

On the spectrum of a translationally invariant Pauli operator

January 31, 2008

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To Mikhail Shlëmovich Birman on his 80-th birthday

ABSTRACT. We consider a class of translationally invariant 3D Pauli operators \mathbf{G} with magnetic fields with circular integral curves. We show that the spectrum of \mathbf{G} is purely absolutely continuous, coincides with $[0, \infty)$, and has an infinite multiplicity.

1. Introduction

In this article we consider the Pauli operator

$$\mathbf{G} := \left(\sum_{j=1,2,3} \hat{\sigma}_j \left(-i \frac{\partial}{\partial x_j} - A_j \right) \right)^2 = (i\nabla + A)^2 I_2 - \sum_{j=1,2,3} B_j \hat{\sigma}_j$$

self-adjoint in $L^2(\mathbb{R}^3)^2$. Here

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are the Pauli matrices, I_2 is the unit 2×2 -matrix, $A = (A_1, A_2, A_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the magnetic potential, and $B = (B_1, B_2, B_3) := \text{curl } A$ is the magnetic field. The operator \mathbf{G} is the Hamiltonian of a three-dimensional $\frac{1}{2}$ -spin non-relativistic quantum particle subject to the magnetic field B .

Let (r, θ, x_3) be the standard cylindrical coordinates in \mathbb{R}^3 . We assume that the magnetic potential A has the form

$$(1.1) \quad A = (0, 0, a), \quad a = a(r).$$

Then we have

$$(1.2) \quad B = a'(r)(\sin \theta, -\cos \theta, 0),$$

i.e. $|B| = |a'(r)|$, and the integral curves of the magnetic field are circles in the planes orthogonal to the Ox_3 axis, centered at the origin.

Applying a partial Fourier transform with respect to x_3 , a decomposition into a Fourier series with respect to θ , and a unitary transform in the spin space \mathbb{C}^2 , we find that the operator \mathbf{G} is unitarily equivalent to

$$(1.3) \quad \bigoplus_{m \in \mathbb{Z}} \int_{\mathbb{R}}^{\oplus} G_m(k) dk$$

where the operator

$$G_m(k) := \begin{pmatrix} H_m(k) - \frac{2m-1}{2r^2} + a' & \\ & H_m(k) - \frac{2m-1}{2r^2} - a' \end{pmatrix}, \quad k \in \mathbb{R}, \quad m \in \mathbb{Z},$$

with

$$H_m(k) := -\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + \frac{m^2}{r^2} + (a(r) - k)^2,$$

is self-adjoint in $L^2(\mathbb{R}_+; r dr)^2$. Assuming that $a \rightarrow \infty$ and $a' = o(a^2)$ as $r \rightarrow \infty$, we find that the spectrum of $G_m(k)$ is discrete, and study the asymptotics as $k \rightarrow \pm\infty$ of its eigenvalues $\mu_{n,m}(k)$ whose graphics as functions of k are known as dispersion curves. First, we check that $\mu_{n,m}(k) \rightarrow \infty$ as $k \rightarrow -\infty$ for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ (see Proposition 4.1). Further, we show that under mild assumptions $\mu_{1,m}(k) \rightarrow 0$ as $k \rightarrow \infty$, $m \in \mathbb{Z}$, even if $a'(r)$ grows (super)exponentially fast as $r \rightarrow \infty$ (see Proposition 4.2). Consequently, the spectrum $\sigma(\mathbf{G})$ of the operator \mathbf{G} is absolutely continuous,

$$(1.4) \quad \sigma(\mathbf{G}) = [0, \infty),$$

and the multiplicity of $\sigma(\mathbf{G})$ is infinite (see Theorem 2.1).

Investigation of the dispersion curves for various translationally invariant quantum Hamiltonians with absolutely continuous spectrum could be found in [11], [8], [5], [9, Chapter 4], [15], [16]. In the recent preprint [16] the 3D magnetic Schrödinger operator $\mathbf{H} := (i\nabla + A)^2$ with potential of the form (1.1) has been considered. It was shown that $\sigma(\mathbf{H}) = [0, \infty)$ if $a'(r) \rightarrow 0$ as $r \rightarrow \infty$, but if $a'(r)$ tends to a positive finite limit or to infinity as $r \rightarrow \infty$, then $\sigma(\mathbf{H}) = [\mathcal{E}, \infty)$ with $\mathcal{E} > 0$. This phenomenon is related to the diamagnetic inequality valid for Schrödinger operators, while the stability of the zero infimum of $\sigma(\mathbf{G})$ is coherent with the paramagnetic effects typical for the Pauli operators (see [7] for a detailed and rigorous discussion of the dia- and paramagnetism for nonhomogeneous magnetic fields).

