $\varepsilon_1 \leq 0$ the quantity $N(2(l + |l| + 1) - \lambda; H_g^{(l)})$ generically grows unboundedly as $\lambda \downarrow 0$.

5.3 Strong magnetic field approximation

5.3.1. In this subsection we consider the case $m = 3$ and restrict our attention to a constant magnetic field and a Coulomb electric potential $V_c(X) := -1/|X|$. We use the same cylindrical coordinates (r, φ, z) as in the previous subsection. Besides, we assume as earlier $b_1 = 2$. Then the operator $H_{1,\mu,1}(A, V_c)$ is unitarily equivalent to the orthogonal sum $\sum_{l \in \mathbf{Z}} \oplus H_c^{(l)}$ where

$$H_c^{(l)} \equiv H_c^{(l)}(\mu) := -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{\partial^2}{\partial z^2} + \frac{l^2}{r^2} + 2\mu l + \mu^2 r^2 - (r^2 + z^2)^{-1/2}, l \in \mathbf{Z}.$$

Denote by $\lambda_l(\mu)$ the minimum eigenvalue of $H_c^{(l)}(\mu)$. The following result concerns the asymptotic expansion of $\lambda_l(\mu)$ with $l \leq 0$ as $\mu \rightarrow \infty$.

Theorem 5.6 [Av.Her.Sim 3, Theorem 2.5] *Under the preceding hypotheses we have*

$$\lambda_l(\mu) = 2\mu - \frac{1}{4}(\log \mu)^2 - (\log \mu)(\log_2 \mu) -$$

$$(\mathcal{C}_l + \log 2) \log \mu - (\log_2 \mu)^2 + 2(\mathcal{C}_l - 1 + \log 2) \log_2 \mu + O(1), l \leq 0, \mu \rightarrow \infty,$$

where $\log_2 \mu := \log \log \mu$, $\mathcal{C}_l := -(\gamma_E + q_{-l})/2$, γ_E is the Euler's constant and $q_p = p^{-1} + q_{p-1}$, $p \geq 1$, $q_0 = 0$.

5.3.2. In this subsection we discuss briefly the approach of B.Helffer and J.Sjöstrand to the strong-magnetic-field approximation of the spectrum of $H_{1,\mu,1}(A, V)$ following their work [Hel.Sjö 5]. A more detailed exposition of these and related results can be found in [Hel.Sjö 1], [Hel.Sjö 3] and [Hel.Sjö 4].

Let $m = 2$. Assume that (2.26) holds. Without any loss of generality we assume $b_1 = 1$. Moreover, we suppose

$$V \in \mathcal{D}_{0,\infty}. \quad (5.48)$$

Our purpose is to investigate the concentration of the spectrum of $H_{1,\mu,1}(A, V)$ around the points $\mu(2q - 1) \equiv \mu\Lambda_q$, $q \geq 1$, (the Landau levels) as $\mu \rightarrow \infty$. More precisely, we shall discuss the behaviour of $\sigma(H_{1,\mu,1}(A, V))$ localized on the closed intervals $I_q(\mu) := [\mu\Lambda_q - C, \mu\Lambda_q + C]$, $C > 0$, $q \geq 1$, as $\mu \rightarrow \infty$. By Lemma 2.26, we conclude that $H_{1,\mu,1}(A, V)$ is unitarily equivalent to $\mathcal{H}_1(\mu B, V)$ (cf. (2.38)). In the case under consideration we have $V_\mu(x, y) = V_{\mathbf{b}}(x, y; \mu) \equiv V(\mu^{-1/2}x, \mu^{-1/2}y)$. Set

$$\mathcal{H}^{(q)}(\mu) := \mu^{-1}\mathcal{H}_{1,1} - (2q - 1) + \mu^{-1}\mathcal{H}_{1,3}, q \geq 1.$$

Note that $\mu^{-1}\mathcal{H}_{1,1} \equiv -\partial^2/\partial x^2 + x^2$ is independent of μ . Obviously, instead of studying the spectrum of $H_{1,\mu,1}(A, V)$ on $I_q(\mu)$, we can investigate the spectrum of $\mathcal{H}^{(q)}(\mu)$ on the interval $[-C/\mu, C/\mu]$.

