Spectral Asymptotics for the Perturbed 2D Pauli Operator with Oscillating Magnetic Fields.
I. Non-Zero Mean Value of the Magnetic Field

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Abstract. We consider the Pauli operator $H(b,V)$ acting in $L^2(\mathbb{R}^2;\mathbb{C}^2)$. We describe a class of oscillating magnetic fields b for which the ground state of the unperturbed operator $H(b,0)$ which coincides with the origin, is an isolated eigenvalue of infinite multiplicity. Under the assumption that the matrix-valued electric potential V has a definite sign and decays at infinity, we investigate the asymptotic distribution of the discrete spectrum of $H(b,V)$ accumulating to the origin. We obtain different asymptotic formulae valid respectively in the cases of power-like decay of V, exponential decay of V, or compact support of V.

1 Introduction

In the present article we consider the two-dimensional Pauli operator $H(b,V) = H(b,0)+V$ which describes a quantum non-relativistic particle of spin $\frac{1}{2}$ subject to a magnetic field b and electric potential V. The unperturbed Pauli operator $H(b,0)$ given by

$$H(b,0) := \begin{pmatrix} (-i\nabla - \mathbf{A})^2 - b & 0 \\ 0 & (-i\nabla - \mathbf{A})^2 + b \end{pmatrix} := \begin{pmatrix} H_1(b) & 0 \\ 0 & H_2(b) \end{pmatrix}, \tag{1.1}$$

acts in $L^2(\mathbb{R}^2;\mathbb{C}^2)$, and is essentially self-adjoint on the Schwartz class $\mathcal{S}(\mathbb{R}^2;\mathbb{C}^2)$. Here $\mathbf{A} = (A_1, A_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is the magnetic potential, and

$$b := \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \tag{1.2}$$

is the magnetic field.
We will say that $b : \mathbb{R}^2 \to \mathbb{R}$ is an *admissible magnetic field* if $b = b_0 + \tilde{b}$ where $b_0 \in \mathbb{R}$ is a constant, and $\tilde{b}$ is such a function that the equation

$$\Delta\tilde{\varphi} = \tilde{b} \tag{1.3}$$

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admits a solution $\tilde{\varphi} \in C^2(\mathbb{R}^d)$ satisfying

$$\sup_{x\in\mathbb{R}^2} |D^\eta \tilde{\varphi}(x)| < \infty, \quad \eta \in \mathbb{Z}_+^2, \quad |\eta| \leq 2. \tag{1.4}$$

A similar class of magnetic fields has been considered in [Ber.Shu, Supplement 3].
Note that if there exists a solution $\tilde{\varphi}$ of (1.3) satisfying (1.4), then Liouville's theorem implies that it is unique up to an additive constant. In order to fix this constant, we suppose that, in addition to (1.3) – (1.4), $\tilde{\varphi}$ obeys

$$\limsup_{T\to\infty} T^{-2} \int_{\left(-\frac{T}{2},\frac{T}{2}\right)^2} \tilde{\varphi} dx = 0. \tag{1.5}$$

If b is admissible, then Green's formula and (1.4) with $|\eta| = 1$ imply

$$\lim_{T\to\infty} T^{-2} \int_{\tau+\left(-\frac{T}{2},\frac{T}{2}\right)^2} b dx = b_0, \quad \tau \in \mathbb{R}^2,$$

so that b_0 can be interpreted as the mean value of b.
Let b be an admissible magnetic field. Set $\varphi_0(x) := b_0|x|^2/4$, $x \in \mathbb{R}^2$, so that we have $\Delta\varphi_0 = b_0$. Pick $\tilde{\varphi}$ satisfying (1.3) – (1.5). Put $\varphi := \varphi_0 + \tilde{\varphi}$ so that $\Delta\varphi = b$, and $\mathbf{A} = (A_1, A_2) := \left(-\frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_1}\right)$. Hence, $\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = \Delta\varphi = b$, i.e. (1.2) holds. Introduce the operators

$$a = a(b) := -2ie^{-\varphi}\frac{\partial}{\partial \overline{z}}\, e^{\varphi}, \; a^* = a(b)^* := -2ie^{\varphi}\frac{\partial}{\partial z}\, e^{-\varphi}, \tag{1.6}$$

where $z := x_1 + ix_2$, $\overline{z} := x_1 - ix_2$. Then we have

$$H_1 = a^*a, \quad H_2 = aa^*, \tag{1.7}$$

where the operators $H_j = H_j(b)$, $j = 1, 2$, are introduced in (1.1). The operators a and a^* defined initially on the Schwartz class $\mathcal{S}(\mathbb{R}^2)$, and then closed in $L^2(\mathbb{R}^2)$, are mutually adjoint. Therefore,

$$\text{Ker } H_1 = \text{Ker } a = \left\{ f \in L^2(\mathbb{R}^2) | f = ge^{-\varphi}, \; \frac{\partial g}{\partial \overline{z}} = 0 \right\}, \tag{1.8}$$

$$\text{Ker } H_2 = \text{Ker } a^* = \left\{ f \in L^2(\mathbb{R}^2) | f = ge^{\varphi}, \; \frac{\partial g}{\partial z} = 0 \right\}, \tag{1.9}$$

$$\text{Ker } H(b, 0) = \{ f = (f_1, f_2) | f_1 \in \text{Ker } H_1, \; f_2 \in \text{Ker } H_2 \}. \tag{1.10}$$

Note that $\text{Ker } H_1$ (respectively, $\text{Ker } H_2$) coincides with the weighted holomorphic (respectively, antiholomorphic) space of Segal-Bargmann type with weight $e^{-\varphi}$ (respectively, e^{φ}) (see e.g. [H, Section 2, Subsection 3.2]).

Proposition 1.1. *Let b be an admissible magnetic field. Then $H_j \geq 0$, $j = 1, 2$, and hence $H(b, 0) \geq 0$. Moreover, if $b_0 > 0$, we have $\dim \text{Ker } H_1 = \infty$, $\dim \text{Ker } H_2 = 0$, if $b_0 < 0$, we have $\dim \text{Ker } H_1 = 0$, $\dim \text{Ker } H_2 = \infty$, while in the case $b_0 = 0$ we have $\dim \text{Ker } H_1 = \dim \text{Ker } H_2 = 0$. Hence, if $b_0 \neq 0$, we have $\dim \text{Ker } H(b, 0) = \infty$, and if $b_0 = 0$, the kernel of $H(b, 0)$ is trivial.*

Proof. The positivity of the operators H_j, $j = 1, 2$, and $H(b, 0)$ follows from (1.7) and (1.1), while the assertions about the deficiency indices are implied by (1.8) – (1.10). □

Since the operators $H(b, 0)$ and $H(-b, 0)$ are anti-unitarily equivalent under the action of the operator $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{C}$ where $\mathbf{C}$ is the complex conjugation, it suffices to consider the cases $b_0 > 0$ and $b_0 = 0$.

Proposition 1.2. *Let b be an admissible magnetic field with $b_0 > 0$. Then $0 = \inf \sigma(H(b, 0))$ is an isolated eigenvalue of infinite multiplicity. More precisely, we have*

$$(0, \lambda_0) \subset \mathbb{R} \setminus \sigma(H(b, 0)) \tag{1.11}$$

with

$$\lambda_0 := 2b_0 \exp\left(-2 \,\text{osc}\, \tilde{\varphi}\right), \tag{1.12}$$

and $\text{osc}\, \tilde{\varphi} := \sup_{x \in \mathbb{R}^2} \tilde{\varphi}(x) - \inf_{x \in \mathbb{R}^2} \tilde{\varphi}(x)$

The proof of Proposition 1.2 is contained in Subsection 3.1. Note that the proof for periodic b with positive mean value can be found in [B], and the proof for general admissible b with $b_0 > 0$ is essentially the same.
If $b_0 = 0$, by Proposition 1.1 we have $\text{Ker } H(b, 0) = \{0\}$. However, in this case it can be shown again that $\inf \sigma(H(b, 0)) = \inf \sigma_{\text{ess}}(H(b, 0)) = 0$.
Introduce the matrix-valued electric potential

$$V(x) := \begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{21}(x) & V_{22}(x) \end{pmatrix}, \quad x \in \mathbb{R}^2,$$

where V_{11} and V_{22} are real-valued functions while V_{12} and $V_{21} = \overline{V_{12}}$ may take complex values. Throughout the paper we assume that the operators $|V_{jk}|^{1/2}(-\Delta+1)^{-1/2}$, $j, k = 1, 2$, are compact in $L^2(\mathbb{R}^2)$, and that b is admissible, and hence bounded. Therefore, the operator $|V|^{1/2}(H(b, 0) + 1)^{-1/2}$ is compact in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. Set

$$H = H(b, V) := H(b, 0) + V$$

where the sum should be understood in the sense of the quadratic forms. Due to H.Weyl's theorem concerning the invariance of the essential spectrum under relatively compact perturbations, we have $\sigma_{\text{ess}}(H(b, V)) = \sigma_{\text{ess}}(H(b, 0))$. In particular, $0 = \inf\ \sigma_{\text{ess}}(H(b, V))$ if $b_0 \geq 0$, and $(0, \lambda_0) \subset \mathbb{R} \setminus \sigma_{\text{ess}}(H(b, V))$ if $b_0 > 0$ (see (1.12)).
The aim of the paper to study the asymptotic distribution of the discrete spectrum of $H(b, V)$ near the origin. The type of the results are quite different in the cases $b_0 > 0$ and $b_0 = 0$. In the present paper we will consider the case $b_0 > 0$, while the results for $b_0 = 0$ we will be published in a future work.

