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Inventiones mathematicae

Eigenvalue asymptotics for the Schrödinger operator with perturbed periodic potential

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Summary. We consider the Schrödinger operator $H = -\Delta + W + V$ acting in $L^2(\mathbb{R}^m)$, $m \geqq 2$, with periodic potential W perturbed by a potential V which decays slowly at infinity. We study the asymptotic behaviour of the discrete spectrum of H near any given boundary point of the essential spectrum.

0 Introduction

Let

$$\Gamma := \{\mathbf{x} \in \mathbb{R}^m \colon \mathbf{x} = \textstyle\sum_{j=1}^m n_j \mathbf{a}_j,\, n_j \in \mathbb{Z},\, j = 1, \ldots, m\}$$

be a lattice associated with the basis $\{\mathbf{a}_j\}_{j=1}^m$ in $\mathbb{R}^m$, $m \geqq 2$. Assume that the function $W \in C^\infty(\mathbb{R}^m)$ is periodic on the lattice Γ, i.e. we have

$$W(\mathbf{x} + \gamma) = W(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^m, \forall \gamma \in \Gamma\,. \tag{0.1}$$

Suppose that the perturbation V satisfies the estimates

$$|D^\beta V(\mathbf{x})| \leqq c_\beta \langle \mathbf{x} \rangle^{-\alpha - |\beta|}, \; \alpha \in (0, 2), \; \langle \mathbf{x} \rangle := (1 + |\mathbf{x}|^2)^{1/2}\,, \tag{0.2}$$

for each $\mathbf{x} \in \mathbb{R}^m$, for each multiindex β, and some constants c_β.

On the Sobolev space $H^2(\mathbb{R}^m)$ introduce the Schrödinger operator

$$\mathscr{H} := \mathscr{H}_0 + V$$

where

$$\mathscr{H}_0 := -\Delta + W\,.$$

Then the operators $\mathscr{H}$ and $\mathscr{H}_0$ are selfadjoint in $L^2(\mathbb{R}^m)$.

Since V decays at infinity, the multiplier by V is a Δ-compact operator. Therefore, the essential spectrum $\sigma_{\mathrm{ess}}(\mathscr{H})$ of the operator $\mathscr{H}$ is independent of V and coincides with $\sigma_{\mathrm{ess}}(\mathscr{H}_0)$. On the other hand, the spectrum $\sigma(\mathscr{H}_0)$ of the operator $\mathscr{H}_0$ is purely continuous, i.e. we have

$$\sigma_{\mathrm{ess}}(\mathscr{H}) = \sigma_{\mathrm{ess}}(\mathscr{H}_0) = \sigma_c(\mathscr{H}_0) = \sigma(\mathscr{H}_0)\,.$$

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Moreover, $\sigma(\mathscr{H}_0)$ has a band structure

$$\sigma(\mathscr{H}_0) = \bigcup_{l=1}^{\infty} I_l \tag{0.3}$$

where I_l are compact intervals which may be pairwise disjoint or overlapping. Denote by $(\mathscr{E}_k^-, \mathscr{E}_k^+)$, $k = 0, 1, \ldots, K$, the gaps in $\sigma(\mathscr{H}_0)$ so that we have

$$\mathbb{R}\backslash\sigma(\mathscr{H}_0) = \bigcup_{k=0}^{K} (\mathscr{E}_k^-, \mathscr{E}_k^+), \mathscr{E}_k^+ < \mathscr{E}_{k+1}^-, \quad k = 0, 1, \ldots, K \leqq \infty .$$

Note that the first gap $(\mathscr{E}_0^-, \mathscr{E}_0^+)$ is semi-infinite, i.e. $\mathscr{E}_0^- = -\infty$. Moreover, the total number K of the gaps $(\mathscr{E}_k^-, \mathscr{E}_k^+)$ is usually finite since we consider the multidimensional case $m \geqq 2$ (see [Skr, Theorem 12.1 and Theorem 15.2]). However, the finiteness of K is of no importance for the problem studied here.

The aim of the paper is to investigate the behaviour of the eigenvalues of the operator $\mathscr{H}$ near a given tip $\mathscr{E}_0^+$ or $\mathscr{E}_k^\pm$, $k = 1, \ldots, K$, of $\sigma_{ess}(\mathscr{H})$. Under some complementary generic assumptions about the structure of $\sigma_{ess}(\mathscr{H})$ we obtain the main asymptotic term of the counting function $\mathscr{N}_k^-(\lambda)$ (or, respectively, $\mathscr{N}_k^+(\lambda)$) describing the distribution of the isolated eigenvalues of $\mathscr{H}$ which accumulate from the right to $\mathscr{E}_k^-$ (or, respectively, from the left to $\mathscr{E}_k^+$).

Our results are related to the ones in [De–He, Al–De–He, Ge–Si, He, Bir and Sob]. The authors of these works, however, investigate the asymptotics as $g \to \infty$ of the number of the eigenvalues of the operator $-\Delta + W \mp tV$, $V \geqq 0$, which cross a given energy level $\mathscr{E} \in [\mathscr{E}_k^-, \mathscr{E}_k^+]$, $k = 1, \ldots, K$, as the coupling constant t grows from zero to the value g, while in the present paper we study the asymptotics as $\lambda \downarrow 0$ of the number $\mathscr{N}_k^\pm(\lambda)$ of the eigenvalues of $\mathscr{H}$ lying on the interval $(\mathscr{E}_k^- + \lambda, \mathscr{E})$ (or, respectively, on $(\mathscr{E}, \mathscr{E}_k^+ - \lambda)$) with fixed $\mathscr{E} \in (\mathscr{E}_k^-, \mathscr{E}_k^+)$.

In the one-dimensional case $m = 1$ the asymptotics of $\mathscr{N}_k^\pm(\lambda)$ have been studied in [Zel] under restrictive assumptions about V. A considerable extension of these one-dimensional results has been announced in [Khr 1]. The results of the present paper have been announced in the author's short communication [Rai 3]. After the submission of the paper the author knew about the article [Khr 2] where results on the asymptotics of $\mathscr{N}_k^\pm(\lambda)$ for $m \geqq 2$ had been announced without proofs; some of these results are similar to the ones established here.

The results of the paper are obtained by means of variational methods. The problem under consideration, however, requires some nontrivial generalizations of these methods since, generically there are segments of $\sigma_{ess}(\mathscr{H})$ lying on both sides of the infinite eigenvalue sequence we study. Similar difficulties arise in the linear magnetohydrodynamic theory (see [Rai 1, Theorem 2.15 and Theorem 2.16]), and in the spectral theory of the Schrödinger operator with constant magnetic field in some even-dimensional cases (see [Rai 2, Theorem 2.6]). Moreover, we apply some techniques typical for the theory of pseudodifferential operators, and use some facts from the theory of Bloch waves. Finally, we employ standard interpolation methods.

1 Statement of the main result

1.1 In order to state our main result we need a more detailed description of $\sigma_{ess}(\mathscr{H})$.

Denote by Ω the fundamental domain of the torus $\mathbb{R}^m/\Gamma$. Further, let $\{\mathbf{b}_j\}_{j=1}^m$ be the basis dual to $\{\mathbf{a}_j\}_{j=1}^m$, i.e. the basis satisfying

$$(\mathbf{a}_i, \mathbf{b}_j) = 2\pi\delta_{ij}, \quad i, j = 1, \ldots, m ,$$

where $(.,.)$ is the inner product in $\mathbb{R}^m$ generated by the euclidean metrics. Set

$$\Gamma^* := \left\{\mathbf{x} \in \mathbb{R}^m : \mathbf{x} = \sum_{j=1}^{m} n_j \mathbf{b}_j,\ n_j \in \mathbb{Z},\ j = 1, \ldots, m\right\}$$

and denote by Ω^* the fundamental domain of the torus $\mathbb{R}^m/\Gamma^*$. The domain Ω^* is known in the physical literature as the Brillouin zone.