In $L^2(\mathbf{R}_{x,y}^2) \oplus L^2(\mathbf{R}_y)$ introduce the selfadjoint operator

$$\mathcal{P}_{\mu,V}^{(q)}(z) = \begin{pmatrix} \mathcal{H}^{(q)}(\mu) - z & \mathcal{R}_q^*(\mu) \\ \mathcal{R}_q(\mu) & 0 \end{pmatrix}, z \in \mathbf{C},$$

where the operator $\mathcal{R}_q(\mu) : L^2(\mathbf{R}_{x,y}^2) \rightarrow L^2(\mathbf{R}_y)$ is defined by

$$(\mathcal{R}_q(\mu)v)(x, y) = f_{q,1}(x)v(y), v \in L^2(\mathbf{R}_y),$$

(see (2.30), and take into account that in the case $m = 2$ all the multiplicities κ_q , $q \geq 1$, equal 1.) If $V = 0$, then it is clear that $\mathcal{P}_{\mu,0}^{(q)}(z)$ is invertible for $|z| < 1$ and its inverse is given by

$$\mathcal{E}_{\mu,0}^{(q)}(z) = \begin{pmatrix} \mathcal{E}_0(z) & \mathcal{E}_0^+ \\ \mathcal{E}_0^- & \mathcal{E}_0^{-+}(z) \end{pmatrix}$$

where $\mathcal{E}_0^+ = \mathcal{R}_q$, $\mathcal{E}_0^- = \mathcal{R}_q^*$, $\mathcal{E}_0^{-+} = z$ and

$$(\mathcal{E}_0(z)u)(x, y) = \sum_{j \geq 1: j \neq q} \frac{f_{j,1}(x)}{(2j - 2q - z)} \int_{\mathbf{R}} f_{j,1}(x')u(x', y) dx'.$$

In the next proposition we describe the operator inverse to $\mathcal{P}_{\mu,V}^{(q)}(z)$ when a nonvanishing electric potential V is present.

Proposition 5.1 [Hel.Sjö 5, Proposition 2.1] *Let V satisfy (5.48), and $|z| < 1$. Then for sufficiently large μ the operator $\mathcal{P}_{\mu,V}^{(q)}(z)$ is invertible and its inverse has the form*

$$\mathcal{E}_{\mu,V}^{(q)}(z) = \begin{pmatrix} \mathcal{E}_V(z, \mu) & \mathcal{E}_V^+(z, \mu) \\ \mathcal{E}_V^-(z, \mu) & \mathcal{E}_V^{-+}(z, \mu) \end{pmatrix}$$

where $z - \mathcal{E}_V^{-+}(z, \mu)$ is a family of h - Ψ DOs with Weyl symbols $e_{\mu,V}(y, h\eta; h, z)$, $h \equiv \mu^{-1}$, whose principal part coincides with $hV(y, \eta)$.

The importance of Proposition 5.1 becomes clear in the context of the following lemma.

Lemma 5.2 [Hel.Sjö 5, Lemma 2.2] *Let $|z| < 1$. Then we have $0 \in \sigma(\mathcal{E}_V^{-+}(z, \mu))$ if and only if $z \in \sigma(\mathcal{H}^{(q)}(\mu))$ (or, equivalently, $\mu(\Lambda_q + z) \in \sigma(H_{1,\mu,1}(A, V))$.)*

In the following example we point to an application of an improved version of Proposition 5.1 and the semiclassical analysis of the Harper's equation (see e.g. [Hel.Sjö 4]).

Example. [Hel.Sjö 5, Corollary 2.3] Let $V(x, y) = \cos x + \cos y$. Assume that h has the form of an infinite fraction

$$h \equiv \mu^{-1} 2\pi / (q_1 + 1 / (q_2 + 1 / (q_3 + \dots))), \quad q_j \in \mathbf{Z}, 1 \leq j < \infty,$$

where $|q_j| \geq C_0$. Then the spectrum of $H_{1,\mu,1}(A, V)$ on $I_q(\mu)$ is a Cantor set of measure zero.

5.4 Weak magnetic field approximation

5.4.1. In the present and the following two paragraphs we consider potentials (A, V) which are similar to the ones treated in Subsection 5.1.

In this paragraph we deal with electric potentials V which decay rapidly at infinity in a certain sense. We say that V satisfies the condition \mathcal{K}_j , $j = 0, 1$, if and only if for each $\varepsilon > 0$ we can write V in the form

$$V = V_1 + V_2. \quad (5.49)$$

where $V_1 \in C_0^\infty(\mathbf{R}^m)$, and the inequality

$$\int_{\mathbf{R}^m} |V_2| |u|^2 dX \leq \varepsilon \int_{\mathbf{R}^m} (|\nabla u|^2 + j|u|^2) dX$$

is valid for each $u \in C_0^\infty(\mathbf{R}^m)$ (the condition \mathcal{K}_0 will be considered only in the case $m \geq 3$).

Remark. If V satisfies the condition \mathcal{K}_1 , then the negative spectrum of the operator $H(0, V)$ is purely discrete and, hence, the quantity $N(-\lambda; H(0, V))$ is finite for each $\lambda > 0$. Moreover, if V satisfies \mathcal{K}_0 , then the negative eigenvalues of $H(0, V)$ do not accumulate to the origin, i.e. we have $N(0; H(0, V)) < \infty$ (see e.g. [Bir 1].)

Example. Let $q = m/2$ if $m \geq 3$, and $q > 1$ if $m = 2$. Assume $V \in L_{loc}^q(\mathbf{R}^m)$ and $\int_{|Y-X|<1} |V(X)|^q dX \rightarrow 0$ as $|Y| \rightarrow \infty$. Then V satisfies the condition \mathcal{K}_1 . Let $m \geq 3$. Assume $V \in L^{m/2}(\mathbf{R}^m)$. Then V satisfies the condition \mathcal{K}_0 .