2 Formulation of the main results

2.1. In order to formulate our main results we need the following notations. Let T be a linear self-adjoint operator in a Hilbert space. Denote by $\mathbb{P}_I(T)$ the spectral projection of T corresponding to the Borel set $I \subset \mathbb{R}$. Set

$$N(\lambda_1, \lambda_2; T) := \text{rank}\, \mathbb{P}_{(\lambda_1,\lambda_2)}(T), \quad \lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 < \lambda_2,$$
$$N(\lambda; T) := \text{rank}\, \mathbb{P}_{(-\infty,\lambda)}(T), \quad \lambda \in \mathbb{R}.$$

If T is compact, we will also use the notations

$$n_\pm(s; T) := \text{rank}\, \mathbb{P}_{(s,\infty)}(\pm T), \quad s > 0. \tag{2.1}$$

Finally, if T is a linear compact operator which is not necessarily self-adjoint, set

$$n_*(s; T) := n_+(s^2; T^*T), \quad s > 0. \tag{2.2}$$

2.2. In the sequel we assume $b_0 > 0$. In the following Theorems 2.1 – 2.4 we will discuss the asymptotic behaviour of the negative or the positive discrete spectrum of the operator $H(b, V)$ which accumulate to the origin respectively on the left or on the right of it. For the sake of the compact formulations of these theorems we set

$$\mathcal{N}_-(\lambda) := N(-\lambda; H(b, V)), \quad \lambda > 0,$$
$$\mathcal{N}_+(\lambda) := N(\lambda, \lambda'; H(b, V)), \quad 0 < \lambda < \lambda' < \lambda_0.$$

Theorems 2.1 – 2.4 below contain independent assertions about the asymptotic behaviour of $\mathcal{N}_-(\lambda)$ or $\mathcal{N}_+(\lambda)$ as $\lambda \downarrow 0$ which correspond respectively to the superior or the inferior sign wherever double signs "$\pm$" are met.

The main asymptotic term of $\mathcal{N}_\pm(\lambda)$ as $\lambda \downarrow 0$ depends only on the entry V_{11} of the matrix V as a consequence of our choice to consider only positive b_0 in the case $b_0 \neq 0$.

Our first theorem treats the case where V_{11} admits a power-like decay at infinity. For the sake of simplicity we will formulate and prove this theorem for a class of almost periodic magnetic fields although, evidently, it remains valid for a larger class of admissible b (see [Bel]).

Let us recall that a bounded continuous function $f : \mathbb{R}^d \to \mathbb{C}$, $d \geq 1$, is called *almost periodic* if there exists a sequence of trigonometric polynomials $\{f_N\}_{N\geq 1}$ where $f_N(x) = \sum_{k=1}^{n_N} f_{k,N} e^{i\gamma_{k,N} x}$, $x \in \mathbb{R}^d$, with $f_{k,N} \in \mathbb{C}$, $\gamma_{k,N} \in \mathbb{R}^d$, such that $\lim_{N\to\infty} \sup_{x\in\mathbb{R}^d} |f(x) - f_N(x)| = 0$. Then we write $f \in CAP(\mathbb{R}^d)$ (see [Shu 1]). The inclusion $f \in CAP(\mathbb{R}^d)$ is equivalent also to the fact that f is bounded and continuous on $\mathbb{R}^d$, and its uniform hull is compact (see e.g. [Shu 1, Proposition 1.2 a]).

We will say that b is an *admissible almost periodic magnetic field* if $b = b_0 + \tilde{b}$, where $b_0 \in \mathbb{R}$ is a constant, and $\tilde{b}(x) := \text{Re}\, \sum_{n=1}^{\infty} b_n e^{i\gamma_n x}$, $x \in \mathbb{R}^2$, with $b_n \in \mathbb{C}$, $\gamma_n \in \mathbb{R}^2 \setminus \{0\}$, $n \in \mathbb{N} := \{1, 2, \ldots\}$, such that $\sum_{n=1}^{\infty} |b_n|(|\gamma_n|^{-2} + 1) < \infty$. Then equation (1.3) admits the solution $\tilde{\varphi}(x) = -\text{Re}\, \sum_{n=1}^{\infty} |\gamma_n|^{-2} b_n e^{i\gamma_n x}$, $x \in \mathbb{R}^2$, satisfying (1.4), and $D^\eta \tilde{\varphi} \in CAP(\mathbb{R}^2)$ for each multi-index $\eta \in \mathbb{Z}_+^2$ with $|\eta| \leq 2$. Finally, the mean value of $\tilde{\varphi}$ is equal to zero, so that (1.5) holds as well.

Theorem 2.1. *Let b be an admissible almost periodic magnetic field with $b_0 > 0$. Assume that the matrix-valued potential V satisfies the following properties:*
a) the matrix $\pm V(x)$, $x \in \mathbb{R}^2$, is non-negative;
b) the operators $|V_{j2}|^{1/2}(-\Delta+1)^{-1/2}$, $j = 1, 2$, are compact in $L^2(\mathbb{R}^2)$;
c) the entry $V_{11} \in C^1(\mathbb{R}^2)$ obeys the asymptotic estimates

$$V_{11}(x) = \pm v_{11}(x/|x|)|x|^{-\alpha}(1+o(1)), \quad |x| \to \infty, \tag{2.3}$$

with $\alpha > 0$, $0 < v_{11} \in C(\mathbb{S}^1)$, and $|\nabla V_{11}(x)| = O(|x|^{-\alpha-1})$ as $|x| \to \infty$. Then we have

$$\mathcal{N}_\pm(\lambda) = \frac{b_0}{2\pi}\left|\left\{x \in \mathbb{R}^2 \,|\pm V_{11}(x) > \lambda\right\}\right|(1+o(1)), \quad \lambda \downarrow 0, \tag{2.4}$$

where $|.|$ denotes the Lebesgue measure.

Remark: Under the hypotheses of Theorem 2.1 asymptotic relation (2.4) is equivalent to

$$\lim_{\lambda \downarrow 0} \lambda^{2/\alpha}\mathcal{N}_\pm(\lambda) = \frac{b_0}{4\pi}\int_{\mathbb{S}^1} v_{11}(s)^{2/\alpha} ds. \tag{2.5}$$

Results quite similar to that of Theorem 2.1 in the case of *constant* magnetic field were obtained in [Rai] (see also [Iv, Chapter 11]) where the Schrödinger and not the Pauli operator was considered; in the case of constant b however this difference is not essential. Let us mention here another result due to A. Iwatsuka and H. Tamura which influenced considerably the present paper. For $x \in \mathbb{R}^2$, $m \in \mathbb{R}$, set $\langle x \rangle := (1+|x|^2)^{1/2}$, $W_m(x) := \langle x \rangle^{-m}$.