It is known that equality (1.4) holds true for a large class of magnetic fields, not necessarily generated by magnetic potentials of the form (1.1). For example, the results of [10] imply that (1.4) is valid for magnetic fields satisfying the following condition. For $x \in \mathbb{R}^3$ set

$$\epsilon_0(x) := |B(x)|, \quad \epsilon_s(x) := \frac{\sum_{|\alpha|=s} |D^\alpha B(x)|}{1 + \sum_{|\alpha|<s} |D^\alpha B(x)|}, \quad s \in \mathbb{N}.$$

Suppose that for some $s \geq 0$, in \mathbb{R}^3 there exists an infinite sequence of disjoint balls whose radii tend to infinity, such that the restriction of the function $\epsilon_s(x)$ onto the union of these balls tends to zero as $|x| \rightarrow \infty$. Then (1.4) holds true. More precisely, the authors of [10] considered the 3D Dirac operator \mathbf{D} self-adjoint in $L^2(\mathbb{R}^3; \mathbb{C}^4)$, and showed that

$$(1.5) \quad \sigma(\mathbf{D}) = \mathbb{R} \setminus (-1, 1).$$

Since we have

$$\mathbf{D}^2 = \begin{pmatrix} (\mathbf{G} + I)I_2 & 0 \\ 0 & (\mathbf{G} + I)I_2 \end{pmatrix},$$

(1.5) implies (1.4).

Note that the magnetic field B in (1.2) does not satisfy the hypotheses of [10] if $a'(r)$ grows exponentially or faster as $r \rightarrow \infty$, but nevertheless (1.4) holds true by our Theorem 2.1.

The paper is organized as follows. In Section 2 we give a precise formulation of our main result, Theorem 2.1, and describe simple sufficient conditions which

guarantee the validity of its assumptions. In Section 3 we define the self-adjoint operator \mathbf{G} , and prove that it is unitarily equivalent to the operator defined in (1.3). In Section 4 we prove Theorem 2.1 following, as explained above, the scheme based on the asymptotic analysis of the eigenvalues $\mu_{n,m}(k)$ as $k \rightarrow \pm\infty$.

I dedicate this article to my teacher Professor Mikhail Birman on the occasion of his 80th birthday. Among his numerous works which have strongly influenced the present paper, I would like to mention especially [1], [2], and [3], where the absolute continuity of the spectrum of various magnetic quantum Hamiltonians with periodic coefficients was proved.

2. Main Result

The main result of the article is the following

THEOREM 2.1. *Let A have the form (1.1). Assume*

$$(2.1) \quad a(r) \rightarrow \infty \quad \text{as } r \rightarrow \infty,$$

$$(2.2) \quad a' \in L_{\text{loc}}^s([0, \infty); r dr), \quad s > 1,$$

$$(2.3) \quad a'(r) = o(a(r)^2), \quad r \rightarrow \infty.$$

Suppose moreover that for each $R > 0$ we have

$$(2.4) \quad \lim_{k \rightarrow \infty} e^{Rk} I(k)^{-1} = 0$$

where

$$(2.5) \quad I(k) := \int_0^\infty e^{-2 \int_0^r a(s) ds + 2kr} dr, \quad k > 0.$$

Then the spectrum $\sigma(\mathbf{G})$ of the operator \mathbf{G} is purely absolutely continuous, (1.4) holds true, and the multiplicity of $\sigma(\mathbf{G})$ is infinite.

Let us discuss briefly the assumptions of Theorem 2.1. First, (2.2) implies that a is absolutely continuous on $\mathbb{R}_+ := (0, \infty)$, and, hence, $a \in L_{\text{loc}}^\infty(\mathbb{R}_+)$. Consequently, (2.3) entails $a' \in L_{\text{loc}}^\infty([R, \infty))$ for $R > 0$ large enough. Further, it follows from (2.2) that a satisfies the estimate

$$|a(r)| = \begin{cases} O(r^{(s-2)/s}), & s \in (1, 2), \\ O(|\ln r|^{1/2}), & s = 2, \\ O(1), & s > 2, \end{cases} \quad r \downarrow 0.$$

Therefore,

$$a \in L_{\text{loc}}^1([0, \infty)) \cap L_{\text{loc}}^2([0, \infty); r dr).$$

In particular, the integral $\int_0^r a(s) ds$ occurring in (2.5) is well-defined.

Evidently, (2.1) and (2.3) are satisfied by a very large class of functions, including, for example, functions which are asymptotically equivalent to positive powers of r , or exponentials of positive powers of r , or finite iterations of exponential functions. The following proposition contains a sufficient condition for the validity of (2.4).

We will say that a given non-negative function $\tau : [0, \infty) \rightarrow [0, \infty)$ belongs to the class \mathcal{K} if:

- $\tau' \in L_{\text{loc}}^s([0, \infty); r dr)$, $s > 1$;
- τ' is continuous and monotonously increasing on $[R_0, \infty)$ for some $R_0 \geq 0$;
- $\tau'(r) \rightarrow \infty$ as $r \rightarrow \infty$;

- for each $R > 0$ there exists $r_0 > 0$ such that

$$(2.6) \quad (R - 2\rho_k)k + 2 \int_0^{\rho_k} \tau(s)ds + \frac{1}{2} \ln \tau'(\rho_k + r_0) \rightarrow -\infty, \quad k \rightarrow \infty,$$

where $\rho_k := \tau^{-1}(k)$ for $k > 0$ large enough.