Theorem 5.7 [Rai 6, Theorem 3.2] *Suppose $A \in L_{loc}^2(\mathbf{R}^m)^m$.*

(i) *Assume that V satisfies the condition \mathcal{K}_1 . Suppose that the number $-\lambda < 0$ is not an eigenvalue of the operator $H(0, V)$. Then we have*

$$N(-\lambda; H_{1,\mu,1}(A, V)) \rightarrow N(-\lambda; H_{1,0,1}(A, V)) \text{ as } \mu \downarrow 0. \quad (5.50)$$

(ii) *Let $m \geq 3$. Assume that V satisfies the condition \mathcal{K}_0 . Suppose that the zero is not an eigenvalue of the operator $H(0, V)$. Then we have*

$$N(0; H_{1,\mu,1}(A, V)) \rightarrow N(0; H_{1,0,1}(A, V)) \text{ as } \mu \downarrow 0. \quad (5.51)$$

Remark. Theorem 5.7 deals with the stability of the isolated eigenvalues of the operator $H(0, V)$ with respect to a perturbation by a weak magnetic field. Related results can be found in [Av.Her.Sim 1, Sect. 6].

Proof of Theorem 5.7. For definiteness we prove the first assertion of the theorem. By Proposition 2.6, we assume without any loss of generality that we have $\operatorname{div} A = 0$ in the distributional sense. Let the multiplier by the real-valued function W is $-\Delta$ -form-compact. Define the compact operator

$$T_{\mu,\lambda}(W) := -(H_{1,\mu,1}(A, 0) + \lambda)^{-1/2} W (H_{1,\mu,1}(A, 0) + \lambda)^{-1/2}, \mu \geq 0.$$

Note that $-\lambda \notin \sigma(H(0, V))$ implies $1 \notin \sigma(T_{0,\lambda}(V))$. Fix $\varepsilon \in (0, 1)$ so that

$$\delta := \varepsilon \min\{1, \lambda\} < \frac{1}{2} \text{dist}(1, \sigma(T_{0,\lambda}(V))) \quad (5.52)$$

and write V in the form (5.49). Set

$$t := \|(H_{1,\mu,1}(A, 0) + \lambda)^{-1/2}|V_1|^{1/2}\|_\infty^2.$$

Then the estimate

$$\|T_{\mu,\lambda}(V_1)\| \leq t \quad (5.53)$$

holds for each $\mu \geq 0$. Hence, we have

$$\pm N(-\lambda; H_{1,\mu,1}(A, V)) = \pm n(1; T_{\mu,\lambda}(V)) \leq$$

$$\pm n(1 \mp \delta; T_{\mu,\lambda}(V_1)) = \pm \mathcal{N}(1 \mp \delta, 1 + t|T_{\mu,\lambda}(V_1)|). \quad (5.54)$$

Since the support of V_1 is compact, we have $T_{0,\lambda}(V_1) \in \mathbf{S}_{2p}$ provided that $p > m/4$. Hence, by Lemma 2.2, we find that $T_{\mu,\lambda}(V_1) \in \mathbf{S}_{2p}$ for each $\mu \geq 0$, and each $p \in \mathbf{N}$ such that $p > m/4$. The inequality (5.53) (resp., (5.52)) entails $1 + t \notin \sigma(T_{0,\lambda}(V_1))$ (resp., $1 \mp \delta \notin \sigma(T_{0,\lambda}(V_1))$). Therefore, it suffices to verify the limiting relations

$$\lim_{\mu \downarrow 0} \text{Tr } T_{\mu,\lambda}(V_1)^n = \text{Tr } T_{0,\lambda}(V_1)^n, \forall n \in \mathbf{N}, n \geq 2p, \quad (5.55)$$

in order to conclude that

$$\lim_{\mu \downarrow 0} \mathcal{N}(1 \mp \delta, 1 + t|T_{\mu,\lambda}(V_1)|) = \mathcal{N}(1 \mp \delta, 1 + t|T_{0,\lambda}(V_1)|) \quad (5.56)$$