Theorem 2.2. *Assume that the inequalities*

$$c_1 \le b(x) \le c_2, \tag{2.6}$$

$$|\nabla b(x)| \le c_3 W_m(x), \tag{2.7}$$

hold for each $x \in \mathbb{R}^2$, some $m \in (0,1]$, and some constants $c_j > 0$, $j = 1, 2, 3$. Let V be multiple of the unit matrix, i.e. $V_{11} = V_{22} = V_0$, and $V_{jk} = 0$, $j \neq k$. Suppose that $0 < \pm V_0 \in C^1(\mathbb{R}^2; \mathbb{R})$ satisfies the asymptotic estimates

$$V_0(x) = \pm v_0(x/|x|)|x|^{-\alpha}(1+o(1)), \quad |x| \to \infty,$$

with $\alpha > 0$, $0 < v_0 \in C(\mathbb{S}^1)$, and $|\nabla V_0(x)| = O(|x|^{-\alpha-1})$, $|x| \to \infty$. Then we have

$$\mathcal{N}_\pm(\lambda) = \frac{1}{2\pi}\int_{\{x \in \mathbb{R}^2 | \pm V_0(x) > \lambda\}} b(x)dx\,(1+o(1)), \quad \lambda \downarrow 0. \tag{2.8}$$

The result of Theorem 2.2 concerning $\mathcal{N}_-(\lambda)$ essentially coincides with [Iw.Tam, Theorem 1(i)], and the one concerning $\mathcal{N}_+(\lambda)$ – with [Iw.Tam, Theorem 1(ii)]. Let us compare Theorem 2.2 with Theorem 2.1. First, (2.6) implies that the magnetic fields considered in Theorem 2.2 should be strictly positive while no such restriction is imposed in Theorem

2.1. Moreover, due to (2.7), Theorem 2.2 does not include the case of almost periodic non-constant magnetic fields. However, if b is an arbitrary admissible magnetic field, and V satisfies the assumptions of Theorem 2.2, then

$$\int_{\{x\in\mathbb{R}^2 \,|\pm V_0(x)>\lambda\}} b(x)dx = b_0 \left|\left\{x \in \mathbb{R}^2 \,|\pm V_0(x) > \lambda\right\}\right| (1+o(1)), \quad \lambda \downarrow 0,$$

i.e. in this case the right-hand sides of (2.8) and (2.4) coincide.
In the following two theorems we treat potentials V which decay exponentially fast or have a compact support.

Theorem 2.3. *Let b be an admissible magnetic field with $b_0 > 0$. Assume that conditions a) and b) of Theorem 2.1 hold. Suppose that $V_{11} \in L^\infty(\mathbb{R}^2)$ obeys the asymptotics*

$$\ln(\pm V_{11}(x)) = -\kappa |x|^{2\beta}(1+o(1)), \quad |x| \to \infty,$$

with some constants $\kappa > 0$ and $\beta > 0$. Then

$$\lim_{\lambda\downarrow 0} \frac{\mathcal{N}_\pm(\lambda)}{|\ln\lambda|^{1/\beta}} = \frac{b_0}{2\kappa^{1/\beta}}, \quad 0<\beta<1, \tag{2.9}$$

$$\lim_{\lambda\downarrow 0} \frac{\mathcal{N}_\pm(\lambda)}{|\ln\lambda|} = \frac{1}{\ln(1+2\kappa/b_0)}, \quad \beta = 1, \tag{2.10}$$

$$\lim_{\lambda\downarrow 0} \frac{\mathcal{N}_\pm(\lambda)}{(\ln|\ln\lambda|)^{-1}|\ln\lambda|} = \frac{\beta}{\beta-1}, \quad \beta > 1. \tag{2.11}$$

Theorem 2.4. *Let b be an admissible magnetic field with $b_0 > 0$. Assume that conditions a) and b) of Theorem 2.1 hold. Suppose that $V_{11} \in L^\infty(\mathbb{R}^2)$ has a compact support, and $\pm V_{11} \geq C > 0$ on an open non-empty subset of $\mathbb{R}^2$. Then we have*

$$\lim_{\lambda\downarrow 0} \frac{\mathcal{N}_\pm(\lambda)}{(\ln|\ln\lambda|)^{-1}|\ln\lambda|} = 1. \tag{2.12}$$

Remark: Under the assumptions of Theorem 2.3 we have

$$\left|\left\{x \in \mathbb{R}^2 \,|\pm V_{11}(x) > \lambda\right\}\right| = \pi \frac{|\ln\lambda|^{1/\beta}}{\kappa^{1/\beta}}(1+o(1)), \quad \lambda \downarrow 0,$$

for each $\beta > 0$, while under the assumptions of Theorem 2.4 we have

$$\left|\left\{x \in \mathbb{R}^2 \,|\pm V_{11}(x) > \lambda\right\}\right| = O(1), \quad \lambda \downarrow 0.$$

Hence, the only asymptotic relation amongst (2.9) – (2.12) which could be re-written in a form analogous to (2.4), is the first one, namely (2.9).
Results similar to those of Theorems 2.3 – 2.4 in the case of the Schrödinger operator with *constant* magnetic field can be found in the recent work [Rai.War].

3 Proof of the main results

3.1. In this subsection we prove Proposition 1.2. Denote by $\mathbf{P} = \mathbf{P}(b) : L^2(\mathbb{R}^2;\mathbb{C}^2) \to L^2(\mathbb{R}^2;\mathbb{C}^2)$ the orthogonal projection onto $\mathrm{Ker}\ H(b,0)$, and by $P = P(b) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ – the orthogonal projection onto $\mathrm{Ker}\ H_1$. Set $\mathbf{Q} := \mathrm{Id} - \mathbf{P}$, and $Q := \mathrm{Id} - P$. Note that $\mathbf{P}$ and $\mathbf{Q}$ admit the matrix representations $\mathbf{P} = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} Q & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$. Further, the restriction of the operator H_1 onto $Q\ \mathrm{Dom}(H_1)$ is unitarily equivalent to the operator H_2 (see (1.7)). Moreover, (1.6) – (1.7) imply

$$\langle H_2(b)u, u\rangle = \|a^*(b)u\|^2_{L^2(\mathbb{R}^2)} = 4\int_{\mathbb{R}^2} e^{2\varphi}\left|\frac{\partial\left(e^{-\varphi}u\right)}{\partial z}\right|^2 dx, \quad u \in \mathrm{Dom}(H_2(b)), \tag{3.1}$$

where $\langle .,.\rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$. Thus we find that in order to prove that (1.11) holds, it suffices to show that the inequality

$$4\int_{\mathbb{R}^2} e^{2\varphi}\left|\frac{\partial\left(e^{-\varphi}u\right)}{\partial z}\right|^2 dx \geq \lambda_0 \int_{\mathbb{R}^2} |u|^2 dx \tag{3.2}$$

is valid for each $u \in \mathrm{Dom}(a^*(b)) = \mathrm{Dom}(H_2(b)^{1/2})$ since $\mathrm{Dom}(H_2(b)) \subset \mathrm{Dom}(a^*(b))$. Moreover, the multiplication by $e^{-\tilde{\varphi}}$ is a bijection from $\mathrm{Dom}(a^*(b))$ onto $\mathrm{Dom}(a^*(b_0))$ (actually, since $\frac{\partial\tilde{\varphi}}{\partial z}$ is bounded, $\mathrm{Dom}(a^*(b))$ and $\mathrm{Dom}(a^*(b_0))$ coincide as functional sets, but we will not use this fact here). Changing the functional variable $u = e^{\tilde{\varphi}}w$ in (3.2), we find that this inequality is equivalent to

$$4\int_{\mathbb{R}^2} e^{2\varphi}\left|\frac{\partial\left(e^{-\varphi_0}w\right)}{\partial z}\right|^2 dx \geq \lambda_0 \int_{\mathbb{R}^2} e^{2\tilde{\varphi}}|w|^2 dx, \quad w \in \mathrm{Dom}(a^*(b_0)).$$

Hence, (3.2) would follow from the inequality

$$4\int_{\mathbb{R}^2} e^{2\varphi_0}\left|\frac{\partial\left(e^{-\varphi_0}w\right)}{\partial z}\right|^2 dx \geq 2b_0 \int_{\mathbb{R}^2} |w|^2 dx, \quad w \in \mathrm{Dom}(a^*(b_0)), \tag{3.3}$$

since $\lambda_0 = 2b_0\frac{\inf_{x\in\mathbb{R}^2} e^{2\tilde{\varphi}(x)}}{\sup_{x\in\mathbb{R}^2} e^{2\tilde{\varphi}(x)}}$ (see (1.12)).
Taking into account (3.1) with $b = b_0$ and the well-known fact that $\inf\sigma(H_2(b_0)) = 2b_0$ (see e.g. [Av.Her.Si]), we immediately get (3.3).
3.2. In this subsection we establish some estimates which allow us to reduce the asymptotic analysis of $\mathcal{N}_\pm(\lambda)$ as $\lambda \downarrow 0$ to the investigation of the discrete spectrum of compact operators of Toeplitz type.

Proposition 3.1. *Let* $\varepsilon \in (0,1)$. *Then under the hypotheses of Theorems 2.1, 2.3, or 2.4, we have*

$$n_-(\lambda; PV_{11}P) \leq \mathcal{N}_-(\lambda) \leq n_-((1-\varepsilon)\lambda; PV_{11}P) + O(1), \quad \lambda \downarrow 0, \tag{3.4}$$

$$n_+((1+\varepsilon)\lambda; PV_{11}P) + O(1) \leq \mathcal{N}_+(\lambda) \leq n_+(\lambda; PV_{11}P), \quad \lambda \downarrow 0, \tag{3.5}$$

$n_\pm$ *being the counting functions defined in* (2.1).