For a fixed $\theta \in \bar{\Omega}^*$ introduce the operator

$$\mathfrak{h}_\theta := (i\nabla - \theta)^2 + W \tag{1.1}$$

with domain the Sobolev space $H^2(\mathbb{R}^m/\Gamma)$. Obviously, the operator $\mathfrak{h}_\theta$ is selfadjoint in $L^2(\mathbb{R}^m/\Gamma)$. Since the manifold $\mathbb{R}^m/\Gamma$ is compact and the operator $\mathfrak{h}_\theta$ is elliptic, the spectrum of $\mathfrak{h}_\theta$ is purely discrete for each $\theta \in \bar{\Omega}^*$. Denote by

$$E_1(\theta) \leqq E_2(\theta) \leqq E_3(\theta) \leqq \ldots$$

the nondecreasing sequence of the eigenvalues of the operator $\mathfrak{h}_\theta$, $\theta \in \bar{\Omega}^*$. It follows from an elementary perturbation-theory argument that each function $E_l(\theta)$, $l \geqq 1$, is continuous over $\bar{\Omega}^*$. Moreover, it can be shown that E_l, $l \geqq 1$, are real-anaytic functions over $\mathbb{R}^m/\Gamma^*$ outside the nullset of the branching points (see [Wilc, Theorem 1] for the case $m = 3$).

Introduce the Hilbert space

$$\mathfrak{L} := \int_{\mathbb{R}^m/\Gamma^*} \oplus L^2(\mathbb{R}^m/\Gamma) \frac{d\theta}{\operatorname{vol} \Omega^*} . \tag{1.2}$$

We can identify $\mathfrak{L}$ with $L^2(\Omega \times \Omega^*; d\mathbf{x}\, d\theta/\operatorname{vol} \Omega^*)$ where $d\mathbf{x}$ and $d\theta$ are the Lebesgue measures respectively over Ω and Ω^*. Introduce the operator

$$\mathfrak{H}_0 := \int_{\mathbb{R}^m/\Gamma^*} \oplus \mathfrak{h}_\theta \frac{d\theta}{\operatorname{vol} \Omega^*} \tag{1.3}$$

which is selfadjoint in $\mathfrak{L}$. Define the operator U: $L^2(\mathbb{R}^m) \to \mathfrak{L}$ by

$$(Uf)(x, \theta) := \sum_{\gamma \in \Gamma} \exp\{-i\theta \cdot (\mathbf{x} + \gamma)\} f(\mathbf{x} + \gamma),\ f \in L^2(\mathbb{R}^m). \tag{1.4}$$

It is well-known that the operator U is an isometric mapping and we have

$$U \mathscr{H}_0 U^* = \mathfrak{H}_0$$

(see [Re–Si IV, Sect. XIII.16]). Hence, the relation (0.3) holds with

$$I_l = \bigcup_{\theta \in \bar{\Omega}^*} \{E_l(\theta)\}, \quad l \geqq 1 .$$

Moreover, the boundary points $\mathscr{E}_k^-$, $k = 1, \ldots, K$, (respectively, $\mathscr{E}_k^+$, $k = 0, \ldots, K$) of $\sigma_{\mathrm{ess}}(\mathscr{H})$ coincide with the maximum (respectively, minimum) value of some function in the sequence $\{E_l(\theta)\}_{l \geqq 1}$.

We shall say that the value $\mathscr{E}_k^{\pm}$ is regular if only if the following conditions are fulfilled:

(i) there is a unique function $E_{k_0}^{\pm}(\theta)$ in the sequence $\{E_l(\theta)\}_{l \geqq 1}$ which takes the value $\mathscr{E}_k^{\pm}$;

(ii) the function $E_{k_0}^{\pm}(\theta)$ takes the value $\mathscr{E}_k^{\pm}$ at finitely many points $\theta = \theta_{k;j}^{\pm} \in \Omega^*$, $j = 1, \ldots, J_k^{\pm} < \infty$;

(iii) the Hesse matrix $M_k^{\pm}(\theta) := \{M_{k;r,s}^{\pm}(\theta)\}_{r,s=1}^{m}$ with entries

$$M_{k;r,s}^{\pm}(\theta) := \pm \partial^2 E_{k_0}^{\pm}(\theta)/\partial\theta_r \partial\theta_s, \quad r, s = 1, \ldots, m,$$

is positively definite at the points $\theta = \theta_{k;j}^{\pm}, j = 1, \ldots, J_k^{\pm}$.

Remark. If the conditions (i) and (ii) hold, then the function $E_{k_0}^{\pm}$ is analytic in a vicinity of each point $\theta_{k;j}^{\pm}$, $j = 1, \ldots, J_k^{\pm}$, where it takes the value $\mathscr{E}_k^{\pm}$ so that the condition (iii) makes sense.

The eigenvalues of the matrix inverse to $M_k^{\pm}(\theta_{k;j}^{\pm})$ are known in the physical literature as the effective masses at the points $\theta_{k;j}^{\pm}$, $j = 1, \ldots, J_k^{\pm}$.

1.2 Let T be a selfadjoint operator in a Hilbert space. Let $I \subset \mathbb{R}$ be an open interval such that the spectrum of T on I is purely discrete. Then $N(I \mid T)$ denotes the number of the eigenvalues of T lying on I, counted with the multiplicities.

Set

$$\mathscr{N}_0^+(\lambda) := N(-\infty, \mathscr{E}_0^+ - \lambda \mid \mathscr{H}), \quad \lambda > 0.$$

Let $(\mathscr{E}_k^-, \mathscr{E}_k^+)$, $k = 1, \ldots, K$, be a bounded gap in $\sigma(\mathscr{H}_0) \equiv \sigma_{\text{ess}}(\mathscr{H})$. Fix $\mathscr{E} \in (\mathscr{E}_k^-, \mathscr{E}_k^+)$ and for $\lambda > 0$ small enough put

$$\mathscr{N}_k^-(\lambda) := N(\mathscr{E}_k^- + \lambda, \mathscr{E} \mid \mathscr{H}), \quad k = 1, \ldots, K,$$

$$\mathscr{N}_k^+(\lambda) := N(\mathscr{E}, \mathscr{E}_k^+ - \lambda \mid \mathscr{H}), \quad k = 1, \ldots, K.$$

Throughout the formulation of the Main Theorem below we follow either the upper or the lower sign whenever double signs " $\pm$ " or " $\mp$ " are met; thus the Main Theorem contains two independent assertions concerning the upper or the lower sign.

Main Theorem. *Let W satisfy* (0.1) *and V satisfy* (0.2). *Moreover, suppose that the estimate*

$$\text{vol}\{\mathbf{x} \in \mathbb{R}^m : \mp V(\mathbf{x}) > \mu\} \geqq c\mu^{-m/\alpha}, \quad c > 0, \tag{1.5$_\pm$}$$

holds for $\mu > 0$ small enough with the same $\alpha \in (0,2)$ as in (0.2).

Let $(\mathscr{E}_k^-, \mathscr{E}_k^+)$ be a gap in $\sigma(\mathscr{H}_0)$. Assume that the value $\mathscr{E}_0^+$ or *$\mathscr{E}_k^{\pm}$, $k = 1, \ldots,$ K, is regular. Then we have*

$$\mathscr{N}_k^{\pm}(\lambda) = \frac{1}{(2\pi)^m} \sum_{j=1}^{J_k^{\pm}} \text{vol}\{(x, \theta) \in \mathbb{R}^{2m} :$$

$$\tfrac{1}{2}(\mathscr{M}_k^{\pm}(\theta_{k;j}^{\pm})\theta, \theta) \pm V(\mathbf{x}) < -\lambda\}(1 + o(1)), \lambda \downarrow 0. \tag{1.6$_\pm$}$$

The estimate (0.2) with $\beta = 0$ combined with the estimate (1.5)$_\pm$ implies that the quantity at the right-hand side of (1.6)$_\pm$ is of order $\lambda^{-m(2-\alpha)/2\alpha}$ as $\lambda \downarrow 0$.