(cf. [Gre.Sze, Sect. 7.1].) Since we have $T_{\mu,\lambda}(V_1)^p \in \mathbf{S}_2$ if $p > m/4$, it is not difficult to verify the validity of the formula

$$\begin{aligned} & \text{Tr } T_{\mu,\lambda}(V_1)^n = \\ & = (-1)^n \int_{\mathbf{R}^{mn}} V_1(X_1)R(X_1, X_2; \mu, \lambda) \dots V_1(X_n)R(X_n, X_1; \mu, \lambda) dX_1 \dots dX_n, \forall \mu \geq 0, \end{aligned}$$

where $R(X, Y; \mu, \lambda)$ is the distributional kernel of the operator $(H_{1,\mu,0} + \lambda)^{-1}$. Taking into account the estimate (2.9) together with the identity (2.10), we find that the inequality

$$|R(X, Y; \mu, \lambda)| \leq R(X, Y; 0, \lambda)$$

holds for each nonnegative μ and almost every $(X, Y) \in \mathbf{R}^{2m}$. Thus we get

$$|\mathrm{Tr} T_{\mu,\lambda}(V_1)^n| \leq \mathrm{Tr} T_{\mu,\lambda}(-|V_1|)^n \equiv \|T_{0,\lambda}(-|V_1|)\|_n^n < \infty, \forall n \geq 2p, n \in \mathbf{Z}, \forall \mu \geq 0,$$

Moreover, the Feynman-Kac-Itô formula (see (2.8)) implies

$$\lim_{\mu \downarrow 0} R(X, Y; \mu, \lambda) = R(X, Y; 0, \lambda)$$

for almost every $(X, Y) \in \mathbf{R}^{2m}$. By the dominated convergence theorem we come to (5.55), and whence to (5.56).

The estimate (5.53), the choice of V_1 , the estimate (5.52) and the Birman-Schwinger principle entail

$$\begin{aligned} \pm \mathcal{N}(1 \mp \delta, 1 + t|T_{0,\lambda}(-V_1)|) &= \pm n(1 \mp \delta; T_{0,\lambda}(V_1)) \leq \\ &\pm n(1 \mp 2\delta; T_{0,\lambda}(V_1)) = \pm \mathcal{N}(0; H(0, V)). \end{aligned} \quad (5.57)$$

Putting together (5.54), (5.56) and (5.57) we come to (5.50).

5.4.2. In this paragraph we consider electric potentials V which decay slowly at infinity.

Theorem 5.8 [Rai 6, Theorem 3.5] *Suppose that (2.26) holds. Assume that $-V \in \mathcal{D}_{-\alpha,1}^+$ with $\alpha \in (0, 2)$. If $k = \dim \mathrm{Ker} B = 0$, assume in addition that the function $\Psi_{-V}(s)$ satisfies the condition \mathcal{T}_0 . Then we have*

$$N(0; H_{1,\mu,1}) = \mu^{m/2} \nu_5(\mu^{-1})(1 + o(1)), \mu \downarrow 0.$$

We omit the proof of Theorem 5.8 since it is quite similar to the the proof of Theorem 5.1.

5.4.3. In this paragraph we formulate a theorem which deals with the asymptotic behaviour as $\mu \downarrow 0$ of the quantity $N(0; H_{1,\mu,1})$ for the case of a constant magnetic field B and an electric potential V satisfying the asymptotics (5.27) with $\alpha = 2$. This is the border-line case between the ones treated respectively in Theorem 5.7 (ii) and Theorem 5.8.

In $L^2(\mathbf{S}^{m-1})$, $m \geq 2$ introduce the selfadjoint operator

$$-\Delta_S - v \quad (5.58)$$

where Δ_S is the standard Beltrami-Laplace operator with domain $W_2^2(\mathbf{S}^{m-1})$, and v is the function appearing in (5.27). Denote by $\{-\lambda_l(v)\}_{l \geq 1}$ the nondecreasing sequence of the negative eigenvalues of the operator defined in (5.58). Evidently, the set $\{-\lambda_l(v)\}_{l \geq 1}$ is finite and not empty.

Theorem 5.9 [Rai 6, Theorem 3.6] *Assume that (2.26) holds and $V \in L^\infty(\mathbf{R}^m)$ satisfies (5.27) with $\alpha = 2$. Then we have*

$$\lim_{\mu \downarrow 0} |\log \mu|^{-1} N(0; H_{1,\mu,1}) = \frac{1}{2\pi} \sum_{l \geq 1} \left(\lambda_l(v) - \frac{(m-2)^2}{4} \right)_+^{1/2}.$$

Moreover, if $\lambda_l(v) < (m-2)^2/4$, we have

$$N(0; H_{1,\mu,1}) = O(1), \mu \downarrow 0.$$

Under the hypotheses of Theorem 5.9 the negative spectrum of the operator $H(0, V)$ is discrete. Moreover, the quantity $N(0; H(0, V))$ is finite if $\lambda_1(v) < (m-2)^2/4$, and infinite if $\lambda_1 > (m-2)^2/4$ (in particular, if $m = 2$, then the quantity $N(0; H(0, V))$ is always infinite.)

For the proof of Theorem 5.9 which employs some of the ideas already used in the proof of Theorem 4.7, we refer the reader to the original work [Rai 6].

5.4.4. In this subsection we sketch the approach of B.Helffer and J.Sjöstrand to the weak-magnetic-field approximation of the spectrum of $H_{1,\mu,1}(A, V)$. As in §4.4.2 we follow the exposition of [Hel.Sjö 5], and refer to [Hel.Sjö 2] and [Hel.Sjö 3] for more details.