Proof. At first we prove (3.4). By the Birman-Schwinger principle

$$\mathcal{N}_-(\lambda) \equiv N(-\lambda; H(b,V)) = n_-(1; (H(b,0)+\lambda)^{-1/2} V (H(b,0)+\lambda)^{-1/2}), \quad \lambda > 0. \quad (3.6)$$

Next, the minimax principle implies

$$n_-(1; (H(b,0)+\lambda)^{-1/2} V (H(b,0)+\lambda)^{-1/2}) \geq$$

$$n_-(1; \mathbf{P}(H(b,0)+\lambda)^{-1/2} V (H(b,0)+\lambda)^{-1/2}\mathbf{P}) =$$

$$n_-(\lambda; \mathbf{P}V\mathbf{P}) = n_-(\lambda; PV_{11}P), \quad \lambda > 0, \quad (3.7)$$

which yields the lower bound in (3.4). On the other hand, since $V \leq 0$, we have

$$n_-(1 : (H(b,0)+\lambda)^{-1/2} V (H(b,0)+\lambda)^{-1/2}) \leq$$

$$n_-(1; (1+\delta)\mathbf{P}(H(b,0)+\lambda)^{-1/2} V (H(b,0)+\lambda)^{-1/2}\mathbf{P}) +$$

$$n_-(1; (1+\delta^{-1})\mathbf{Q}(H(b,0)+\lambda)^{-1/2} V (H(b,0)+\lambda)^{-1/2}\mathbf{Q}), \quad \lambda > 0, \quad \delta > 0. \quad (3.8)$$

Fix $\varepsilon \in (0,1)$ and pick $\delta = \varepsilon/(1-\varepsilon)$. Then we get

$$n_-(1; (1+\delta)\mathbf{P}(H(b,0)+\lambda)^{-1/2} V (H(b,0)+\lambda)^{-1/2}\mathbf{P}) = n_-((1-\varepsilon)\lambda; PV_{11}P), \quad (3.9)$$

for each $\lambda > 0$, $\varepsilon \in (0,1)$. Finally, we show that

$$n_-(1; (1+\delta^{-1})\mathbf{Q}(H(b,0)+\lambda)^{-1/2} V (H(b,0)+\lambda)^{-1/2}\mathbf{Q}) = O(1), \quad \lambda \downarrow 0. \quad (3.10)$$

for each $\delta > 0$. Evidently, there exists a constant C independent of $\lambda > 0$ such that $\|\mathbf{Q}(H(b,0)+\lambda)^{-1/2}(H(b,0)+1)^{1/2}\| \leq C$. Therefore,

$$n_-(t; \mathbf{Q}(H(b,0)+\lambda)^{-1/2} V (H(b,0)+\lambda)^{-1/2}\mathbf{Q}) \leq$$

$$n_-(t; C^2(H(b,0)+1)^{-1/2} V (H(b,0)+1)^{-1/2}), \quad t > 0. \quad (3.11)$$

Since the operator $|V|^{1/2}(H(b,0)+1)^{-1/2}$ is compact, (3.11) implies (3.10). Now the upper bound in (3.4) follows from (3.6), (3.8), (3.9), and (3.10).
Let us now prove (3.5). The generalized Birman-Schwinger principle (see e.g. [Al.Dei.Hem, Theorem 1.3]) entails

$$\mathcal{N}_-(\lambda) \equiv N(\lambda, \lambda'; H(b,V)) = n_+(1; V^{1/2}(\lambda - H(b,0))^{-1} V^{1/2}) -$$

$$n_+(1; V^{1/2}(\lambda' - H(b,0))^{-1} V^{1/2}) - \dim \operatorname{Ker}(H(b,0) - \lambda'), \quad 0 < \lambda < \lambda' < \lambda_0. \quad (3.12)$$

The compactness of the operator $V^{1/2}(H(b,0)+1)^{-1/2}$ easily implies that the second and the third term on the right-hand side of (3.12) which are independent of $\lambda > 0$, are finite. Further, since $(\lambda - H(b,0))^{-1} \leq \mathbf{P}(\lambda - H(b,0))^{-1} = \lambda^{-1}\mathbf{P}$, we have

$$n_+(1; V^{1/2}(\lambda - H(b,0))^{-1} V^{1/2}) \leq n_+(\lambda; V^{1/2}\mathbf{P}V^{1/2})$$

$$= n_+(\lambda; \mathbf{P}V\mathbf{P}) = n_+(\lambda; PV_{11}P), \quad \lambda > 0. \tag{3.13}$$

On the other hand, the H. Weyl's inequalities for the eigenvalues of compact self-adjoint operators imply

$$n_+(1; V^{1/2}(\lambda - H(b,0))^{-1}V^{1/2}) \geq$$
$$n_+((1+\varepsilon); V^{1/2}\mathbf{P}(\lambda - H(b,0))^{-1}V^{1/2}) - n_-(\varepsilon; V^{1/2}\mathbf{Q}(\lambda - H(b,0))^{-1}V^{1/2}) =$$
$$n_+((1+\varepsilon)\lambda; PV_{11}P) - n_+(\varepsilon; \mathbf{Q}|H(b,0) - \lambda|^{-1/2}V|H(b,0) - \lambda|^{-1/2}\mathbf{Q}). \tag{3.14}$$

By analogy with (3.10), we can show that

$$n_+(\varepsilon; \mathbf{Q}|H(b,0) - \lambda|^{-1/2}V|H(b,0) - \lambda|^{-1/2}\mathbf{Q}) = O(1), \quad \lambda \downarrow 0. \tag{3.15}$$

Putting together (3.12) – (3.15), we get (3.5). □

3.3. In this subsection we prove Theorems 2.3 and 2.4.

Proposition 3.2. *Let b be an admissible magnetic field with $b_0 > 0$. Let $U : \mathbb{R}^2 \to [0, +\infty)$. Assume that the operator $U^{1/2}(-\Delta + 1)^{-1/2}$ is compact. Then the operators $P(b)UP(b)$ and $P(b_0)UP(b_0)$ are compact as well, and we have*

$$n_+(\exp(2\,\mathrm{osc}\,\tilde{\varphi})\lambda; P(b_0)UP(b_0)) \leq n_+(\lambda; P(b)UP(b)) \leq$$
$$n_+(\exp(-2\,\mathrm{osc}\,\tilde{\varphi})\lambda; P(b_0)UP(b_0)), \quad \lambda > 0. \tag{3.16}$$

Proof. The compactness of the operators follows easily from

$$U^{1/2}P(b) = U^{1/2}(H_1(b)+1)^{-1/2}P(b), \quad U^{1/2}P(b_0) = U^{1/2}(H_1(b_0)+1)^{-1/2}P(b_0),$$

and the diamagnetic inequality (see [Av.Her.Si, Theorem 2.2]).
Let us prove (3.16). The minimax principle implies that $n_+(\lambda; P(b)UP(b))$, $\lambda > 0$, coincides with the maximum dimension of the linear subsets of $\mathrm{Ker}\ (H_1(b)) = \mathrm{Ker}\ a(b)$ (see (1.6)) whose non-zero elements u satisfy the inequality

$$\int_{\mathbb{R}^2} U|u|^2 dx > \lambda \int_{\mathbb{R}^2} |u|^2 dx.$$

Since the multiplication by $e^{\tilde{\varphi}}$ is a bijection from $\mathrm{Ker}\ a(b)$ onto $\mathrm{Ker}\ a(b_0)$, we find that $n_+(\lambda; P(b)UP(b))$, $\lambda > 0$, coincides with the maximum dimension of the linear subsets of $\mathrm{Ker}\ H(b_0) = \mathrm{Ker}\ a(b_0)$ (see (1.6)) whose non-zero elements w satisfy the inequality

$$\int_{\mathbb{R}^2} Ue^{-2\tilde{\varphi}}|w|^2 dx > \lambda \int_{\mathbb{R}^2} e^{-2\tilde{\varphi}}|w|^2 dx. \tag{3.17}$$