For example, all non-negative functions τ with $\tau' \in L^s_{\text{loc}}([0, \infty); r dr)$, $s > 1$, which for large $r > 0$ are proportional to e^{r^γ} , $\gamma > 0$, or to finite iterations of exponential functions $\exp(\exp(\dots(\exp r)))$, are in the class \mathcal{K} .

PROPOSITION 2.1. *Suppose that there exists a function $\tau \in \mathcal{K}$ such that*

$$(2.7) \quad a(r) \leq \tau(r), \quad r \in [0, \infty).$$

Then (2.4) holds true.

The proof of Proposition 2.1 is contained in Subsection 4.4.

3. Representation of the Hamiltonians as direct integrals of operators with discrete spectra

Assume $A \in L^2_{\text{loc}}(\mathbb{R}^3)^3$, $B = \text{curl } A \in L^1_{\text{loc}}(\mathbb{R}^3)^3$. Let \mathbf{G} be the self-adjoint operator generated in $L^2(\mathbb{R}^3)^2$ by the closure of the quadratic form

$$g[\mathbf{u}] := \int_{\mathbb{R}^3} \left| \sum_{j=1,2,3} \hat{\sigma}_j \left(i \frac{\partial}{\partial x_j} + A_j \right) \mathbf{u} \right|^2 dx = \sum_{l=1,2} \int_{\mathbb{R}^3} |(i\nabla + A)u_l|^2 dx - \sum_{j=1,2,3} \int_{\mathbb{R}^3} B_j \langle \hat{\sigma}_j \mathbf{u}, \mathbf{u} \rangle dx, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in C_0^\infty(\mathbb{R}^3)^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{C}^2 . Assume now that $A = (0, 0, a)$ with $a = a(x_1, x_2)$ satisfying $a \in L^2_{\text{loc}}(\mathbb{R}^2)$, $\nabla a \in L^1_{\text{loc}}(\mathbb{R}^2)^2$. Evidently, in this case $C_0^\infty(\mathbb{R}^2_{x_1, x_2}; \mathcal{S}(\mathbb{R}_{x_3}))^2$ where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz class, is also a form core for the operator \mathbf{G} .

Further, for $k \in \mathbb{R}$ introduce the quadratic form

$$g_k[\mathbf{u}] := \int_{\mathbb{R}^2} \left| i \sum_{j=1,2} \hat{\sigma}_j \frac{\partial \mathbf{u}}{\partial x_j} + \hat{\sigma}_3 (a - k) \mathbf{u} \right|^2 dx = \int_{\mathbb{R}^2} \left(\left| 2i \frac{\partial u_2}{\partial z} + (a - k)u_1 \right|^2 + \left| 2i \frac{\partial u_1}{\partial \bar{z}} - (a - k)u_2 \right|^2 \right) dx = \int_{\mathbb{R}^2} \left(\sum_{l=1,2} (|\nabla u_l|^2 + (a - k)^2 |u_l|^2) - 4 \text{Re } i \frac{\partial a}{\partial z} u_2 \bar{u}_1 \right) dx, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in C_0^\infty(\mathbb{R}^2)^2,$$

where $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$. Denote by $G(k)$ the self-adjoint operator generated in $L^2(\mathbb{R}^2)$ by the closure of g_k , $k \in \mathbb{R}$. Note that $C_0^\infty(\mathbb{R}^2 \setminus \{0\})^2$ is also a form core for the operator $G(k)$.

Let \mathcal{F} be the partial Fourier transform with respect to x_3 , i.e.

$$(\mathcal{F}u)(x_1, x_2, k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix_3 k} u(x_1, x_2, x_3) dx_3.$$

Since $C_0^\infty(\mathbb{R}_{x_1, x_2}^2; \mathcal{S}(\mathbb{R}_{x_3}))$ is invariant with respect to \mathcal{F} , we have

$$(3.1) \quad \mathcal{F}G\mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} G(k)dk.$$

Further, assume that a has the form (1.1), and satisfies (2.2). For $m \in \mathbb{Z}$ define the matrix

$$M_m := \frac{1}{2} \begin{pmatrix} m^2 + (m-1)^2 & m^2 - (m-1)^2 \\ m^2 - (m-1)^2 & m^2 + (m-1)^2 \end{pmatrix}.$$

Note that M_m has eigenvalues m^2 and $(m-1)^2$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. For $k \in \mathbb{R}$ and $m \in \mathbb{Z}$ introduce the quadratic form

$$g_{k,m}[\mathbf{w}] := \sum_{l=1,2} \int_0^\infty (|w'_l|^2 + (a-k)^2 |w_l|^2 - (-1)^l a' |w_l|^2) r dr + \int_0^\infty \langle M_m \mathbf{w}, \mathbf{w} \rangle r^{-1} dr$$

defined originally on $C_0^\infty(\mathbb{R}_+)^2$, and then closed in $L_2(\mathbb{R}_+; r dr)^2$. Let $G_m(k)$ be the self-adjoint operator generated in $L^2(\mathbb{R}_+; r dr)^2$ by $g_{k,m}$.