Moreover, under the hypotheses of the Main Theorem the asymptotics (1.6)$_\pm$ can be re-written in the form

$$\mathscr{N}_k^{\pm}(\lambda) = (2\pi)^{-m}\text{vol}\{\mathbf{x} \in \mathbb{R}^m, \theta \in \Omega^* :$$

$$\pm \mathscr{E}_{k_0}^{\pm}(\theta) \pm V(\mathbf{x}) < \pm \mathscr{E}_k^{\pm} - \lambda\}(1 + o(1)), \lambda \downarrow 0,$$

or in the form

$$\mathcal{N}_k^{\pm}(\lambda) = \mathscr{C}_k^{\pm} \int_{\mathbb{R}^m} (\pm V(\mathbf{x}) + \lambda)_{-}^{m/2} d\mathbf{x}(1 + o(1)), \; \lambda \downarrow 0 \,,$$

with

$$\mathscr{C}_k^{\pm} := v_m/(2\pi^2)^{m/2} \mathfrak{M}_k^{\pm} \,, \tag{1.7$_\pm$}$$

where v_m is the volume of the unit ball in $\mathbb{R}^m$, and

$$\frac{1}{\mathfrak{M}_k^{\pm}} := \sum_{j=1}^{J_k^{\pm}} (\det \mathcal{M}_k^{\pm}(\theta_{k;j}^{\pm}))^{-1/2}.$$

Furthermore, if we assume that the perturbation V obeys the asymptotics

$$V(\mathbf{x}) = \frac{\Phi(\hat{\mathbf{x}})}{|\mathbf{x}|^{\alpha}}(1 + o(1)), \; |\mathbf{x}| \to \infty, \; \hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}| \,,$$

in addition to the hypotheses of the Main Theorem, we can further simplify the relations $(1.6)_{\pm}$, writing them in the form

$$\lim_{\lambda \downarrow 0} \lambda^{m(2-\alpha)/2\alpha} \mathcal{N}_k^{\pm}(\lambda) = \mathscr{C}_k^{\pm} B(m(2-\alpha)/2\alpha, 1 + m/2) \int_{\mathbb{S}^{m-1}} \Phi_{\mp}^{m/\alpha}(\omega)\, d\sigma(\omega)/\alpha$$

where $B(p, q)$, $p > 0$, $q > 0$, is the Euler beta function, and $d\sigma$ is the canonical measure over $\mathbb{S}^{m-1}$.

The proof of the Main Theorem can be found in Sect. 3, while Sect. 2 contains some necessary auxiliary results.

1.3 There exists a wide literature on the eigenvalue asymptotics of the same type for the case $W \equiv 0$ and $m \geqq 1$, i.e. the asymptotics of the quantity $N(-\infty, -\lambda | -\Delta + V)$ as $\lambda \downarrow 0$ (see [Roz, Tam]). In the following proposition we formulate a result of this type which will be used further on.

Proposition 1.1 *Let $\mathcal{M} : \mathbb{R}^m \to \mathbb{R}^m$, $m \geqq 2$, be a symmetric positively definite matrix. Assume that V satisfies* (0.2) *and* $(1.5)_{+}$. *Then we have*

$$N(-\infty, -\lambda | -(\mathcal{M}\nabla, \nabla) + V) = (2\pi)^{-m} \mathrm{vol}\{(\mathbf{x}, \theta) \in \mathbb{R}^{2m}: (\mathcal{M}\theta, \theta) + V(\mathbf{x}) < -\lambda\}(1 + o(1)), \quad \lambda \downarrow 0. \tag{1.8}$$

The asymptotics (1.8) would have been contained as a special case in the result of [Roz, Theorem 1] if we have assumed in addition $m \geqq 3$, $\mathcal{M} = \mathrm{Id}$ and $V \leqq 0$. However, it is not hard to verify (1.8) under the hypotheses of Proposition 1.1.

2 Auxiliary results

2.1 In this subsection we formulate several simple assertions from the abstract eigenvalue perturbation theory.

We denote by $\mathfrak{S}_\infty$ the space of compact linear operators acting in a given Hilbert space. If $T \in \mathfrak{S}_\infty$ and $\mu > 0$, then $\nu(\mu; T) \equiv N(\mu^2 | T^*T)$ is the number of the singular values of the operator T larger than μ counted with the multiplicities. Obviously, if $T = T^*$, we have

$$N(\mu, \infty | \pm T) \leqq \nu(\mu; T), \quad \forall \mu > 0. \tag{2.1$_\pm$}$$

Moreover, if $T_i \in \mathfrak{S}_\infty$, $i = 1, 2$, then the Weyl–Ky Fan inequality

$$\nu(\mu_1 + \mu_2; T_1 + T_2) \leqq \nu(\mu_1; T_1) + \nu(\mu_2; T_2) \tag{2.2}$$

holds for each $\mu_i > 0$, $i = 1, 2$, (see [Goh–Kr, Chap. II, Subsect. 2.3]). We would also mention some versions of this inequality concerning the eigenvalues of selfadjoint operators. Namely, if $T_i = T_i^* \in \mathfrak{S}_\infty$, $i = 1, 2$, then the inequalities

$$\pm N(\mu, \infty | T_1 + T_2) \leqq \pm N(\mu(1 \mp \tau), \infty | T_1) + N(\mu\tau, \infty; \pm T_2) \qquad (2.3)_\pm$$

hold for each $\mu > 0$ and $\tau \in (0, 1)$ (see [Goh–Kr, Chap. II, Subsect. 2.3]).

Lemma 2.1 [Bir–Sol 2, Chap. 9, Sect. 3, Theorem 3] *Let T_i, $i = 0, 1$, be selfadjoint operators in Hilbert space. Assume* $\operatorname{rank} T_1 < \infty$. *Then for each* $-\infty < \mu_1 < \mu_2 < \infty$ *we have*

$$|N(\mu_1, \mu_2 | T_0) - N(\mu_1, \mu_2 | T_0 + T_1)| \leqq \operatorname{rank} T_1 .$$

Lemma 2.2 [Bir–Sol 2, Chap. 9, Sect. 4, Lemma 1] *Let T_i, $i = 0, 1$, be selfadjoint operators in Hilbert space. Suppose that T_1 is bounded and $\sigma(T_1) \subseteq [\tau_1, \tau_2]$. Then the estimates*

$$N(\mu_1, \mu_2 | T_0) \leqq N(\mu_1 + \tau_1, \mu_2 + \tau_2 | T_0 + T_1)$$

hold for each $-\infty < \mu_1 < \mu_2 < \infty$.

2.2 This subsection contains some necessary estimates of the spectrum of integral operators.

In what follows we denote by $\mathfrak{S}_p$, $1 \leqq p < \infty$, the space of compact operators such that the norm

$$\| T \|_{\mathfrak{S}_p} := (\operatorname{Tr} | T |^p)^{1/p}$$

is finite (see e.g. [Bir–Sol 2, Chap. 11]). Obviously, if $T \in \mathfrak{S}_p$, $1 \leqq p < \infty$, then the elementary estimate

$$\nu(\mu; T) \leqq \mu^{-p} \| T \|^p_{\mathfrak{S}_p} \tag{2.4}$$

is valid for each $\mu > 0$.