Evidently, (3.17) implies the inequality $\int_{\mathbb{R}^2} U|w|^2 dx > \lambda e^{-2\,\mathrm{osc}\,\tilde{\varphi}} \int_{\mathbb{R}^2} |w|^2\, dx$, and follows from the inequality $\int_{\mathbb{R}^2} U|w|^2 dx > \lambda e^{2\,\mathrm{osc}\,\tilde{\varphi}} \int_{\mathbb{R}^2} |w|^2\, dx$. Now, the validity of (3.16) is a direct consequence of the minimax principle, and the fact that $n_+(t; P(b_0)UP(b_0))$, $t > 0$, coincides with the maximum dimension of the linear subsets of $\mathrm{Ker}\ H_1(b_0) = \mathrm{Ker}\ a(b_0)$ whose non-zero elements w satisfy the inequality $\int_{\mathbb{R}^2} U|w|^2 dx > t \int_{\mathbb{R}^2} |w|^2 dx$. □

Proposition 3.3. [Rai.War, Theorem 2.1, Proposition 3.1 with $q = 0$] *Let* $b_0 > 0$. *Suppose that* $0 \leq U \in L^\infty(\mathbb{R}^2)$ *obeys the asymptotics* $\ln U(x) = -\kappa|x|^{2\beta}(1 + o(1))$ *as* $|x| \to \infty$ *with some constant* $\kappa > 0$ *and* $\beta > 0$. *Then we have*

$$\lim_{\lambda \downarrow 0} \frac{n_+(\lambda; P(b_0)UP(b_0))}{|\ln \lambda|^{1/\beta}} = \frac{b_0}{2\kappa^{1/\beta}}, \quad 0 < \beta < 1,$$

$$\lim_{\lambda \downarrow 0} \frac{n_+(\lambda; P(b_0)UP(b_0))}{|\ln \lambda|} = \frac{1}{\ln(1 + 2\kappa/b_0)}, \quad \beta = 1,$$

$$\lim_{\lambda \downarrow 0} \frac{n_+(\lambda; P(b_0)UP(b_0))}{(\ln|\ln \lambda|)^{-1}|\ln \lambda|} = \frac{\beta}{\beta - 1}, \quad \beta > 1.$$

Proposition 3.4. [Rai.War, Theorem 2.2, Proposition 3.2 with $q = 0$] *Let* $b_0 > 0$. *Suppose that* $0 \leq U \in L^\infty(\mathbb{R}^2)$ *has a compact support, and* $U \geq C > 0$ *on an open non-empty subset of* $\mathbb{R}^2$. *Then we have*

$$\lim_{\lambda \downarrow 0} \frac{n_+(\lambda; P(b_0)WP(b_0))}{(\ln|\ln \lambda|)^{-1}|\ln \lambda|} = 1.$$

Now the result of Theorem 2.3 follows from Propositions 3.1, 3.2, and 3.3, while the result of Theorem 2.4 is a corollary from Propositions 3.1, 3.2, and 3.4.

3.4. This subsection contains some auxiliary results which will be essentially used in the following one where we prove Theorem 2.1.

For $T > 0$ and $\tau \in \mathbb{R}^2$ denote by $H^{\mathcal{D}}_{T,\tau}$ (respectively, $H^{\mathcal{N}}_{T,\tau}$) the self-adjoint operator $(-i\nabla - \mathbf{A})^2 - b$ defined on the Sobolev space $\mathrm{H}^2\left(\tau + \left(-\frac{T}{2}, \frac{T}{2}\right)^2\right)$ with Dirichlet (respectively, Neumann) boundary conditions. The non-decreasing function $\varrho_b : \mathbb{R} \to [0, \infty)$ is called *integrated density of states (IDS)* for the operator $H_1(b)$ if it satisfies

$$\varrho_b(t) = \lim_{T \to \infty} T^{-2} N(t; H^{\mathcal{D}}_{T,\tau}), \quad \tau \in \mathbb{R}^2, \tag{3.18}$$

in the vague sense, i.e. at its continuity points $t \in \mathbb{R}$ (see [P.Fi], [K]). If $b = b_0 > 0$, then the IDS ϱ_{b_0} exists and its explicit expression is well-known (see e.g. [CdV]):

$$\varrho_{b_0}(t) = \frac{b_0}{2\pi} \sum_{k=0}^{\infty} \theta(t - 2kb_0), \quad t \in \mathbb{R}, \tag{3.19}$$

where $\theta(t) := \begin{cases} 0 & \text{if} \quad t < 0, \\ 1 & \text{if} \quad t > 0, \end{cases}$ is the Heaviside function.

Lemma 3.1. [D.Iw.Mi, Theorem 1.2] *Let* b *be an admissible magnetic field. Then the existence of the vague limit* (3.18) *is equivalent to the existence of any of the following two limits*

$$\varrho_b(t) = \lim_{T \to \infty} T^{-2} N(t; H^{\mathcal{N}}_{T,\tau}), \quad \tau \in \mathbb{R}^2, \tag{3.20}$$

$$\varrho_b(t) = \lim_{T\to\infty} T^{-2} \operatorname{Tr} \chi_{T,\tau}\mathbb{P}_{(-\infty,t)}(H_1)\, \chi_{T,\tau}, \quad \tau \in \mathbb{R}^2, \tag{3.21}$$

at the continuity points of $t \in \mathbb{R}^2$ of $\varrho_b(t)$. Here $\chi_{T,\tau}$ denotes the characteristic function of the domain $\tau + \left(-\frac{T}{2}, \frac{T}{2}\right)^2$, and as indicated in Subsection 2.1, $\mathbb{P}_{(-\infty,t)}(H_1)$ denotes the spectral projection of the operator H_1 (see (1.1)*) corresponding to $(-\infty, t)$, $t \in \mathbb{R}$.*

Lemma 3.2. *Let b be an admissible almost periodic magnetic field with $b_0 > 0$. Then the IDS ϱ_b exists, and we have*

$$\varrho_b(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ \frac{b_0}{2\pi} & \text{if} \quad 0 < t < \lambda_0. \end{cases} \tag{3.22}$$

Proof. At first we prove the existence of the IDS.
Let Ω be the Bohr compactification of $\mathbb{R}^2$ (see [Shu 1, Section 1]). We recall that Ω is a compact Abelian group, and there exists a continuous homomorphism $\iota : \mathbb{R}^2 \to \Omega$ such that $\iota(\mathbb{R}^2)$ is dense in Ω. As in $\mathbb{R}^2$, we will denote by "+" the group operation in Ω, and by $-\omega$ – the element inverse to $\omega \in \Omega$. Note that ι induces an isomorphism between $CAP(\mathbb{R}^2)$ and $C(\Omega)$. In particular, for each $f \in CAP(\mathbb{R}^2)$ there exists a unique $g \in C(\Omega)$ such that $f(x) = g(\iota(x))$, $x \in \mathbb{R}^2$; we will call g the extension by continuity of f.
Let μ be the normalized Haar measure on Ω, and $\mathcal{F}$ be the σ-algebra of the corresponding μ-measurable subsets of Ω. Then $(\Omega, \mathcal{F}, \mu)$ is a probabilistic space.
Set $\mathbf{A_0} = (A_{0,1}, A_{0,2}) := \left(-\frac{\partial\varphi_0}{\partial x_2}, \frac{\partial\varphi_0}{\partial x_1}\right) = \left(-\frac{b_0 x_2}{2}, \frac{b_0 x_1}{2}\right)$, $\tilde{\mathbf{A}} = (\tilde{A}_1, \tilde{A}_2) := \left(-\frac{\partial\tilde{\varphi}}{\partial x_2}, \frac{\partial\tilde{\varphi}}{\partial x_1}\right)$. Note that $\tilde{A}_j \in CAP(\mathbb{R}^2)$, $j = 1, 2$, $b \in CAP(\mathbb{R}^2)$. Denote by $\hat{\alpha}_j$, $j = 1, 2$, and $\hat{\beta}$, the extension by continuity of the functions $\tilde{A}_j$, $j = 1, 2$, and b, respectively. Set

$$\alpha_{\omega,j}(x) := \hat{\alpha}_j(\omega + \iota(x)), \quad j = 1, 2, \quad \beta_\omega(x) := \hat{\beta}(\omega + \iota(x)), \quad \omega \in \Omega, \quad x \in \mathbb{R}^2,$$

as well as $\alpha_\omega := (\alpha_{\omega,1}, \alpha_{\omega,2})$. On $D(H_1)$ define the operator

$$\mathcal{H}_\omega := (-i\nabla - \mathbf{A_0} - \alpha_\omega)^2 - \beta_\omega =$$

$$(-i\nabla - \mathbf{A_0})^2 - 2\alpha_\omega.(-i\nabla - \mathbf{A_0}) + |\alpha_\omega|^2 - \beta_\omega, \quad \omega \in \Omega.$$

Evidently, the operator family $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ is continuous with respect to $\omega \in \Omega$ in the norm resolvent sense, and $\mathcal{H}_{\iota(0)} = H_1$.
Further, for $\xi \in \mathbb{R}^2$ introduce the unitary operator $\mathcal{U}_\xi : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by

$$(\mathcal{U}_\xi f)(x) = e^{\frac{i}{2} b_0(\xi_1 x_2 - x_1 \xi_2)} f(x - \xi), \quad x \in \mathbb{R}^2, \quad f \in L^2(\mathbb{R}^2). \tag{3.23}$$

The operators $\mathcal{U}_\xi$, $\xi \in \mathbb{R}^2$, commute with $-i\partial/\partial x_j - A_{0,j}$, $j = 1, 2$. Hence,

$$\mathcal{U}_\xi \mathcal{H}_\omega \mathcal{U}_\xi^* = \mathcal{H}_{\mathcal{T}_\xi \omega} \tag{3.24}$$

where $\mathcal{T}_\xi \omega := \omega - \iota(\xi)$, $\omega \in \Omega$, $\xi \in \mathbb{R}^2$. It is clear that $\{\mathcal{T}_\xi\}_{\xi\in\mathbb{R}^2}$ is an ergodic family of measure preserving automorphisms of Ω (see [P.Fi], [K]).