PROPOSITION 3.1. *Let a satisfy (2.2). Then the operator $G(k)$, $k \in \mathbb{R}$, is unitarily equivalent to the orthogonal sum*

$$(3.2) \quad \bigoplus_{m \in \mathbb{Z}} G_m(k).$$

PROOF. Pick $\mathbf{u} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})^2$. Passing to polar coordinates (r, θ) , and setting $\mathbf{v}(r, \theta) := \mathbf{u}(r \cos \theta, r \sin \theta)$, we find that

$$(3.3) \quad g_k[\mathbf{u}] = \tilde{g}_k[\mathbf{v}], \quad \|\mathbf{u}\|_{L^2(\mathbb{R}_+; r dr)^2}^2 = \sum_{l=1,2} \int_0^\infty \int_0^{2\pi} |v_l(r, \theta)|^2 d\theta r dr,$$

where

$$\begin{aligned} \tilde{g}_k[\mathbf{v}] := & \sum_{l=1,2} \int_0^\infty \int_0^{2\pi} \left(\left| \frac{\partial v_l}{\partial r} \right|^2 + \frac{1}{r^2} \left| \frac{\partial v_l}{\partial \theta} \right|^2 + (a-k)^2 |v_l|^2 \right) d\theta r dr + \\ & \int_0^\infty \int_0^{2\pi} a'(r) \langle \mathcal{M}(\theta) \mathbf{v}, \mathbf{v} \rangle d\theta r dr, \end{aligned}$$

and

$$\mathcal{M}(\theta) = \begin{pmatrix} 0 & -ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}.$$

The eigenvalues of the matrix $\mathcal{M}(\theta)$ are equal to 1 and -1 , and the corresponding orthonormal eigenvectors can be chosen as

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -ie^{-i\theta} \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} ie^{-i\theta} \\ 1 \end{pmatrix}.$$

Set

$$\mathcal{U} := \frac{1}{\sqrt{2}} \begin{pmatrix} -ie^{-i\theta} & ie^{-i\theta} \\ 1 & 1 \end{pmatrix}.$$

Then

$$\mathcal{U}^* = \mathcal{U}^{-1} := \frac{1}{\sqrt{2}} \begin{pmatrix} ie^{i\theta} & 1 \\ -ie^{i\theta} & 1 \end{pmatrix}, \quad \mathcal{U}^* \mathcal{M}(\theta) \mathcal{U} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Set $\mathbf{w} := \mathcal{U}^* \mathbf{v}$. We have

$$\begin{aligned} \tilde{g}_{k,m}[\mathbf{v}] &= \sum_{l=1,2} \int_0^\infty \int_0^{2\pi} \left(\left| \frac{\partial w_l}{\partial r} \right|^2 + (a-k)^2 |w_l|^2 - (-1)^l a' |w_l|^2 \right) d\theta r dr + \\ (3.4) \quad & \frac{1}{2} \int_0^\infty \int_0^{2\pi} \left(\left| i \frac{\partial w_1}{\partial \theta} + w_1 - i \frac{\partial w_2}{\partial \theta} - w_2 \right|^2 + \left| \frac{\partial w_1}{\partial \theta} + \frac{\partial w_2}{\partial \theta} \right|^2 \right) d\theta r^{-1} dr. \end{aligned}$$

Decomposing \mathbf{w} into a Fourier series

$$\mathbf{w}(r, \theta) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} \mathbf{w}_m(r) e^{im\theta},$$

with $\mathbf{w}_m(r) = \begin{pmatrix} w_{m,1}(r) \\ w_{m,2}(r) \end{pmatrix}$, $m \in \mathbb{Z}$, and bearing in mind (3.3) - (3.4), we find that

$$g_k[\mathbf{u}] = \sum_{m \in \mathbb{Z}} g_{k,m}[\mathbf{w}_m], \quad \|\mathbf{u}\|_{L^2(\mathbb{R}^2)^2}^2 = \sum_{m \in \mathbb{Z}} \|\mathbf{w}_m\|_{L^2(\mathbb{R}_+; r dr)^2}^2, \quad k \in \mathbb{R},$$

which implies the unitary equivalence of $G(k)$ and the orthogonal sum in (3.2). \square

Combining (3.1) with Proposition 3.1, we find that \mathbf{G} is unitarily equivalent to the operator defined in (1.3).

Suppose now that (2.1) - (2.3) hold true. Then for each $m \in \mathbb{Z}$ and $k \in \mathbb{R}$, the spectrum of the operator $G_m(k)$ is discrete. Moreover, since the quadratic form $\int_0^\infty a|u|^2 r dr$ is relatively compact with respect to the closed quadratic form $g_{0,m}[u]$, we find easily that $\{G_m(k)\}_{k \in \mathbb{R}}$ is a Kato analytic family of type (B) (see [12, Chapter VII, Theorem 4.8]). Let $\{\mu_{n,m}(k)\}_{n \in \mathbb{N}}$ be the non-increasing sequence of the eigenvalues of the operator $G_m(k)$, $m \in \mathbb{Z}$, $k \in \mathbb{R}$. Evidently, for each $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ we have $\mu_{n,m} \in C(\mathbb{R})$.

4. Proof of the main result

4.1. In this subsection we investigate the asymptotic behaviour as $k \rightarrow -\infty$ of the eigenvalues $\mu_{n,m}(k)$, $n \in \mathbb{N}$, $m \in \mathbb{Z}$.