Let $\mathscr{D} \subseteq \mathbb{R}^m$ be an open set, and G: $L^2(\mathbb{R}^m) \to L^2(\mathscr{D})$ be a bounded integral operator with kernel $g \in L^\infty(\mathscr{D} \times \mathbb{R}^m)$. Let $f \in L^p(\mathbb{R}^m)$, $a \in L^{2p}(\mathscr{D})$, $b \in L^{2p}(\mathscr{D})$, $p \in [1, \infty]$; we denote the multipliers by a, $\bar{b}$ and f by the same symbols. Define the operator $T_1 : L^2(\mathscr{D}) \to L^2(\mathscr{D})$ by

$$T_1 := aGfG^*\bar{b} . \tag{2.5}$$

Lemma 2.3 *Let $p \in [1, \infty)$. Then the operator T_1 defined in* (2.5) *is compact, and the estimate*

$$\nu(\mu; T_1) \leqq C_p \mu^{-p} \| a \|^p_{L^{2p}} \| b \|^p_{L^{2p}} \| f \|^p_{L^p}, \quad \forall \mu > 0 , \tag{2.6}$$

holds with $C_p := \| g \|^{2/p}_{L^\infty} \| G \|^{2(p-1)/p}$.

Proof. The estimate (2.6) will follow from (2.4) and the estimate

$$\| T_1 \|^p_{\mathfrak{S}_p} \leqq C_p \| a \|^p_{L^{2p}} \| b \|^p_{L^{2p}} \| f \|^p_{L^p}, \; p \in [1, \infty) . \tag{2.7}$$

In order to verify (2.7), first of all note that we have

$$\| T_1 \| \leqq \| G \|^2 \| a \|_{L^\infty} \| b \|_{L^\infty} \| f \|_{L^\infty} . \tag{2.8}$$

Further, assume $p = 1$ and write $T_1 = T_1' T_1''$ where $T_1' : L^2(\mathscr{D}_\theta) \to L^2(\mathbb{R}_{\mathbf{x}}^m)$ is an integral operator with kernel $a(\theta)\, g(\theta, \mathbf{x})\, |f(\mathbf{x})|^{1/2}$, and $T_1'' : L^2(\mathbb{R}_{\mathbf{x}}^m) \to L^2(\mathscr{D}_\theta)$ is an integral operator with kernel $g(\theta, \mathbf{x})(\operatorname{sign} f(\mathbf{x}))|f(\mathbf{x})|^{1/2}\overline{b(\theta)}$. Combining the obvious inequalities

$$\| T_1 \|_{\mathfrak{S}^1} \leqq \| T_1' \|_{\mathfrak{S}^2} \| T_1'' \|_{\mathfrak{S}^2}$$

and

$$\| T_1' \|_{\mathfrak{S}^2}^2 \leqq \| g \|_{L^\infty}^2 \| a \|_{L^2}^2 \| f \|_{L^1} ,$$
$$\| T_1'' \|_{\mathfrak{S}^2}^2 \leqq \| g \|_{L^\infty}^2 \| b \|_{L^2}^2 \| f \|_{L^1} ,$$

we obtain

$$\| T_1 \|_{\mathfrak{S}^1} \leqq \| g \|_{L^\infty}^2 \| a \|_{L^2} \| b \|_{L^2} \| f \|_{L^1} . \tag{2.9}$$

Applying standard polylinear interpolation (see e.g. [Ber–Löf, Sect. 4.4]), we find that the estimate (2.7) follows from (2.8)–(2.9). □

Remark. The result of Lemma 2.3 and its various generalizations are well-known for the case $g(\theta, \mathbf{x}) = \exp\{i\mathbf{x} \cdot \theta\}$, i.e. for the case where G is the Fourier transform (see [Bir–Sol 1, Theorem 6.3] or [Re–Si III, Theorem XI.20]). The author is obliged to Prof. M.S. Birman for drawing his attention to the possible extensions for general G.

Now, let $T_2 : L^2(\mathscr{D}) \to L^2(\mathscr{D})$ be an integral operator with kernel

$$a(\theta)\overline{b(\theta')}t(\theta, \theta')|\theta - \theta'|^{-m+\delta}, \quad \theta, \theta' \in \mathscr{D} .$$

Lemma 2.4 [Kos, Theorem 1] *Assume that $\mathscr{D}$ is bounded. Let $0 < \delta < m/2$, $a \in L^p(\mathscr{D})$, $b \in L^p(\mathscr{D})$, with $p > 2m/\delta$, and $t \in L^\infty(\mathscr{D} \times \mathscr{D})$. Then the operator T_2 is compact and we have*

$$\nu(\mu; T_2) \leqslant c\mu^{-m/\delta} \| a \|_{L^p}^{m/\delta} \| b \|_{L^p}^{m/\delta} \| t \|_{L^\infty}^{m/\delta}, \ \forall \mu > 0 ,$$

where the constant c is independent of μ, a, b and t.

Lemma 2.5 *Let $m/2 < \delta < m$, $a \in L^p(\mathscr{D})$, $b \in L^p(\mathscr{D})$, with $p = 2m/\delta$, and $t \in L^\infty(\mathscr{D} \times \mathscr{D})$. Then the operator T_2 is compact and we have*

$$\nu(\mu; T_2) \leqq c\mu^{-2} \| a \|_{L^p}^2 \| b \|_{L^p}^2 \| t \|_{L^\infty}^2, \quad \forall \mu > 0 , \tag{2.10}$$

where the constant c is again independent of μ, a, b and t.

Proof. The estimate (2.10) follows from (2.4) with $p = 2$ combined with the identity

$$\| T_2 \|_{\mathfrak{S}^2}^2 = \int_{\mathscr{D}} \int_{\mathscr{D}} |a(\theta)|^2 |b(\theta')|^2 |t(\theta, \theta')|^2 |\theta - \theta'|^{-2m+2\delta} d\theta\, d\theta'$$

and the well-known Sobolev inequality (see e.g. [Re–Si II, Sect. IX.4, Example 3]), written in the form

$$\int_{\mathscr{D}} \int_{\mathscr{D}} |a(\theta)|^2 |b(\theta')|^2 |\theta - \theta'|^{-2m+2\delta} d\theta\, d\theta' \leqq c'_{m,\delta} \| a \|_{L^p}^2 \| b \|_{L^p}^2 .$$ □

3 Proof of the Main Theorem

3.1 For definiteness we shall prove the Main Theorem for the case of a minimum $\mathscr{E}_k^+$, $k = 1, \ldots, K$. The proof in the case of a maximum $\mathscr{E}_k^-$ is analogous, and the minor modifications needed are quite obvious.

For simplicity, we assume that the value $\mathscr{E}_k^+$ is taken by the function $E_{k_0}^+(\theta)$ at a unique point $\theta_k \in \Omega^*$. Evidently, no loss of generality follows from this assumption.

First of all we use the operator U (see (1.4)) in order to show that the operator $\mathscr{H}$ is unitarily equivalent to the operator

$$\mathfrak{H} := \mathfrak{H}_0 + \mathfrak{B}$$

where $\mathfrak{H}_0$ is introduced in (1.3), and $\mathfrak{B} := UVU^*$ is defined by

$$(\mathfrak{B}u)(\mathbf{x}, \theta) = \frac{1}{\operatorname{vol} \Omega^*} \sum_{\gamma \in \Gamma} V(\mathbf{x} + \gamma) \int_{\mathbb{R}^m/\Gamma^*} \exp\{i(\theta' - \theta)\cdot(\mathbf{x} + \gamma)\} u(\mathbf{x}, \theta')\, d\theta', \quad u \in \mathfrak{L},$$

(see (1.2)). Then we have

$$\mathscr{N}_k^+(\lambda) = N(\mathscr{E}, \mathscr{E}_k^+ - \lambda | \mathfrak{H}) . \tag{3.1}$$

3.2 Let $\mathcal{O} \subset \Omega^*$ be an open ball centred at θ_k such that $E_{k_0}^+(\theta)$ is a simple eigenvalue of the operator $\mathfrak{h}_\theta$ (see (1.1)) for each $\theta \in \bar{\mathcal{O}}$. Note that we have

$$\inf_{\theta \in \Omega^* \setminus \mathcal{O}} (E_{k_0}^+(\theta) - \mathscr{E}_k^+) > 0 . \tag{3.2}$$

Let $\psi(\mathbf{x}, \theta)$ be the eigenfunction of the operator $\mathfrak{h}_\theta$ corresponding to the eigenvalue $E_{k_0}^+(\theta)$, $\theta \in \bar{\mathcal{O}}$. We assume that ψ possesses the following properties:

a) $\int |\psi(\mathbf{x}, \theta)|^2 \, d\mathbf{x} = 1, \ \forall \theta \in \mathcal{O}$;

b) $\psi \in L^\infty(\Omega \times \mathcal{O})$;

c) the mapping $\theta \to \psi(\cdot, \theta)$ is C^2-smooth with respect to $\theta \in \bar{\mathcal{O}}$ in the $L^2(\Omega)$-topology.