For $T > 0$ and $\tau \in \mathbb{R}^2$ denote by $\mathcal{H}^{\mathcal{D}}_{\omega,T}$ the self-adjoint in $L^2\left(\tau + \left(-\frac{T}{2}, \frac{T}{2}\right)^2\right)$ operator $(-i\nabla - \mathbf{A_0} - \alpha_\omega)^2 - \beta_\omega$ with domain $\mathrm{Dom}\left(H^{\mathcal{D}}_{1,T}\right)$. Arguing as in [K, Section 7] (see also [U 2, Lemma 3.1]), we find that the vague limit $\varrho_b := \lim_{T\to\infty} T^{-2} N\left(.; \mathcal{H}^{\mathcal{D}}_{\omega,T,\tau}\right)$ exists for μ-almost every $\omega \in \Omega$ and every $\tau \in \mathbb{R}^2$, and is independent of ω and τ.
Due to the norm resolvent continuity of $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ with respect to $\omega \in \Omega$, and the fact that $\iota(\mathbb{R}^2)$ is dense in Ω, we easily find that the IDS ϱ_b exists for *every* $\omega \in \Omega$.
Let us now prove (3.22). Set $b_s := b_0 + s\tilde{b}$, $s \in [0, 1]$, so that we have $b = b_1$. Note that b_s is an admissible almost periodic magnetic field whose mean value is equal to b_0 for each $s \in [0, 1]$. Fix $s \in [0, 1]$. The fact that $\varrho_{b_s}(t) = 0$ if $t < 0$ follows immediately from (3.21) and the fact that $\inf \ \sigma(H_1(b_s)) = 0$. Similarly, (3.21) implies that ϱ_{b_s} is constant on the interval $(0, \lambda_0)$ since $\sigma(H_1(b_s))$ does not intersect with it. Let us prove now that

$$\varrho_{b_s}(t) > 0, \quad t \in (0, \lambda_0), \quad s \in [0, 1]. \tag{3.25}$$

To this end, we will show that

$$\lim_{T\to\infty} T^{-2}\mathrm{Tr}\ \chi_{T,\tau}\mathbb{P}_{(-\infty,t)}(H_1(b_s))\chi_{T,\tau} \geq \frac{b_0}{2\pi} e^{-2s\,\mathrm{osc}\,\tilde{\varphi}},\ t \in (0, \lambda_0),\ s \in [0, 1],\ \tau \in \mathbb{R}^2. \tag{3.26}$$

Recall that $\mathbb{P}_{(-\infty,t)}(H_1(b_s)) = \mathbb{P}_{\{0\}}(H_1(b_s))$ is the orthogonal projection to the weighted holomorphic subspace of $L^2(\mathbb{R}^2)$ defined in (1.8). It is well-known that its integral kernel $K_{b_s}(x, y)$ is a continuous function of $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ (see e.g. [H, Theorem 2.3]). Hence, for $t \in (0, \lambda_0)$ and $s \in [0, 1]$, we have

$$\mathrm{Tr}\ \chi_{T,\tau}\mathbb{P}_{(-\infty,t)}(H_1(b_s))\chi_{T,\tau} = \mathrm{Tr}\ \chi_{T,\tau}\mathbb{P}_{\{0\}}(H_1(b_s))\chi_{T,\tau} = \int_{\tau+\left(-\frac{T}{2},\frac{T}{2}\right)^2} K_{b_s}(x, x)\, dx. \tag{3.27}$$

We will prove that

$$K_{b_s}(x, x) \geq \frac{b_0}{2\pi} e^{-2s\,\mathrm{osc}\,\tilde{\varphi}}, \quad x \in \mathbb{R}^2, \quad s \in [0, 1], \quad \tau \in \mathbb{R}^2, \tag{3.28}$$

which will imply (3.26), and hence (3.25). Introduce the functions

$$\phi_k(x) := \sqrt{\frac{b_0}{2\pi\Gamma(k + 1)}} \left(\frac{b_0}{2}\right)^{k/2} (x_1 + ix_2)^k\, e^{-\varphi_0(x)}, \quad k \in \mathbb{Z}_+, \quad x \in \mathbb{R}^2, \tag{3.29}$$

which constitute an orthonormal in $L^2(\mathbb{R}^2)$ basis of $\mathrm{Ker}\ H_1(b_0) = \mathrm{Ker}\ a(b_0)$ (see e.g. [Rai.War]). Since the multiplication by $e^{-s\tilde{\varphi}}$ is a bijection from $\mathrm{Ker}\ a(b_0)$ on $\mathrm{Ker}\ a(b_s)$, $s \in [0, 1]$, the functions $\left\{e^{-s\tilde{\varphi}}\phi_k\right\}_{k=0}^{\infty}$ constitute a basis in $\mathrm{Ker}\ a(b_s)$, $s \in [0, 1]$. Let $M = M(s) : l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+)$ be the operator given in the canonic basis of $l^2(\mathbb{Z}_+)$ by the matrix $\{m_{jk}\}_{j,k=0}^{\infty}$ with $m_{jk} := \int_{\mathbb{R}^2} e^{-2s\tilde{\varphi}}\phi_j\overline{\phi_k}\, dx$, $j, k \in \mathbb{Z}_+$, $s \in [0, 1]$. It is easy to see that M is self-adjoint, and

$$\inf_{y\in\mathbb{R}^2} e^{-2s\tilde{\varphi}(y)} \leq \inf \sigma(M(s)) \leq \sup \sigma(M(s)) \leq \sup_{y\in\mathbb{R}^2} e^{-2s\tilde{\varphi}(y)}, \quad s \in [0, 1]. \tag{3.30}$$

Set $R(s) := M(s)^{-1/2}$, $s \in [0,1]$. Then (3.30) entails

$$\inf \sigma(R(s)) \geq \inf_{y \in \mathbb{R}^2} e^{s\tilde{\varphi}(y)}, \quad s \in [0,1]. \tag{3.31}$$

Let $\{r_{jk}\}_{j,k=0}^{\infty}$ be the matrix of the operator R in the canonic basis of $l^2(\mathbb{Z}_+)$. Put

$$\psi_j(x) := e^{-s\,\tilde{\varphi}(x)} \sum_{k=0}^{\infty} r_{jk}\phi_k(x), \quad x \in \mathbb{R}^2, \quad j \in \mathbb{Z}_+.$$

Then $\{\psi_j\}_{j=0}^{\infty}$ is an orthonormal in $L^2(\mathbb{R}^2)$ basis of $\mathrm{Ker}\ a(b_s)$, and

$$K_{b_s}(x,x) = \sum_{j=0}^{\infty} |\psi_j(x)|^2 = e^{-2s\,\tilde{\varphi}(x)} \|R\phi(x)\|^2_{l^2(\mathbb{Z}_+)} \tag{3.32}$$

where $\phi(x) := \{\phi_k(x)\}_{k=0}^{\infty} \in l^2(\mathbb{Z}_+)$, with $x \in \mathbb{R}^2$ being fixed (see [H, Theorem 2.4]). Combining (3.32) with (3.31) and the obvious equality $\sum_{k=0}^{\infty} |\phi_k(x)|^2 = \frac{b_0}{2\pi}$, valid for each $x \in \mathbb{R}^2$, we get

$$K_{b_s}(x,x) \geq \inf_{y \in \mathbb{R}^2} e^{-2s\tilde{\varphi}(y)} \|R\phi(x)\|^2_{l^2(\mathbb{Z}_+)} \geq \inf_{y \in \mathbb{R}^2} e^{-2s\tilde{\varphi}(y)} \inf_{y \in \mathbb{R}^2} e^{2s\tilde{\varphi}(y)} \ \|\phi(x)\|^2_{l^2(\mathbb{Z}_+)} =$$

$$e^{-2s\,\mathrm{osc}\,\tilde{\varphi}} \sum_{k=0}^{\infty} |\phi_k(x)|^2 = e^{-2s\,\mathrm{osc}\,\tilde{\varphi}} \frac{b_0}{2\pi}, \quad x \in \mathbb{R}^2.$$

Thus we obtain (3.28), and hence (3.25). In particular, we have found that the interval $(0, \lambda_0)$ is a part of a gap in the support of ϱ_{b_s}, and the lower end of this gap coincides with the origin for all $s \in [0,1]$.