Let $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\beta : \mathbb{R}^2 \rightarrow \mathbb{R}$, satisfy

$$(4.1) \quad \alpha_+ \in L_{\text{loc}}^2(\mathbb{R}^2), \alpha_- \in L^2(\mathbb{R}^2), (\alpha^2 - \beta)_+ \in L_{\text{loc}}^1(\mathbb{R}^2), (\alpha^2 - \beta)_- \in L^s(\mathbb{R}^2), s > 1.$$

Then for each $p \geq 0$ the quadratic form

$$(4.2) \quad \int_{\mathbb{R}^2} (|\nabla u|^2 + ((p + \alpha)^2 - \beta) |u|^2) dx, \quad u \in C_0^\infty(\mathbb{R}^2),$$

is lower bounded in $L^2(\mathbb{R}^2)$. Denote by $\mathcal{L}(p)$ the self-adjoint operator generated in $L^2(\mathbb{R}^2)$ by the closure of (4.1).

LEMMA 4.1. *Assume that α and β satisfy (4.1). Then we have*

$$(4.3) \quad \liminf_{p \rightarrow \infty} p^{-2} \inf \sigma(\mathcal{L}(p)) \geq 1.$$

PROOF. If α is essentially lower-bounded, then (4.3) is trivial. Assume that α is not essentially lower-bounded, and pick an arbitrary $t > 0$. Then we have

$$(p + \alpha)^2 - \beta \geq p^2 - 2pt - 2p(\alpha + t)_- - (\alpha^2 - \beta)_-.$$

Hence,

$$(4.4) \quad \int_{\mathbb{R}^2} (|\nabla u|^2 + ((p + \alpha)^2 - \beta) |u|^2) dx \geq \int_{\mathbb{R}^2} (|\nabla u|^2 + ((p^2 - 2pt) - 2p(\alpha + t)_- - (\alpha^2 - \beta)_-) |u|^2) dx, \quad u \in C_0^\infty(\mathbb{R}^2).$$

Denote by $\mathcal{L}_1 = \mathcal{L}_1(p, t)$ the self-adjoint operator generated in $L^2(\mathbb{R}^2)$ by the closure of the quadratic form

$$\int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla u|^2 - 2p(\alpha + t)_- |u|^2 \right) dx, \quad u \in C_0^\infty(\mathbb{R}^2),$$

and by \mathcal{L}_2 the self-adjoint operator generated in $L^2(\mathbb{R}^2)$ by the closure of the quadratic form

$$\int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla u|^2 - (\alpha^2 - \beta)_- |u|^2 \right) dx, \quad u \in C_0^\infty(\mathbb{R}^2).$$

The essential spectra of \mathcal{L}_j , $j = 1, 2$, coincide with $[0, \infty)$, while their discrete spectra are finite. Moreover, for every fixed $t > 0$ and sufficiently large $p > 0$, the negative spectrum of $\mathcal{L}_1(p, t)$ is not empty. Let $\{-\lambda_j(p, t)\}_{j=1}^N$ be the negative eigenvalues of $\mathcal{L}_1(p, t)$ counted with the multiplicities and enumerated in a non-decreasing order. Then (4.4) implies

$$(4.5) \quad \inf \sigma(\mathcal{L}(p)) \geq p^2 - 2pt - \lambda_1(p, t) + \inf \sigma(\mathcal{L}_2).$$

On the other hand, the well known Lieb-Thirring inequality (see [13]) yields

$$(4.6) \quad \lambda_1(p, t) \leq \sum_{j=1}^N \lambda_j(p, t) \leq Cp^2 \int_{\mathbb{R}^2} (\alpha + t)_-^2 dx$$

with $C > 0$ independent of p , α , and t . Combining (4.5) and (4.6), we get

$$(4.7) \quad \liminf_{p \rightarrow \infty} p^{-2} \inf \sigma(\mathcal{L}(p)) \geq 1 - C \int_{\mathbb{R}^2} (\alpha + t)_-^2 dx.$$

Since $\alpha_- \in L^2(\mathbb{R}^2)$, we have $\lim_{t \rightarrow \infty} \int_{\mathbb{R}^2} (\alpha + t)_-^2 dx = 0$, and (4.3) follows from (4.7). \square

Remark: We formulate Lemma 4.1 in a form suitable for our purposes. A related result is contained in [16, Proposition 3.2].

PROPOSITION 4.1. *Let $n \in \mathbb{N}$, $m \in \mathbb{Z}$. Assume that (2.1) – (2.3) hold. Then we have*

$$(4.8) \quad \mu_{n,m}(k) = k^2(1 + o(1)), \quad k \rightarrow -\infty.$$

PROOF. Evidently, $\mu_{n,m}(k) \geq \inf \sigma(G(k))$, $k \in \mathbb{R}$. On the other hand, for $k \leq 0$ we have $G(k) \geq \tilde{H}(k)I_2$ where $\tilde{H}(k) := \mathcal{L}(-k)$ with $\alpha = a$ and $\beta = |a'|$. Then (4.3) entails

$$(4.9) \quad \liminf_{k \rightarrow -\infty} k^{-2} \mu_{n,m}(k) \geq 1.$$

For $\varepsilon > 0$ and $m \in \mathbb{Z}$ denote by $Q_m(\varepsilon)$ the self-adjoint operator generated in $L^2(\mathbb{R}_+; r dr)$ by the closure of the quadratic form

$$\int_0^\infty \left(|u'|^2 + \left(\frac{m^2}{r^2} + (1 + \varepsilon^{-1})a^2 + |a'| \right) |u|^2 \right) r dr, \quad u \in C_0^\infty(\mathbb{R}_+).$$