The existence of an eigenfunction ψ with the properties a)–c) follows easily from [Ka, Chap. II, Sect. 5, Theorem 5.16] and the fact that the eigenvalue $E_{k_0}^+(\theta)$ is simple for $\theta \in \bar{\mathcal{O}}$ (see also [Wilc]). As a matter of fact the function ψ is smoother than it is pointed in the conditions b) and c). We state these conditions in a form which is sufficient for our purposes.

For further references we establish here some properties of the function $\psi(\mathbf{x}, \theta)$ which is extended when necessary by periodicity for all $\mathbf{x} \in \mathbb{R}^m$. Set

$$\psi_{\gamma'}(\theta) := \int_\Omega \exp\{i\gamma' \cdot \mathbf{x}\} \psi(\mathbf{x}, \theta)\, d\mathbf{x}, \quad \gamma' \in \Gamma^*, \theta \in \bar{\Omega}^*.$$

Note that we have

$$\psi(\mathbf{x}, \theta) = \frac{1}{\operatorname{vol} \Omega} \sum_{\gamma' \in \Gamma^*} \exp\{-i\gamma' \cdot \mathbf{x}\} \psi_{\gamma'}(\theta), \quad \mathbf{x} \in \Omega, \theta \in \mathcal{O} .$$

Hence, the property a) implies

$$\sum_{\gamma' \in \Gamma^*} |\psi_{\gamma'}(\theta)|^2 = \operatorname{vol} \Omega, \quad \theta \in \Omega . \tag{3.3}$$

Put

$$\varphi(\mathbf{x}, \theta) = \exp\{i\mathbf{x} \cdot \theta\} \psi(\mathbf{x}, \theta), \quad \mathbf{x} \in \mathbb{R}^m, \theta \in \bar{\mathcal{O}} .$$

Define the operator $\Phi : L^2(\mathbb{R}^m) \to L^2(\mathcal{O})$ by

$$(\Phi u)(\theta) := \int_{\mathbb{R}^m} \overline{\varphi(\mathbf{x}, \theta)} u(\mathbf{x})\, d\mathbf{x}, \quad u \in L^2(\mathbb{R}^m).$$

Lemma 3.1 *The operator Φ is bounded.*

Proof. Obviously we have

$$(\Phi u)(\theta) = \frac{1}{\operatorname{vol}\Omega} \sum_{\gamma' \in \Gamma^*} \overline{\psi_{\gamma'}(\theta)}\check{u}(\theta + \gamma'), \quad u \in \mathscr{S}(\mathbb{R}^m),$$

where

$$\check{u}(\mathbf{y}) := \int_{\mathbb{R}^m} \exp\{-i\mathbf{x}\cdot\mathbf{y}\} u(\mathbf{x})\, d\mathbf{x}, \quad y \in \mathbb{R}^m.$$

By the Schwarz inequality, the identity (3.3) and the Parseval equality, we get

$$\int_{\mathcal{O}} |(\Phi u)(\theta)|^2\, d\theta \leqq \frac{1}{\operatorname{vol}\Omega} \sum_{\gamma' \in \Gamma^*} \int_{\mathcal{O}} |\check{u}(\theta + \gamma')|^2\, d\theta \leqq$$

$$\leqq \frac{1}{\operatorname{vol}\Omega} \sum_{\gamma' \in \Gamma^*} \int_{\Omega^*} |\check{u}(\theta + \gamma')|^2\, d\theta = \frac{1}{\operatorname{vol}\Omega} \int_{\mathbb{R}^m} |\check{u}(\theta)|^2\, d\theta = \operatorname{vol}\Omega^* \int_{\mathbb{R}^m} |u(\mathbf{x})|^2\, d\mathbf{x};$$

in the last inequality we have also used the identity

$$\operatorname{vol}\Omega \operatorname{vol}\Omega^* = (2\pi)^m. \tag{3.4}$$

Thus we find that the norm of the operator Φ is upper bounded by $(\operatorname{vol}\Omega^*)^{1/2}$. □

Further, let $\zeta(\theta)$, $\theta \in \Omega^*$, be the characteristic function of the ball $\mathcal{O}$. Define the orthogonal projection $P: \mathfrak{L} \to \mathfrak{L}$ by

$$(Pu)(\mathbf{x}, \theta) = \zeta(\theta)\psi(\mathbf{x}, \theta) \int_{\Omega} \overline{\psi(\mathbf{x}', \theta)} u(\mathbf{x}', \theta)\, d\mathbf{x}', \quad u \in \mathfrak{L}.$$

Set $Q := \operatorname{Id} - P$.

Lemma 3.2 *Let $f \in L^p(\mathbb{R}^m)$, $p \in [2, \infty)$. Then we have*

$$PUfU^* \in \mathfrak{S}_p.$$

Proof. Assume at first $f \in L^\infty(\mathbb{R}^m)$. Obviously, we have

$$\|PUfU^*\| \leqq \|f\|_{L^\infty}.$$

Assume now $f \in L^2(\mathbb{R}^m)$. Then we have

$$\|PUfU^*\|_{\mathfrak{S}_2}^2 = \operatorname{Tr} PUfU^* U\bar{f}U^* P = \operatorname{Tr} PU|f|^2 U^* =$$

$$= \frac{1}{\operatorname{vol}\Omega^*} \int_{\mathbb{R}^m} \int_{\mathcal{O}} |\psi(\mathbf{x}, \theta)|^2 |f(\mathbf{x})|^2\, d\mathbf{x}\, d\theta \leqq \|\psi\|_{L^\infty(\Omega\times\mathcal{O})}^2 \|f\|_{L^2(\mathbb{R}^m)}^2.$$

Using a standard interpolation technique, we obtain the estimate

$$\|PUfU^*\|_{\mathfrak{S}_p}^p \leqq \|\psi\|_{L^\infty}^2 \|f\|_{L^p}^p, \quad p \in [2, \infty).$$ □

In the sequel we shall say that a function $f \in C^\infty(\mathbb{R}^m)$ is in the class $\mathscr{S}_\alpha$, $\alpha > 0$, if the estimates

$$|D^\beta f(\mathbf{x})| \leqq c_\beta \langle \mathbf{x} \rangle^{-\alpha - |\beta|}, \quad \forall \mathbf{x} \in \mathbb{R}^m,$$

hold for each multiindex β; thus, the estimates (0.2) are equivalent to the inclusion

$$V \in \mathscr{S}_\alpha, \quad \alpha \in (0, 2).$$

Moreover, we shall say that a function $f \in L^\infty(\mathbb{R}^m)$ is in the class $\mathring{\mathscr{S}}_\alpha$, $\alpha > 0$, if we have

$$|f(\mathbf{x})| \leqq c_0 \langle \mathbf{x} \rangle^{-\alpha}, \quad \forall \mathbf{x} \in \mathbb{R}^m .$$

Corollary 3.3 *Assume* $f \in \mathring{\mathscr{S}}_\alpha$, $\alpha > 0$. *Then we have* $f \in L^\infty \cap L^r$ *with* $r > \max\{1, m/\alpha\}$ *and, hence,* $PUfU^* \in \mathfrak{S}_p$ *with* $\max\{2, r\} < p < \infty$.