Let $m = 2$. We assume that (2.26) holds, and $b_1 = 1$. Moreover, we suppose that V is C^∞ -smooth and 2π -periodic, i.e.

$$V(X + 2\pi\alpha) = V(X), \forall X \in \mathbf{R}^2, \forall \alpha \in \mathbf{Z}^2.$$

Recall the operator $H_{\mathbf{T}}(-\Xi, V)$ with $\mathbf{T} = \mathbf{R}^2/(2\pi\mathbf{Z})^2$ (see §2.4.2, assuming $A \equiv 0$), and the nondecreasing sequence $\{E_l(\Xi)\}_{l \geq 1}$ of its eigenvalues. Assume that for some $l_0 \geq 1$ the segment

$$I_{l_0} := \bigcup_{\Xi \in [-\frac{1}{2}, \frac{1}{2}]^2} E_{l_0}(\Xi)$$

is simple, i.e. $E_l(\Xi) \notin I_{l_0}$ if $l \neq l_0$ for any $\Xi \in [-\frac{1}{2}, \frac{1}{2}]^2$. In particular, E_{l_0} is a simple eigenvalue of $H_{\mathbf{T}}(-\Xi, V)$ for any fixed $\Xi \in [-\frac{1}{2}, \frac{1}{2}]^2$.

Under these hypotheses $\Xi \rightarrow E_{l_0}(\Xi)$ is an analytic even 2π -periodic function. Moreover, we can choose the eigenfunction $\varphi_{l_0}(X; \Xi)$ of the operator $e^{iX \cdot \Xi} H_{\mathbf{T}}(-\Xi, V) e^{-iX \cdot \Xi}$ corresponding to the eigenvalue $E_{l_0}(\Xi)$ so that the following properties are fulfilled:

- (i) $\Xi \rightarrow \varphi_{l_0}(X; \Xi)$ is an analytic function;
- (ii) $\varphi_{l_0}(X; \Xi + 2\pi\alpha) = \varphi_{l_0}(X; \Xi)$, $\forall \alpha \in \mathbf{Z}^2$;
- (iii) $\varphi_{l_0}(X; \Xi) = \overline{\varphi_{l_0}(X; -\Xi)}$;
- (iv) $\varphi_{l_0}(X - 2\pi\alpha; \Xi) = e^{-i2\pi\alpha \cdot \Xi} \varphi_{l_0}(X; -\Xi)$, $\forall \alpha \in \mathbf{Z}^2$.

Introduce the Wannier functions

$$\psi_0(X) := \int_{[-\frac{1}{2}, \frac{1}{2}]^2} \varphi_{l_0}(X; \Xi) d\Xi,$$

$$\psi_\alpha(X) := \psi_0(X - 2\pi\alpha), \forall \alpha \in \mathbf{Z}^2.$$

Note that $\{\psi_\alpha\}_{\alpha \in \mathbf{Z}^2}$ is an orthonormal basis in the invariant spectral subspace for $H(0, V)$ which corresponds to the segment I_{l_0} .

Further, introduce the magnetic Wannier functions

$$\psi_\alpha^\mu := \exp\{i\pi\mu(\alpha_2 X_1 - \alpha_1 X_2)\} \psi_0(X - 2\pi\alpha), \alpha \in \mathbf{Z}^2.$$

For $\mu \neq 0$ the magnetic Wannier functions are not orthogonal but remain linearly independent. Let $\{e_\alpha^\mu\}_{\alpha \in \mathbf{R}^2}$ be the orthogonal system obtained from $\{\psi_\alpha^\mu\}_{\alpha \in \mathbf{R}^2}$ by the standard Gram procedure.

By analogy with §5.3.2, in $L^2(\mathbf{R}^2) \oplus l^2(\mathbf{Z}^2)$ introduce the selfadjoint operator

$$\mathcal{P}_\mu(z) = \begin{pmatrix} H_{1, \mu, 1} - z & \mathcal{R}_\mu^* \\ \mathcal{R}_\mu & 0 \end{pmatrix}, z \in \mathbf{C}.$$

where the operator $\mathcal{R}_\mu : L^2(\mathbf{R}^2) \rightarrow l^2(\mathbf{Z}^2)$ is defined by

$$(\mathcal{R}_\mu u)_\alpha = \int_{\mathbf{R}^2} e_\alpha^\mu(X) u(X) dX, u \in L^2(\mathbf{R}^2).$$

It turns that the operator $\mathcal{P}_\mu(z)$ is invertible for z in a small complex neighbourhood of I_{l_0} and its inverse has the form

$$\mathcal{E}_\mu(z) = \begin{pmatrix} \mathcal{E}_\mu^+(z) & \mathcal{E}_\mu^+(z) \\ \mathcal{E}_\mu^-(z) & \mathcal{E}_\mu^{-+}(z) \end{pmatrix}. \quad (5.59)$$

A simple calculation yields

$$(\mathcal{E}_\mu^{-+}(z))_{\alpha\beta} = z - \hat{E}(\alpha - \beta), \alpha, \beta \in \mathbf{Z}^2,$$

where

$$\hat{E}(\gamma) := \int_{[-\frac{1}{2}, \frac{1}{2}]^2} \exp\{-i\gamma \cdot \Xi\} E_{l_0}(\Xi) d\Xi, \gamma \in \mathbf{Z}^2,$$

is the Fourier coefficient of $E_{l_0}(\Xi)$.