The spectrum of $Q_m(\varepsilon)$ is discrete and simple; let $\{\nu_{n,m}(\varepsilon)\}_{n \in \mathbb{N}}$ be the increasing sequence of its eigenvalues. It is easy to check that

$$G_m(k) \leq (Q_{m^*}(\varepsilon) + (1 + \varepsilon)k^2) I_2$$

where $m^* := \max\{|m|, |m - 1|\}$. By the minimax principle

$$\mu_{n,m}(k) \leq \nu_{\lfloor \frac{n-1}{2} \rfloor + 1, m^*}(\varepsilon) + (1 + \varepsilon)k^2,$$

and, hence,

$$\limsup_{k \rightarrow -\infty} k^{-2} \mu_{n,m}(k) \leq 1 + \varepsilon, \quad \forall \varepsilon > 0,$$

which combined with (4.9), yields (4.8). \square

4.2. In this subsection we study the behaviour of the eigenvalues $\mu_{1,m}(k)$, $m \in \mathbb{Z}$, as $k \rightarrow \infty$.

PROPOSITION 4.2. *Assume the hypotheses of Theorem 2.1. Then we have*

$$(4.10) \quad \lim_{k \rightarrow \infty} \mu_{1,m}(k) = 0, \quad m \in \mathbb{Z}.$$

PROOF. In order to prove (4.10), it suffices to show that for each $\varepsilon > 0$ and $k \in \mathbb{R}$ there exists $\mathbf{u}_{k,\varepsilon} \in D(g_{k,m})$, the domain of the closed quadratic form $g_{k,m}$, such that

$$(4.11) \quad \lim_{\varepsilon \downarrow 0} \limsup_{k \rightarrow \infty} \frac{g_{k,m}[\mathbf{u}_{k,\varepsilon}]}{\|\mathbf{u}_{k,\varepsilon}\|_{L^2(\mathbb{R}_+; r dr)}^2} = 0.$$

Set

$$\phi_k(r) := e^{-\int_0^r a(s) ds + kr}, \quad r > 0, \quad k \in \mathbb{R}.$$

By (2.1) – (2.3), the functions $\phi_k(r)$ and $\phi_k'(r) = -(a(r) - k)\phi_k(r)$ decay superexponentially as $r \rightarrow \infty$. Hence, $\phi_k \in L^2(\mathbb{R}_+)$ for each $k \in \mathbb{R}$ and we have

$$\int_0^\infty \phi_k(r)^2 dr = I(k), \quad k \in \mathbb{R},$$

the integral $I(k)$ being defined in (2.5). Moreover, ϕ_k satisfies the differential equation

$$(4.12) \quad -\phi_k'' + (a(r) - k)^2 - a'(r)\phi_k(r) = 0.$$

Further, pick a function $\zeta \in C^\infty([0, \infty))$ such that $\zeta(r) = 0$ if $r \in [0, 1/2]$, $0 \leq \zeta(r) \leq 1$ if $r \in [1/2, 1]$, and $\zeta(r) = 1$ if $r \in [1, \infty)$. Fix $\varepsilon > 0$ and for $r > 0$ set

$$\zeta_\varepsilon(r) := \zeta(\varepsilon^{1/2}r), \quad u_{k,\varepsilon}(r) := r^{-1/2}\zeta_\varepsilon(r)\phi_k(r), \quad \mathbf{u}_{k,\varepsilon}(r) := \begin{pmatrix} 0 \\ u_{k,\varepsilon}(r) \end{pmatrix}.$$

Since $\text{supp } u_{k,\varepsilon} \subseteq [R/2, \infty)$ where $R = R(\varepsilon) := \varepsilon^{-1/2}$, and

$$u_{k,\varepsilon} \in L^2(\mathbb{R}_+; (a^2 + r^{-2})r dr), \quad u_{k,\varepsilon}' \in L^2(\mathbb{R}_+; r dr),$$

we easily find that $\mathbf{u}_{k,\varepsilon} \in D(g_{k,m})$. Integrating by parts, and taking into account (4.12), we get

$$(4.13) \quad \begin{aligned} g_{k,m}[\mathbf{u}_{k,\varepsilon}] &= \int_0^\infty \phi_k(r)^2 \zeta_\varepsilon'(r)^2 dr + \frac{2m^2 + 2(m-1)^2 - 1}{4} \int_0^\infty \frac{\phi_k(r)^2 \zeta_\varepsilon(r)^2}{r^2} dr \\ &\leq \varepsilon \left(\max_{r \in \mathbb{R}_+} \zeta'(r)^2 \int_0^R \phi_k(r)^2 dr + (2m^2 + 2(m-1)^2 - 1) \int_{R/2}^\infty \phi_k(r)^2 dr \right) \\ &\leq \text{const. } \varepsilon I(k). \end{aligned}$$