In particular, we find that $PUfU^*$ is compact. Therefore, if V satisfies (0.2), then the operator $P\mathfrak{B} \equiv PUVU^*$ is compact.

At the end of this subsection we introduce the integral

$$\mathscr{I}_p(\lambda; r_0, r_1) := \int_{r_0}^{r_1} \frac{r^{m-1}\, dr}{(r^2 + \lambda)^p}, \quad p \in [1, \infty),\ \lambda > 0,\ r_0 \geqq 0,\ r_1 \leqq \infty .$$

For further references we state here the following elementary asymptotic estimates

$$\mathscr{I}_p(\lambda; 0, r_1) = O(1),\ r_1 < \infty,\ 2p < m , \tag{3.5}$$

$$\mathscr{I}_p(\lambda; 0, r_1) = O(|\ln \lambda|),\ r_1 < \infty,\ 2p = m, \tag{3.6}$$

$$\mathscr{I}_p(\lambda; 0, r_1) = O(\lambda^{-(2p-m)/2}),\ r_1 < \infty,\ 2p > m , \tag{3.7}$$

$$\mathscr{I}_p(\lambda; r_0, \infty) = O(1),\ r_0 > 0,\ 2p > m , \tag{3.8}$$

which all hold as $\lambda \downarrow 0$.

3.3 For $\theta \in \bar{\mathcal{O}}$ put

$$\chi(\theta; \lambda) := (E_{k_0}^+(\theta) - \mathscr{E}_k^+ + \lambda)^{-1/2},\ \lambda > 0 .$$

For a given function $f \in \mathring{\mathscr{S}}_\delta$, $\delta > 0$, define the operator $\mathscr{F}(f): L^2(\mathcal{O}) \to L^2(\mathcal{O})$ as the operator T_1 defined in (2.5) with $\mathscr{D} = \mathcal{O}$, $a = b \equiv 1$ and $g(\theta, \mathbf{x}) = \overline{(\operatorname{vol} \Omega^*)^{-1/2} \varphi(\mathbf{x}, \theta)}$. Set

$$\mathscr{R}(f; \lambda) = -\chi(\lambda)\mathscr{F}(f)\chi(\lambda) .$$

Note that the operator $\mathscr{R}(f; \lambda): L^2(\mathcal{O}) \to L^2(\mathcal{O})$ can be regarded as a pseudodifferential operator with an amplitude

$$A(\theta, \theta', \mathbf{x}; \lambda) = -\operatorname{vol}\Omega\, \chi(\theta, \lambda)\overline{\psi(-\mathbf{x}, \theta)} f(-x)\psi(-\mathbf{x}, \theta')\chi(\theta', \lambda)$$

(see [Tay, Chap. II, Definition 3.5]).

Proposition 3.4 *The estimates*

$$\pm N(\mathscr{E}, \mathscr{E}_k^+ - \lambda | \mathfrak{H}) \leqq \pm N(1 \mp \tau, \infty | \mathscr{R}(V, \lambda)) + o(\lambda^{-m(2-\alpha)/2\alpha}), \quad \lambda \downarrow 0 , \tag{3.9$_\pm$}$$

hold for each $\tau \in (0, 1)$.

Proof. First of all, write the identity

$$\mathfrak{H} = (P + Q)\mathfrak{H}(P + Q) \equiv P\mathfrak{H}P + Q\mathfrak{H}Q + P\mathfrak{B}Q + Q\mathfrak{B}P . \tag{3.10}$$

Next, introduce the selfadjoint operator

$$\mathscr{K} := -i(P\mathfrak{B} - \mathfrak{B}P).$$

In view of Corollary 3.3, $\mathscr{K}$ is compact. Fix some real $\varepsilon \neq 0$ and denote by $\Pi \equiv \Pi_\varepsilon$ the spectral projection of $\mathscr{K}$ corresponding to the set $(-\infty, -\varepsilon^2) \cup (\varepsilon^2, \infty)$. Set $\mathscr{K}_0 := \Pi\mathscr{K}$, $\mathscr{K}_1 := \mathscr{K} - \mathscr{K}_0$. Obviously, we have

$$\operatorname{rank} \mathscr{K}_0 = N(-\infty, -\varepsilon^2 | \mathscr{K}) + N(\varepsilon^2, \infty | \mathscr{K}) , \tag{3.11}$$

and $\|\mathscr{K}_1\| \leqq \varepsilon^2$. Denote by $Y_\pm(\varepsilon)$ the operator generated in $\mathfrak{L}$ by the quadratic form

$$\|i\varepsilon Qu \pm \varepsilon^{-1}\mathscr{K}_1 Pu\|^2, \quad u \in \mathfrak{L} .$$

Obviously, we have

$$\sigma(Y_\pm(\varepsilon)) \subseteq [0, 4\varepsilon^2] .$$

Now we can re-write the identity (3.10) in the form

$$\mathfrak{H} = P(\mathfrak{H} \pm \varepsilon^{-2}\mathscr{K}_1^2)P + Q(\mathfrak{H} \pm \varepsilon^2)Q \mp Y_\mp(\varepsilon) + i(P\mathscr{K}_0 Q - Q\mathscr{K}_0 P) .$$

Note that the rank of the last term at the right-hand side of this identity does not exceed 2 rank $\mathscr{K}_0$. Applying Lemmas 2.1 and 2.2, we get

$$\pm N(\mathscr{E}, \mathscr{E}_k^+ - \lambda|\mathfrak{H}) \leqq \pm N(\mathscr{E} \mp 4\varepsilon^2, \mathscr{E}_k^+ - \lambda|P(\mathfrak{H} \pm \varepsilon^{-2}\mathscr{K}_1^2)P + Q(\mathfrak{H} \pm \varepsilon^2)Q) + + 2\operatorname{rank}\mathscr{K}_0 . \qquad (3.12)_\pm$$

The last term at the right-hand side of $(3.12)_\pm$ is independent of λ, and, by virtue of (3.11), is finite.

Denote by $Z_1(\delta)$, $\delta \in \mathbb{R}$, the operator $P(\mathfrak{H} + \delta\mathscr{K}_1^2)P$ with domain $PD(\mathfrak{H}) \equiv P\mathfrak{L}$, and by $Z_2(\delta)$ – the operator $Q(\mathfrak{H} + \delta)Q$ with domain $QD(\mathfrak{H})$. Then we have

$$N(\mathscr{E} \mp 4\varepsilon^2, \mathscr{E}_k^+ - \lambda|P(\mathfrak{H} \pm \varepsilon^{-2}\mathscr{K}_1^2)P + Q(\mathfrak{H} \pm \varepsilon^2)Q) =$$
$$= N(\mathscr{E} \mp 4\varepsilon^2, \mathscr{E}_k^+ - \lambda|Z_1(\mp\varepsilon^{-2})) + N(\mathscr{E} \mp 4\varepsilon^2, \mathscr{E}_k^+ - \lambda|Z_2(\mp\varepsilon^2)) . \qquad (3.13)_\pm$$

Obviously the inequalities

$$N(\mathscr{E} \mp 4\varepsilon^2, \mathscr{E}_k^+ - \lambda|Z_2(\mp\varepsilon^2)) \leqq N(\mathscr{E} \mp 3\varepsilon^2, \mathscr{E}_k^+ \pm \varepsilon^2|Z_2(0)) \qquad (3.14)_\pm$$

hold for all $\lambda \geqq 0$. Since the value $\mathscr{E}_k^+$ is regular and (3.2) is valid, we have

$$\sigma_{\text{ess}}(Z_2(0)) \cap [\mathscr{E}, \mathscr{E}_k^+] = \varnothing .$$

Therefore, the quantity at the right-hand side of $(3.14)_\pm$ is finite for $|\varepsilon|$ small enough. Hence, the second term at the right-hand side of $(3.13)_\pm$ remains uniformly bounded as $\lambda \downarrow 0$.