In order to complete the proof of (3.22), it suffices to show that the value J_1 of the jump of $\varrho_b(t)$ at $t = 0$ is equal to $\frac{b_0}{2\pi}$. Using the resolvent identity, it is not difficult to check that the operator family $H_1(b_s)$, $s \in [0,1]$, is continuous in the norm-resolvent sense. Then, a gap-labelling lemma of J.Bellissard (see [Bel, Proposition 4.2.5]) implies that the value J_s of the jump of $\varrho_{b_s}(t)$ at $t = 0$ is independent of $s \in [0,1]$. In particular, $J_1 = J_0$, and $J_0 = \frac{b_0}{2\pi}$ by (3.19). $\square$

Remarks: i) In the case where b is a Γ-periodic magnetic field with $b_0 > 0$ such that the flux $(2\pi)^{-1} \int_{\mathbb{R}^2/\Gamma} b dx$ is rational, the existence of the IDS ϱ_b as well as the validity of (3.22) follows directly from the results of [Dub.Nov].

ii) The existence of the IDS of general elliptic partial differential with smooth almost periodic coefficients has been proved in [Shu 2]; note, however, that not all the coefficients of H_1 are almost periodic because of the linear part $\mathbf{A_0}$ of the magnetic potential.

The existence of the IDS for various Schrödinger operators with ergodic electric potentials, and deterministic or random magnetic fields has been considered in [Bel], [Ma], [U 1], [Hu.Le.M.W], [U 2].

As a by-product of Lemma 3.2 we obtain the following result concerning the theory of the weighted holomorphic spaces.

Corollary 3.1. *Let b be as in Lemma 3.2. Let, as above, K_b be the integral kernel of the spectral projection $\mathbb{P}_{\{0\}}(H_1(b))$ which coincides with the orthogonal projection onto* $\mathrm{Ker}\ a(b)$ *(see* (1.8)*). Then we have*

$$\lim_{T\to\infty} T^{-2} \int_{\tau+\left(-\frac{T}{2},\frac{T}{2}\right)^2} K_b(x,x)\,dx = \frac{b_0}{2\pi}, \quad \tau\in\mathbb{R}^2.$$

Proof. It suffices to combine (3.21), (3.27) with $s=1$, and (3.22). □

Lemma 3.3. *Assume that the hypotheses of Theorem 2.1 hold. Let $E\in(0,\lambda_0)$. Then*

$$\lim_{g\to\infty} g^{-2/\alpha} n_+(g^{-1};|V_{11}|^{1/2}(E-H_1)^{-1}|V_{11}|^{1/2}) =$$

$$\int_{-\infty}^{E} \left|\left\{x\in\mathbb{R}^2\,\middle|\,v_{11}(x/|x|)|x|^{-\alpha} > E-t\right\}\right|\, d\varrho_b(t) = E^{-2/\alpha}\frac{b_0}{4\pi}\int_{\mathbb{S}^1} v_{11}(s)^{2/\alpha}ds. \quad (3.33)$$

Sketch of the proof: The generalized Birman-Schwinger principle entails

$$n_+(g^{-1};|V_{11}|^{1/2}(E-H_1)^{-1}|V_{11}|^{1/2}) = \sum_{0<g'<g} \mathrm{dim\ Ker}\ \left(H_1+g'|V_{11}|-E\right). \quad (3.34)$$

Hence, (3.33) could be regarded as a result on the large coupling constant asymptotics of the discrete spectrum of a positively perturbed Schrödinger operator, situated in a gap of its essential spectrum. There are numerous results in this respect due to S. Alama, P. Deift, R. Hempel, and more recently, to S. Z. Levendorskii. In general, these results can be formulated as follows. Let $\mathcal{S}$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$, $d\geq 2$, for which the IDS ρ exists. Let $E\in\mathbb{R}\setminus\sigma(\mathcal{S})$. Assume that $W\in L^\infty(\mathbb{R}^d)$ satisfies $W\geq 0$ and

$$W(x) = |x|^{-\alpha}w(x/|x|)(1+o(1)), \quad |x|\to\infty,$$

with some $\alpha>0$ and $0<w\in C(\mathbb{S}^{d-1})$. Then we have

$$\lim_{g\to\infty} g^{-d/\alpha} \sum_{0<g'<g} \mathrm{dim\ Ker}\ \left(\mathcal{S}+g'W-E\right) =$$

$$\int_{-\infty}^{E} \left|\left\{x\in\mathbb{R}^d\,\middle|\,w(x/|x|)|x|^{-\alpha} > E-t\right\}\right|\, d\rho(t). \quad (3.35)$$

The first result of this type was published in [Al.Dei.Hem] for the case where $\mathcal{S}=-\Delta+\mathcal{V}$ with bounded electric potential $\mathcal{V}$. In [Lev, Theorem 1.2] asymptotic relation (3.35) was proved for $\mathcal{S}=(-i\nabla-\mathcal{A})^2+\mathcal{V}$ with bounded $\mathcal{V}$ and magnetic potential $\mathcal{A}$ which generates a non-zero constant magnetic field. In [Hem.Lev] this result was implicitly extended to magnetic potentials $\mathcal{A}$ with bounded first-order derivatives: – although (3.35) was not stated explicitly, the main lemma needed for the proofs in [Lev] was generalized for this class of magnetic potentials (compare [Lev, Lemma 2.9] with [Hem.Lev, Lemma 2.3]). Note that thanks to Lemma 3.1, and especially to (3.20), the argument in [Lev] could be simplified. Taking into account (3.34), we find that (3.33) is a special case of (3.35) with $\mathcal{S} = H_1(b) = (-i\nabla - \mathbf{A})^2 - b$ and $W = |V_{11}|$. Since $\sup_{x\in\mathbb{R}^2} |D^\eta \varphi(x)|$ is finite for $|\eta| = 2$, and by Lemma 3.2 the IDS $\rho = \varrho_b$ exists, we can apply the general results of [Lev] and [Hem.Lev]. Finally, in order to evaluate the first integral in (3.33), we note that

$$\left|\left\{x \in \mathbb{R}^2 \,\middle|\, v_{11}(x/|x|)|x|^{-\alpha} > E - t\right\}\right| = \frac{(E-t)^{-2/\alpha}}{2} \int_{\mathbb{S}^1} v_{11}(s)^{2/\alpha} ds, \quad t < E,$$

and then apply (3.22). □

Lemma 3.4. [Iw.Tam, Lemma 1.4] *Let b be an admissible magnetic field. Let $U : \mathbb{R}^2 \to \mathbb{R}$ satisfy $|U(x)| \leq CW_m(x)$ and $|\nabla U(x)| \leq CW_{m+1}(x)$ for each $x \in \mathbb{R}^2$ and some $m > 0$, $C > 0$. Then the commutator $\mathcal{C} := [P(b), U] := P(b)U - UP(b)$ admits the representation*

$$\mathcal{C} = \mathcal{B}W_{m+1} \tag{3.36}$$

where $\mathcal{B} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a bounded operator.