On the other hand, we have

(4.14)

$$\|\mathbf{u}_{k,\varepsilon}\|_{L^2(\mathbb{R}_+; r dr)}^2 = \int_0^\infty \phi_k(r)^2 \zeta_\varepsilon(r)^2 dr \geq I(k) - \int_0^R \phi_k(r)^2 dr \geq I(k) - ce^{2kR}$$

with $c := \max_{r \in [0, R]} \exp(-2 \int_0^r a(s) ds)$, provided that $k \geq 1/2$. Combining (4.13) with (4.14), we obtain

$$(4.15) \quad \frac{g_{k,m}[\mathbf{u}_{k,\varepsilon}]}{\|\mathbf{u}_{k,\varepsilon}\|_{L^2(\mathbb{R}_+; r dr)}^2} \leq \frac{\text{const. } \varepsilon}{1 - ce^{2kR} I(k)^{-1}}.$$

Now (4.11) follows from (4.15) and (2.4). \square

Remark: The method of the proof of Proposition 4.2 was inspired by the supersymmetric model

$$S = \begin{pmatrix} LL^* & 0 \\ 0 & L^*L \end{pmatrix} := \begin{pmatrix} -\frac{d^2}{dx^2} + q^2 + q' & 0 \\ 0 & -\frac{d^2}{dx^2} + q^2 - q' \end{pmatrix}$$

considered in [6] (see also [4, Section 6.3]). Here $q : \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial, and the operators

$$L := \frac{d}{dx} + q, \quad L^* := -\frac{d}{dx} + q,$$

with appropriate (coinciding) domains are mutually adjoint in $L^2(\mathbb{R})$. Then we have

$$\text{Ker } S = \left\{ \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mid u_1 \in \text{Ker } L^*, u_2 \in \text{Ker } L \right\},$$

and the kernels $\text{Ker } L$ and $\text{Ker } L^*$ admit an explicit and simple description.

4.3. In this subsection we prove Theorem 2.1.

Since \mathbf{G} is unitarily equivalent to the operator defined in (1.3), the absolute continuity of $\sigma(\mathbf{G})$ will follow from the absolute continuity of $\sigma(G_m)$ where $G_m := \int_{\mathbb{R}}^{\oplus} G_m(k) dk$, $m \in \mathbb{Z}$. Recall that $\{G_m(k)\}_{k \in \mathbb{R}}$ is a Kato analytic family of type (B). By an appropriate version of the Rellich theorem (see [14, Theorem XII.13], [12, Chapter VII, Remark 4.22]), there exists a family of analytic functions $\mathbb{R} \ni k \mapsto \tilde{\mu}_{n,m}(k)$, $n \in \mathbb{N}$, such that the set of the eigenvalues of the operator $G_m(k)$, $k \in \mathbb{R}$, counted with the multiplicities, coincides with $\{\tilde{\mu}_{n,m}(k)\}_{n \in \mathbb{N}}$. By the general spectral theory of direct integrals of self-adjoint operators with discrete spectra (see e.g. [14, Theorem XIII.86]), it suffices to check that among $\{\tilde{\mu}_{n,m}(k)\}_{n \in \mathbb{N}}$ there are no functions which are identically constant. This, however, follows immediately from the inequality $\tilde{\mu}_{n,m}(k) \geq \mu_{1,m}(k)$, $n \in \mathbb{N}$, $k \in \mathbb{R}$, and the relation $\mu_{1,m}(k) \rightarrow \infty$ as $k \rightarrow -\infty$ implied by Proposition 4.1.

Further, since $\mathbf{G} \geq 0$, we have

$$(4.16) \quad \overline{\mu_{1,m}(\mathbb{R})} \subseteq \sigma(\mathbf{G}) \subseteq [0, \infty), \quad m \in \mathbb{Z}.$$

By Propositions 4.1 and 4.2 and the continuity of $\mu_{1,m}$, we have $\overline{\mu_{1,m}(\mathbb{R})} = [0, \infty)$ for each $m \in \mathbb{Z}$, so that (4.16) entails (1.4) and the infinite multiplicity of the spectrum of the operator \mathbf{G} .

4.4. In this subsection we prove Proposition 2.1.

Our argument will be close to the well known Laplace method for approximate evaluation of integrals depending on a large parameter.