Now we are ready to formulate the analogues of Proposition 5.1 and Lemma 5.2.

Proposition 5.2 [Hel.Sjö 5, Proposition 3.3] *Under the preceding hypotheses the operator $\mathcal{P}_\mu(z)$ admits an inverse $\mathcal{E}_\mu(z)$ in the form (5.59) for z belonging to a sufficiently small vicinity of I_{l_0} , and $\mu > 0$ small enough. In particular, the operator $\mathcal{E}_\mu^{-+}(z)$ appearing in (5.59) has the following properties:*

(i) there exist positive constants c_j , $j = 0, 1$, such that the estimates

$$|(E_\mu^{-+}(z))_{\alpha\beta}| \leq c_0 \exp\{-c_1|\alpha - \beta|\}$$

are valid;

(ii) there exists a classical symbol $f(\gamma, \mu, z)$, $\gamma \in \mathbf{Z}^2$, which admits the asymptotic expansion

$$f(\gamma, \mu, z) \sim \sum_{j \in \mathbf{N}} f_j(\gamma, z) \mu^j, \mu \downarrow 0,$$

with $|f_j(\gamma, z)| \leq c_j \exp\{-c_1|\gamma|\}$, and satisfies the identity

$$(\mathcal{E}_\mu^{-+}(z))_{\alpha\beta} = \exp\{i\mu 2\pi^2(\alpha_1\beta_2 - \alpha_2\beta_1)\} f(\alpha - \beta; \mu, z),$$

$$f_0(\gamma, z) = z - \hat{E}(\gamma).$$

Lemma 5.3 [Hel.Sjö 5, Lemma 3.4] *For μ and z as in the preceding proposition, the conditions $z \in \sigma(H_{1,\mu,1}(A, V))$ and $0 \in \sigma(\mathcal{E}_\mu^{-+}(z))$ are equivalent.*

5.5 Semiclassical approximation

5.5.1. In this paragraph we assume that the resolvent of the operator $H_{h,1,1}(A, 0)$ is compact and discuss the asymptotics of the quantity $N(\lambda; H_{h,1,1}(A, 0))$ as $h \downarrow 0$, the real number λ being fixed.

Theorem 5.10 *Assume that the hypotheses of Theorem 4.1 hold. In addition, assume that the function $\nu_1(s)$ (see paragraph §4.1.1) satisfies the condition \mathcal{T}_0 as $s \rightarrow \infty$. Then for any fixed $\lambda \in \mathbf{R}$ we have*

$$N(\lambda; H_{h,1,1}(A, 0)) = h^{-m/2} \nu_1(h^{-1}\lambda)(1 + o(1)), h \downarrow 0. \quad (5.60)$$

We omit the proof since it is quite similar to the proof of Theorem 4.1. The asymptotic formula (5.60) remains valid under slightly different assumptions. For example, an analogous result for the case $m = 3$ has been established in [Tam 1, Theorem 2] under the following hypotheses:

- (i) $|B(X)| \rightarrow \infty$ as $|X| \rightarrow \infty$;
- (ii) $A \in C^2(\mathbf{R}^3)^3$;
- (iii) there exists a constant C such that we have $\tilde{\Psi}_B(\lambda) < C\tilde{\Psi}_B(\lambda)$ for sufficiently large λ (cf. (4.1));
- (iv) $|D^\alpha A_j(X)| = o(|B(X)|^2)$ as $|X| \rightarrow \infty$ for $|\alpha| = 2$ and $j = 1, 2, 3$.

5.5.2. In this paragraph we consider potentials (A, V) which are similar to the ones studied in Subsection 5.1.

In this subsection we deal with arbitrary A and electric potentials V which decay rapidly at infinity.

Theorem 5.11 [Rai 6, Theorem 2.1] *Let $m \geq 3$. Suppose that $A \in L_{loc}^m(\mathbf{R}^m; \mathbf{R}^m)$, $V_+ \in L_{loc}^1(\mathbf{R}^m)$. Fix $\lambda \leq 0$ and assume that $(V + \lambda)_- \in L^{m/2}(\mathbf{R}^m)$. In addition, assume that exists an open set Ω_λ such that $V(X) + \lambda \leq 0$ if $X \in \Omega_\lambda$, and $V(X) + \lambda \geq 0$ if $X \notin \Omega_\lambda$. Then we have*

$$\lim_{h \downarrow 0} h^m N(-\lambda; H_{h,1,1}(A, V)) = \omega_m \int_{\mathbf{R}^m} (V + \lambda)_-^{m/2} dX / (2\pi)^m. \quad (5.61)$$

We omit the proof of Theorem 5.11 since is quite the the same as the proof of Theorem 5.1.