Lemma 3.5. *Let b be an admissible magnetic field with $b_0 > 0$. Let $U : \mathbb{R}^2 \to \mathbb{R}$ satisfy $|U(x)| \leq CW_m(x)$, $x \in \mathbb{R}^2$, for some $m > 0$, $C > 0$. Then the operator $P(b)UP(b)$ is compact, and we have*

$$n_\pm(t; P(b)UP(b)) = O(t^{-2/m}), \quad t \downarrow 0. \tag{3.37}$$

Proof. Bearing in mind (3.16), and applying the minimax principle, we find that it suffices to show that

$$n_+(t; P(b_0)W_mP(b_0)) = O(t^{-2/m}), \quad t \downarrow 0. \tag{3.38}$$

Due to the radial symmetry of the function W_m, the non-zero eigenvalues of the operator $P(b_0)W_mP(b_0)$ coincide with $\nu_k := \langle W_m\phi_k, \phi_k\rangle$, $k \in \mathbb{Z}_+$, where the functions ϕ_k, $k \in \mathbb{Z}_+$ are defined in (3.29). Hence,

$$\nu_k = \frac{1}{\Gamma(k+1)} \left(\frac{b_0}{2}\right)^{k+1} \int_0^\infty e^{-b_0r/2} r^k (1+r)^{-m/2}\, dr, \quad k \in \mathbb{Z}_+,$$

which implies $\nu_k \leq \left(\frac{2}{b_0}\right)^{-m/2} \frac{\Gamma(k+1-m/2)}{\Gamma(k+1)}$ for $k \in \mathbb{Z}_+$ such that $k > \frac{m}{2} - 1$. Utilizing the asymptotic relation $\lim_{k\to\infty} k^{m/2} \frac{\Gamma(k+1-m/2)}{\Gamma(k+1)} = 1$ (see [Ab.St, Eq. 6.1.46]), we find that for each $\varepsilon > 0$ there exists $\mathbb{Z} \ni K_\varepsilon > \max\left\{0, \frac{m}{2} - 1\right\}$ such that we have $\nu_k \leq (1+\varepsilon)(2k/b_0)^{-m/2}$ provided that $k \geq K_\varepsilon$. Therefore,

$$\begin{aligned} n_\pm(t; P(b)UP(b)) &\leq \#\{k \in \mathbb{N} | (1+\varepsilon)(2k/b_0)^{-m/2} > t\} + K_\varepsilon \\ &\leq (1+\varepsilon)^{2/m} b_0\, t^{-2/m}/2 + K_\varepsilon, \quad t > 0, \end{aligned}$$

which implies (3.38). □

Remark: A result similar to Lemma 3.5 is contained in [Iw.Tam, Proposition 2.3]. There, however, the assumptions on b are essentially different from ours.

Corollary 3.2. *Let b be an admissible magnetic field with $b_0 > 0$. Let $U : \mathbb{R}^2 \to \mathbb{R}$ satisfy the hypotheses of Lemma 3.4. Then we have*

$$n_*(t; P(b)UQ(b)) = O(t^{-2/(m+1)}), \quad t \downarrow 0, \tag{3.39}$$

n_* being the counting function defined in (2.2).

Proof. Evidently,

$$n_*(t; P(b)UQ(b)) = n_*(t; P(b)\mathcal{C}Q(b)) \le n_*(t; P(b)\mathcal{C}) = n_*(t; \mathcal{C}P(b)), \quad t > 0.$$

Applying Lemma 3.4, we find that the following estimates

$$n_*(t; \mathcal{C}P(b)) = n_*(t; \mathcal{B}W_{m+1}P(b)) \le$$

$$n_*(t; \|\mathcal{B}\|W_{m+1}P(b)) = n_+(t^2; \|\mathcal{B}\|^2 P(b)W_{2m+2}P(b))$$

are valid for each $t > 0$. Bearing in mind (3.37), we get (3.39). □

3.5. In this section we complete the proof of Theorem 2.1.

Proposition 3.5. *Assume that the hypotheses of Theorem 2.1 hold. Then we have*

$$\lim_{\lambda \downarrow 0} \lambda^{2/\alpha} n_+(\lambda; P(b)|V_{11}|P(b)) = \frac{b_0}{4\pi} \int_{\mathbb{S}^1} v_{11}(s)^{2/\alpha} ds. \tag{3.40}$$

Proof. Fix $E \in (0, \lambda_0)$. Then (3.33) implies

$$\lim_{\lambda \downarrow 0} \lambda^{2/\alpha} n_+(\lambda/E; |V_{11}|^{1/2}(E - H_1)^{-1}|V_{11}|^{1/2}) = \frac{b_0}{4\pi} \int_{\mathbb{S}^1} v_{11}(s)^{2/\alpha} ds. \tag{3.41}$$

As in the proof of (3.13) we get

$$n_+(\lambda/E; |V_{11}|^{1/2}(E - H_1)^{-1}|V_{11}|^{1/2}) \le n_+(\lambda; P(b)|V_{11}|P(b)), \quad \lambda > 0. \tag{3.42}$$

Combining (3.41) and (3.42), we obtain

$$\frac{b_0}{4\pi} \int_{\mathbb{S}^1} v_{11}(s)^{2/\alpha} ds \le \liminf_{\lambda \downarrow 0} \lambda^{2/\alpha} n_+(\lambda; P(b)|V_{11}|P(b)). \tag{3.43}$$

Further, the minimax principle and the Weyl inequalities imply

$$n_+(\lambda/E; |V_{11}|^{1/2}(E - H_1)^{-1}|V_{11}|^{1/2}) \ge n_+(\lambda/E; P(b)|V_{11}|^{1/2}(E - H_1)^{-1}|V_{11}|^{1/2}P(b)) \ge$$

$$n_+((1+\varepsilon)\lambda; P(b)|V_{11}|^{1/2}P(b)|V_{11}|^{1/2}P(b)) -$$

$$n_-(\varepsilon\lambda/E; P(b)|V_{11}|^{1/2}Q(b)(E-H_1)^{-1}|V_{11}|^{1/2}P(b)). \tag{3.44}$$

Let us estimate at first the second term at the r.h.s. of (3.44). Obviously, for $t>0$ we have

$$n_-(t; P(b)|V_{11}|^{1/2}Q(b)(E-H_1)^{-1}|V_{11}|^{1/2}P(b)) =$$

$$n_+(t; |H_1-E|^{-1/2}Q(b)|V_{11}|^{1/2}P(b)|V_{11}|^{1/2}Q(b)|H_1-E|^{-1/2}). \tag{3.45}$$

Since $\||H_1-E|^{-1/2}Q(b)\| \leq (\lambda_0-E)^{-1/2}$, we have

$$n_+(t; |H_1-E|^{-1/2}Q(b)|V_{11}|^{1/2}P(b)|V_{11}|^{1/2}Q(b)|H_1-E|^{-1/2}) \leq$$

$$n_*(((\lambda_0-E)t)^{1/2}; P(b)|V_{11}|^{1/2}Q(b)), \quad t>0. \tag{3.46}$$

The combination of (3.45), (3.46), and (3.39) with $m=\alpha/2$, yields

$$n_-(\varepsilon\lambda/E; P(b)|V_{11}|^{1/2}Q(b)(E-H_1)^{-1}|V_{11}|^{1/2}P(b)) = O(\lambda^{-2/(2+\alpha)}) = o(\lambda^{-2/\alpha}), \quad \lambda\downarrow 0. \tag{3.47}$$

Now we estimate the first term at the r.h.s. of (3.44). For $t>0$ and $\varepsilon>0$ we have

$$n_+(t; P(b)|V_{11}|^{1/2}P(b)|V_{11}|^{1/2}P(b)) = n_+(t^{1/2}; P(b)|V_{11}|^{1/2}P(b)) =$$

$$n_*(t^{1/2}; P(b)|V_{11}|^{1/2}P(b)) \leq n_*((1+\varepsilon)t^{1/2}; P(b)|V_{11}|^{1/2}) + n_*(\varepsilon t^{1/2}; P(b)|V_{11}|^{1/2}Q(b)). \tag{3.48}$$

Evidently,

$$n_*((1+\varepsilon)t^{1/2}; P(b)|V_{11}|^{1/2}) = n_+((1+\varepsilon)^2 t; P(b)|V_{11}|P(b)), \quad t>0, \quad \varepsilon>0. \tag{3.49}$$

On the other hand, Corollary 3.2 implies

$$n_*(\varepsilon t^{1/2}; P(b)|V_{11}|^{1/2}Q(b)) = O(t^{-2/(2+\alpha)}) = o(t^{-2/\alpha}), \quad t\downarrow 0. \tag{3.50}$$

Putting together (3.41), (3.44), and (3.47) – (3.50), we get

$$\limsup_{\lambda\downarrow 0}\lambda^{2/\alpha} n_+(\lambda; P(b)|V_{11}|P(b)) \leq (1+\varepsilon)^{6/\alpha}\frac{b_0}{4\pi}\int_{\mathbb{S}^1} v_{11}(s)^{2/\alpha}ds, \quad \varepsilon>0. \tag{3.51}$$

Letting $\varepsilon\downarrow 0$ in (3.51), and combining this upper bound with the corresponding lower bound (3.43), we get (3.40). □

In order to complete the proof of Theorem 2.1 it remains to note that asymptotic relations (2.5) which are equivalent to (2.4), now follow from from (3.4) – (3.5), and (3.40).

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