Fix R and pick $r_0 > 0$ such that (2.6) holds true. Also, assume that k so large that $R_0 \leq \rho_k$, and hence τ' is continuous and monotonous on $[\rho_k, \infty)$. Then we have

$$I(k) \geq \int_0^\infty \exp(-2 \int_0^r \tau(s) ds + 2kr) dr \geq \int_{\rho_k}^{\rho_k+r_0} \exp(-2 \int_0^r \tau(s) ds + 2kr) dr.$$

A second order Taylor expansion of the function $-2 \int_0^r \tau(s) ds + 2kr$ at ρ_k , and the monotonicity of $\tau'(r)$ yield

$$\begin{aligned} -2 \int_0^r \tau(s) ds + 2kr &= -2 \int_0^{\rho_k} \tau(s) ds + 2k\rho_k - 2 \int_{\rho_k}^r \tau'(s)(r-s) ds \geq \\ &-2 \int_0^{\rho_k} \tau(s) ds + 2k\rho_k - \tau'(\rho_k+r_0)(r-\rho_k)^2, \quad r \in [\rho_k, \rho_k+r_0]. \end{aligned}$$

Therefore,

$$I(k) \geq \exp(-2 \int_0^{\rho_k} \tau(s) ds + 2k\rho_k) \int_{\rho_k}^{\rho_k+r_0} \exp(-\tau'(\rho_k+r_0)(r-\rho_k)^2) dr.$$

Changing the variable $r = \tau'(\rho_k+r_0)^{-1/2}s + \rho_k$, and taking into account that $\tau'(r) \rightarrow \infty$ as $r \rightarrow \infty$, we get

$$(4.17) \quad I(k) \geq \frac{\exp(-2 \int_0^{\rho_k} \tau(s) ds + 2k\rho_k)}{\tau'(\rho_k+r_0)^{1/2}} \left(\frac{\sqrt{\pi}}{2} + o(1) \right), \quad k \rightarrow \infty.$$

Bearing in mind (2.6), we find that (4.17) implies (2.4).

Acknowledgements. The author thanks D. R. Yafaev who drew his attention to the spectral properties of a special class of magnetic quantum Hamiltonians. Acknowledgements are due to László Erdős and Xue Ping Wang for useful e-mail exchanges. The financial support of the Chilean Science Foundation *Fondecyt* under Grant 1050716, and of the University of Rennes I during author's visits in 2005 and 2007 is gratefully acknowledged.

References

- [1] M. SH. BIRMAN, T. A. SUSLINA, *The two-dimensional periodic magnetic Hamiltonian is absolutely continuous* (Russian) *Algebra i Analiz* **9** (1997), 32–48; English translation in *St. Petersburg Math. J.* **9** (1998), 21–32.
- [2] M. SH. BIRMAN, T. A. SUSLINA, *Absolute continuity of a two-dimensional periodic magnetic Hamiltonian with discontinuous vector potential* (Russian) *Algebra i Analiz* **10** (1998), 1–36; English translation in *St. Petersburg Math. J.* **10** (1999), 579–601.
- [3] M. SH. BIRMAN, T. A. SUSLINA, *The periodic Dirac operator is absolutely continuous*, *Integral Equations Operator Theory* **34** (1999), 377–395.
- [4] H. CYCON, R. FROESE, W. KIRSCH, B. SIMON, *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Texts and Monographs in Physics, Springer-Verlag, Berlin, Heidelberg, New York, 1987.
- [5] S. DE BIÈVRE, J. V. PULÉ, *Propagating edge states for a magnetic Hamiltonian*, *Math. Phys. Electron. J.* **5** (1999), Paper 3, 17 pp.
- [6] P. A. DEIFT, *Applications of a commutation formula*, *Duke Math. J.* **45** (1978), 267–310.

- [7] L. ERDŐS, *Dia- and paramagnetism for nonhomogeneous magnetic fields*, J. Math. Phys. **38** (1997), 1289–1317.
- [8] V. GEĬLER, M. SENATOROV, *The structure of the spectrum of the Schrödinger operator with a magnetic field in a strip, and finite-gap potentials*, Mat. Sb. **188** (1997), 21–32 (Russian); English translation in Sb. Math. **188** (1997), 657–669.
- [9] B. HELFFER, *Introduction to Semi-Classical Methods for the Schrödinger Operator with Magnetic Field. Vienna Version*, Lecture notes of a course given at the ESI, 2006, available at <http://www.math.u-psud.fr/~helffer/syrievienne2006.pdf>.
- [10] B. HELFFER, J. NOURRIGAT, X. P. WANG, *Sur le spectre de l'équation de Dirac (dans \mathbb{R}^3 ou \mathbb{R}^2) avec champ magnétique*, Ann. Sci. École Norm. Sup. (4) **22** (1989), 515–533.
- [11] A. IWATSUKA, *Examples of absolutely continuous Schrödinger operators in magnetic fields*, Publ. Res. Inst. Math. Sci. **21** (1985), 385–401.
- [12] T. KATO, *Perturbation Theory for Linear Operators*, 2nd Edition, Die Grundlehren der mathematischen Wissenschaften, **132** Springer-Verlag New York, Inc., New York 1980.
- [13] E. H. LIEB, W. THIRRING, *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, In: Studies in Mathematical Physics (Essays in Honor of Valentine Bargmann), pp. 269–303. Princeton Univ. Press, Princeton, NJ, 1976.
- [14] M. REED, B. SIMON, *Methods of Modern Mathematical Physics, IV. Analysis of Operators*, Academic Press, New York, 1978.
- [15] D. YAFAEV, *A particle in a magnetic field of an infinite rectilinear current*, Math. Phys. Anal. Geom. **6** (2003), 219–230.
- [16] D. YAFAEV, *On spectral properties of translationally invariant magnetic Schrödinger operators*, ArXiv Preprint: math.SP 0705.2871.

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