Further, write the obvious identities

$$N(\mathscr{E} \mp 4\varepsilon^2, \mathscr{E}_k^+ - \lambda|Z_1(\mp\varepsilon^{-2})) = N(-\infty, 0|Z_1(\mp\varepsilon^{-2}) - \mathscr{E}_k^+ + \lambda) -$$
$$- \lim_{\tau \downarrow 0} N(-\infty, \mathscr{E} \mp 4\varepsilon^2 + \tau|Z_1(\mp\varepsilon^{-2})) . \qquad (3.15)_\pm$$

The operator $Z_1(\delta)$ is bounded for every fixed $\delta \in \mathbb{R}$, and we have

$$\inf \sigma_{\text{ess}}(Z_1(\delta)) = \mathscr{E}_k^+, \quad \forall \delta \in \mathbb{R} .$$

Hence, the second term of the right-hand side of $(3.15)_\pm$ is finite for $|\varepsilon| \neq 0$ small enough.

Next, denote by $\tilde{Z}$ the operator PUV^2U^*P with domain $P\mathfrak{L}$. Applying the elementary estimates

$$(P\mathscr{K}_1^2 Pu, u) = \|\Pi\mathscr{K}Pu\|^2 \leqq \|\mathscr{K}Pu\|^2 \leqq 2\|\mathfrak{B}PPu\|^2 + 2\|P\mathfrak{B}Pu\|^2$$
$$\leqq 4\|\mathfrak{B}Pu\|^2 = 4\|VU^*Pu\|^2 ,$$

we obtain the operator inequalities

$$\pm Z_1(\mp \varepsilon^{-2}) \geqq \pm Z_1(0) - 4\varepsilon^{-2}\tilde{Z} .$$

The operator $Z_1(0) + \delta\tilde{Z}$, $\delta \in \mathbb{R}$, is unitarily equivalent to the operator

$$E^{+}_{k_0} + \mathscr{F}(V) + \delta\mathscr{F}(V^2) \equiv \chi^{-2}(\lambda) + \mathscr{E}^{+}_{k} - \lambda + \mathscr{F}(V) + \delta\mathscr{F}(V^2) .$$

Hence, we have

$$\pm N(-\infty, 0 | Z_1(\mp \varepsilon^{-2}) - \mathscr{E}^{+}_{k} + \lambda) \leqq \pm N(-\infty, 0 | \chi^{-2}(\lambda) + \mathscr{F}(V) \mp \mp 4\varepsilon^{-2}\mathscr{F}(V^2)) . \qquad (3.16)_{\pm}$$

The Birman–Schwinger principle yields

$$N(-\infty, 0 | \chi^{-2}(\lambda) + \mathscr{F}(V^2) \mp 4\varepsilon^{-2}\mathscr{F}(V)) = N(1, \infty | \mathscr{R}(V;\lambda) \mp 4\varepsilon^{-2}\mathscr{R}(V^2;\lambda)) . \qquad (3.17)_{\pm}$$

In view of $(2.3)_{\pm}$ we have

$$\pm N(1, \infty | \mathscr{R}(V;\lambda) \mp \mathscr{R}(V^2;\lambda)) \leqq \pm N(1 \mp \tau, \infty | \mathscr{R}(V;\lambda)) + + N(\tau\varepsilon^2/4, \infty | -\mathscr{R}(V^2;\lambda)), \quad \forall \tau \in (0,1) . \qquad (3.18)_{\pm}$$

Apply Lemma 2.3 with $\mathscr{D} = \mathcal{O}$, $g(\theta, \mathbf{x}) = (\operatorname{vol}\Omega^*)^{-1/2}\overline{\varphi(\mathbf{x}, \theta)}$, $a = b = \chi(\lambda)$, and $f = V^2$, bearing in mind Lemma 3.1. Taking into account $(2.1)_{\pm}$, we obtain the estimate

$$N(\tau\varepsilon^2/4, \infty | -\mathscr{R}(V^2;\lambda)) \leqq c_0\mathscr{I}_p(\lambda; O, r_1) \qquad (3.19)_{\pm}$$

with $p = m/2\alpha + \delta$, $\delta > 0$ being arbitrarily small, if $m \geqq 2\alpha$, $p = 1$, if $m < 2\alpha$, and some numbers c_0 and r_1 independent of λ. Employing the estimates (3.5)–(3.7), we find that the quantity at the right-hand side has order $o(\lambda^{-m(2-\alpha)/2\alpha})$ as $\lambda \downarrow 0$. Combining $(3.12)_{\pm}$–(3.19), we come to $(3.9)_{\pm}$. □

3.4 In what follows we choose the coordinates on $\mathcal{O}$ so that θ_k coincides with the origin. For $\theta \in \mathbb{R}^m$ set

$$h(\theta; \lambda) := (\tfrac{1}{2}(\mathscr{M}^{+}_{k}(\theta_k)\theta, \theta) + \lambda)^{-1/2}, \ \lambda > 0 ,$$

and define the operator $\mathfrak{R}(\lambda) : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ by

$$\mathfrak{R}(\lambda) := -h(\lambda)_{|\mathcal{O}}\mathscr{F}(V)h(\lambda)_{|\mathcal{O}} .$$

Proposition 3.5 *The estimates*

$$\pm N(1 \mp \tau, \infty | \mathscr{R}(V;\lambda)) \leqq \pm N(1 \mp \tau', \infty | \mathfrak{R}(\lambda)), \quad \forall \lambda > 0 , \qquad (3.20)_{\pm}$$

hold for each $\tau \in (0, 1)$, $\tau' \in (\tau, 1)$, *and sufficiently small* diam $\mathcal{O}$.

Proof. Fix arbitrary $\varepsilon \in (0, 1)$ and choose diam $\mathcal{O}$ so small that the inequalities

$$\pm h(\theta; \lambda) \leqq \pm (1 \mp \varepsilon)^{1/2}\chi(\theta; \lambda),$$

hold for each $\theta \in \bar{\mathcal{O}}$ and each $\lambda > 0$. Then the minimax principle implies

$$\pm N(1 \mp \tau, \infty | \mathscr{R}(V;\lambda)) \leqq \pm N((1 \mp \tau)(1 \mp \varepsilon), \infty | \mathfrak{R}(\lambda)), \forall \lambda > 0 , \qquad (3.21)_{\pm}$$

Fix the pair $\tau \in (0, 1)$, $\tau' \in (\tau, 1)$ and choose $\varepsilon \in (0, 1)$ so that we have $\pm(1 \mp \tau') \leqq \pm(1 \mp \tau)(1 \mp \varepsilon)$. Then $(3.21)_{\pm}$ entails $(3.20)_{\pm}$. □

3.4 Let $f \in \mathscr{S}_\delta$, $\delta > 0$. Set

$$\hat{f}(\theta) := \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \exp\{i\mathbf{x}\cdot\theta\} f(\mathbf{x})\, d\mathbf{x} \tag{3.22}$$

where the integral should be understood in the distribution sense.

Lemma 3.6 [Tay, Chap. XII, Lemma 3.1] *Let $f \in \mathscr{S}_\delta$, $\delta > 0$, $\delta \neq m$. Then the Fourier transform defined in* (3.22) *satisfies the estimates*

$$|\hat{f}(\theta)| \leqq C\,,$$

if $|\theta| \geqq 1$, and

$$\hat{f}(\theta) \leqq \begin{cases} C|\theta|^{-m+\delta}, & \delta < m\,, \\ C, & \delta > m\,, \end{cases}$$

if $|\theta| \leqq 1$, with a constant C which is independent of θ.