Asymptotic formulae which are formally identical to (5.61) can be found in [Ale] and [Com.Sc.Sei]. However, in [Ale] it is supposed that $A \in L^m(\mathbf{R}^m)^m$ while in the hypotheses of Theorem 5.11 we assume just the validity of the *local* condition $A \in L_{loc}^m(\mathbf{R}^m)^m$ without imposing *any* restrictions on the behaviour of A at infinity. On the other hand, the authors of [Com.Sc.Sei] assume that the resolvent of the operator $H(A, V)$ is compact, while the hypotheses of Theorem 5.11 imply the discreteness of the spectrum of $H(A, V)$ only below the point $\lambda \leq 0$.

Next, we deal with constant magnetic fields B and electric potentials V which decay slowly at infinity, i.e. satisfy the condition $-V \in \mathcal{D}_{-\alpha}^+$ with $\alpha \in (0, 2]$ (see §4.1.2.)

Theorem 5.12 [Rai 6, Theorem 2.4] *Assume that (2.26) holds and $-V \in \mathcal{D}_{-\alpha,1}^+$ with $\alpha \in (0, 2]$. If $k \equiv \dim \text{Ker } B = 0$ and $\alpha \neq 2$, assume in addition that the function $\Psi_{-V}(s)$ satisfies the condition \mathcal{T}_0 as $s \downarrow 0$. Then we have*

$$N(0; H_{h,1,1}(A, V)) = h^{-m/2} \nu_5(h^{-1})(1 + o(1)), h \downarrow 0. \quad (5.62)$$

Results related to Theorem 5.12 can be found in [Sob 5] and [Moh 2].

We omit the proof of Theorem 5.12 since it is quite similar as the proof of Theorem 5.3.

5.5.3. In this subsection we present some typical results on the semiclassical analysis of the Aharonov-Bohm effect for a bound state (see §2.2.3.) Let the manifold M , the potentials (A, V) and the operator $H_M(A, V)$ be the same as at the beginning of §2.2.1. For $h > 0$ and $\mu \in \mathbf{R}$ set

$$H_{h,\mu;M} := h^2 H_M(h^{-1}\mu A, h^{-2}V) \equiv (ih\nabla + \mu A)^2 + V.$$

Assume that the lower bound of the spectrum of $H_{h,\mu;M}$ is an isolated eigenvalue denoted by $\lambda_\mu \equiv \lambda_\mu(h) \equiv \lambda_\mu(h; V)$. Denote by $u_{h,\mu}$ any fixed normalized eigenfunction corresponding to λ_μ . As we saw in Corollary 2.2, the inequality $\lambda_\mu - \lambda_0 \geq 0$ holds for each $\mu \in \mathbf{R}$.

For simplicity assume that the curvature of M is identically zero, i.e. the Riemannian metric coincides with the Euclidean one. Suppose that $B \equiv \text{curl } A = 0$. Moreover, assume that the potential V has a unique nondegenerate minimum in M coinciding with a domain $\Omega \subset \mathbf{R}^2$ taken at some point $\tilde{X} \in M$; for definiteness we assume $V(\tilde{X}) = 0$. Introduce the Agmon metric $V_+ dX^2$ where dX^2 is the Euclidean metric in \mathbf{R}^2 , and define the Agmon distance $d_V(X, Y)$ as the infimum of the Agmon lengths of all piecewise C^1 -smooth paths connecting the points X and Y in M .

Then the rough estimate

$$\lambda_\mu(h) - \lambda_0(h) \leq C \exp\{-\delta h\}, \delta > 0, \quad (5.63)$$

can be established by means of the following simple argument.

First, note that the identity

$$\begin{aligned} & \int_M \left\{ |(ih\nabla + A)(\exp(\Phi/h)u)|^2 + (V - |\nabla\Phi|^2) |\exp(\Phi/h)u|^2 \right\} dX = \\ & = \text{Re} (\exp(2\Phi/h) H_{h,\mu;M} u, u), \forall u \in C_0^\infty(M), \forall \Phi = \bar{\Phi} \in \text{Lip}(M), \end{aligned}$$

entails the estimate

$$\| \exp\{(1 - \varepsilon)d_V(X, \tilde{X})\} u_{h,M} \|_{W_2^1(M)} \leq C, \varepsilon > 0.$$

Further, let $R > 0$ satisfy $\mathbf{B}(\tilde{X}; R) \subset M$. Choose $\varphi \in C_0^\infty(M)$ so that $\text{supp } \varphi \subset \mathbf{B}(\tilde{X}; R)$ and $\varphi \equiv 1$ on $\mathbf{B}(\tilde{X}; R/2)$. Let Φ be a real-valued function such that $\nabla\Phi = A$ in $\mathbf{B}(\tilde{X}; R)$. Then there exists a $\delta > 0$ satisfying

$$\| (H_{h,0;M} - \lambda_\mu(h)) e^{i\mu\Phi} \varphi u_{h,\mu} \| \leq C e^{-\delta h},$$

$$\| (H_{h,\mu;M} - \lambda_0(h)) e^{-i\mu\Phi} \varphi u_{h,0} \| \leq C e^{-\delta h}.$$

So, it is easy to see that $\lambda_\mu(h), \mu \in \mathbf{R}$, is equal to $h\hat{\Lambda}_1 + o(h)$, $h \downarrow 0$, $\hat{\Lambda}_1$ being the first eigenvalue of the harmonic oscillator $-\Delta + \frac{1}{2} \sum_{j,k=1}^2 (\partial_j \partial_k V(\tilde{X})) X_j X_k$ defined in $L^2(T_{\tilde{X}}^* M)$, and the distance from the rest of the spectrum is not less than Ch . Combining the results above, we come to (5.63).

In the two examples below we show that under some additional assumptions we can obtain an asymptotic expansion of $\lambda_\mu(h) - \lambda_0(h)$ as $h \downarrow 0$, thus providing more precise semiclassical analysis of the Aharonov-Bohm effect.

In the first example we assume that M is the unit circle \mathbf{S}^1 , and A is constant. Then the spectrum of $H_{h,M;\mathbf{S}^1}(A, 0)$ coincides with the set $\cup_{k \in \mathbf{Z}} (hk - \mu A)^2$. Thus we find that $\lambda_\mu(h; 0) = \lambda_0(h; 0)$ if and only if $\mu A/h \in \mathbf{Z}$.

Theorem 5.13 [Hel, Theorem 7.2.2.1] *Under the preceding hypotheses we have*

$$\lambda_\mu(h; V) - \lambda_0(h; V) =$$

$$[1 - \cos 2\pi\mu A/h] h^{1/2} |a(h)| \exp(-S_0/h) + O(\exp\{-(S_0 + \varepsilon_0)/h\}), h \downarrow 0,$$

where $S_0 = \int_0^{2\pi} V^{1/2} dX$, the number ε_0 is positive, $a(h) \sim \sum_{j \geq 0} a_j h^j$ as $h \downarrow 0$ with $a_0 > 0$, and the order relation is uniform with respect to μ .

The proof is an elementary application of the asymptotic expansion of the first eigenvalue of the Schrödinger operator defined on $W_2^2(0, 2\pi)$ with quasiperiodic boundary conditions obtained by Outassourt in [Out] (see [Hel, Sects. 6.2 and 7.2.2].)

In the second example we take $M \equiv \Omega := \{X \in \mathbf{R}^2 : |X| > \delta\}$ with some $\delta \in (0, 1)$, and $V(X) \rightarrow \infty$ as $|X| \rightarrow \infty$. Besides, for definiteness, we suppose $\tilde{X} = (1, 0)$. Finally, we assume for simplicity that the function V is even with respect to the variable X_2 . Put $S_0 = d_V(\tilde{X}, \partial\Omega)$ and $S_1 = \inf_{\gamma \in \Gamma} \{\text{Agmon length of } \gamma\}$ where Γ is the set of all closed paths in Ω which contain the point \tilde{X} and are not homotopic to a point. We assume for simplicity that the minimum S_1 is taken along a unique path γ_1 which is nondegenerate in the classical sense (see [Hel, p.54]). Our main geometrical assumption is that the inequality $S_1 < 2S_0$ holds; this means that the effect of the boundary is weak because V creates a barrier between \tilde{X} and $\partial\Omega$. This geometrical condition is expressed analytically by the inequality $0 < \partial^2 d_V(X, Y) / \partial X_1^2|_{X=\tilde{Y}}$ where \tilde{Y} is the intersection point of γ_1 with the semiaxis $\{X_2 = 0, X_1 < 0\}$.

Set $\Phi(h) = (2\pi h)^{-1} \int_{\partial\Omega} \mathcal{A}(A)$. Making use of Theorem 2.6, we find that $\lambda_\mu(h; V) - \lambda_0(h; V)$ is strictly positive if and only if $\mu\Phi \notin \mathbf{Z}$.

Theorem 5.14 [Hel, Theorem 7.2.2.2] *Under the preceding hypotheses we have*

$$\lambda_\mu(h; V) - \lambda_0(h; V) =$$

$$[1 - \cos 2\pi\mu\Phi(h)] h^{1/2} |a(h)| \exp(-S_1/h) + O(\exp\{-(S_1 + \varepsilon_0)/h\}), h \downarrow 0,$$

where ε_0 is positive, $a(h)$ admits the asymptotic expansion $\sum_{j \geq 0} a_j h^j$ as $h \downarrow 0$ with $a_0 > 0$, and the order relation is uniform with respect to μ .

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