Let $\mathscr{D} \subseteq \mathbb{R}^m$ be an open set and $f \in L^p(\mathbb{R}^m)$, $p \in [1, \infty)$. Define the operator $F(f; \mathscr{D})$ as the operator (2.5) with $a = b \equiv 1$ and $g(\theta, \mathbf{x}) = (2\pi)^{-m/2}\exp\{-i\mathbf{x}\cdot\theta)$. In particular, if $f \in \mathscr{S}_\delta$, $\delta > 0$, then $F(f; \mathscr{D})$ is defined by

$$(F(f;\mathscr{D})u)(\theta) = \int_{\mathscr{D}} \hat{f}(\theta' - \theta) u(\theta')\, d\theta', \quad u \in L^2(\mathscr{D})\,.$$

Set

$$R(f; \lambda, \mathscr{D}) := -h(\lambda)_{|\mathscr{D}} F(f; \mathscr{D}) h(\lambda)_{|\mathscr{D}}.$$

Write the operator $\mathscr{F}(V)$ in the form

$$\mathscr{F}(V) = F(V; \mathscr{O}) + F_1 + F_2$$

where the operators F_j, $j = 1, 2$, are defined by

$$(F_1 u)(\theta) := \frac{1}{\operatorname{vol}\Omega} \int_{\mathscr{O}} \left(\sum_{\gamma' \in \Gamma^*} \overline{\psi_{\gamma'}(\theta)} \psi_{\gamma'}(\theta') - \operatorname{vol}\Omega \right) \hat{V}(\theta' - \theta) u(\theta')\, d\theta'$$

and

$$(F_2 u)(\theta) := \frac{1}{\operatorname{vol}\Omega} \sum_{\substack{\gamma' \in \Gamma^*, \gamma'' \in \Gamma^* \\ \gamma' \neq \gamma''}} \int_{\mathscr{O}} \overline{\psi_{\gamma'}(\theta)} \psi_{\gamma''}(\theta') \hat{V}(\theta' - \theta + \gamma' - \gamma'') u(\theta')\, d\theta'\,.$$

Here we have used the relations (3.3) and (3.4). Set

$$R_j(\lambda) := -h(\lambda)_{|\mathscr{O}} F_j h(\lambda)_{|\mathscr{O}}, \quad j = 1, 2, \lambda > 0\,.$$

Then we have

$$\mathfrak{R}(\lambda) = R(V; \lambda, \mathscr{O}) + R_1(\lambda) + R_2(\lambda)\,.$$

Lemma 3.7 *For every fixed $\mu > 0$ we have*

$$N(\mu, \infty | \pm R_1(\lambda)) = o(\lambda^{-m(2-\alpha)/2\alpha}), \quad \lambda \downarrow 0\,. \tag{3.23$_\pm$}$$

Proof. First of all note the identity

$$\sum_{\gamma' \in \Gamma^*} \overline{\psi_{\gamma'}(\theta)} \psi_{\gamma'}(\theta') = \operatorname{vol}\Omega \int_{\Omega} \overline{\psi(\mathbf{x}, \theta)} \psi(\mathbf{x}, \theta')\, d\mathbf{x}\,.$$

Applying a second-order Taylor expansion of $\overline{\psi(\mathbf{x}, \theta)}$ and $\psi(\mathbf{x}, \theta')$ with respect to $\theta - \theta'$, we find the operator F_1 can be written in the form

$$F_1 = \sum_{j=1}^{m} (F_{1,1}^{(j)} + F_{1,1}^{(j)*}) + F_{1,2}$$

where

$$(F_{1,1}^{(j)} u)(\theta) := -\int_{\mathcal{O}} (\theta'_j - \theta_j)\hat{V}(\theta' - \theta)\overline{\omega_j(\theta')}u(\theta')\,d\theta' \equiv$$

$$\equiv \int_{\mathcal{O}} \widehat{D_j V}(\theta' - \theta)\overline{\omega_j(\theta')}u(\theta')\,d\theta'$$

with

$$\omega_j(\theta) := \tfrac{1}{2}\int_{\Omega} \frac{\partial\psi(\mathbf{x}, \theta)}{\partial\theta_j}\overline{\psi(\mathbf{x}, \theta)}\,d\mathbf{x}, \quad j = 1, \ldots, m,$$

and

$$(F_{1,2} u)(\theta) := \int_{\mathcal{O}} |\theta' - \theta|^2 B(\theta, \theta')\hat{V}(\theta' - \theta)u(\theta')\,d\theta'$$

with

$$B(\theta, \theta') := \tfrac{1}{2}|\theta' - \theta|^{-2} \sum_{j=1}^{m}\sum_{l=1}^{m} (\theta'_j - \theta_j)(\theta'_l - \theta_l)$$

$$\int_{\Omega}\int_0^1 (1 - t)\left(\frac{\partial^2\psi(\mathbf{x}, \theta + t(\theta' - \theta))}{\partial\theta_j\partial\theta_l}\overline{\psi(\mathbf{x}, \theta)} + \overline{\frac{\partial^2\psi(\mathbf{x}, \theta' + t(\theta - \theta'))}{\partial\theta_j\partial\theta_l}}\psi(\mathbf{x}, \theta')\right)dt\,d\mathbf{x}\,.$$

Note that $B(\theta, \theta')$ is uniformly bounded on $\mathcal{O} \times \mathcal{O}$.

Set

$$R_{1,1}^{(j)}(\lambda) := -h(\lambda)_{|\mathcal{O}} F_{1,1}^{(j)}\, h(\lambda)_{|\mathcal{O}}\,, \quad j = 1, \ldots, m\,,$$
$$R_{1,2}(\lambda := -h(\lambda)_{|\mathcal{O}} F_{1,2} h(\lambda)_{|\mathcal{O}}\,.$$

Then we have

$$R_1(\lambda) = \sum_{j=1}^{m} (R_{1,1}^{(j)}(\lambda) + R_{1,1}^{(j)*}) + R_{1,2}(\lambda)\,.$$

In view of the inequalities

$$N(\mu, \infty| \pm R_1(\lambda)) \leqq \nu(\mu; R_1(\lambda)) \leqq$$

$$\leqq 2\sum_{j=1}^{m} \nu(\mu/(2m + 1); R_{1,1}^{(j)}(\lambda)) + \nu(\mu/(2m + 1); R_{1,2}(\lambda)), \quad \forall\mu > 0\,,$$

(which follow from $(2.1)_\pm$ and (2.2)), it suffices to check the estimates

$$\nu(\tau; R_{1,1}^{(j)}(\lambda)) = o(\lambda^{-m(2-\alpha)/2\alpha}), \quad \lambda \downarrow 0, \forall\tau > 0, j = 1, \ldots, m\,, \tag{3.24}$$

and

$$\nu(\tau; R_{1,2}(\lambda)) = o(\lambda^{-m(2-\alpha)/2\alpha}), \quad \lambda \downarrow 0, \forall\tau > 0\,, \tag{3.25}$$

in order to verify $(3.23)_\pm$.

In order to check (3.24), apply Lemma 2.3. with $\mathscr{D} = \mathcal{O}$, $a = h(\lambda)$, $b = h(\lambda)\omega_j$, $g(\theta, \mathbf{x}) = (2\pi)^{-m/2}\exp\{-i\mathbf{x}\cdot\theta\}$ and $f = -D_j V$, $j = 1, \ldots, m$. Thus we get the inequality

$$\nu(\tau; R^{(j)}_{1,1}(\lambda)) \leqq c_1 \mathscr{I}_p(\lambda; 0, r_1),\ j = 1, \ldots, m\,, \tag{3.26}$$

with $p = \delta + m/(\alpha + 1)$, $\delta > 0$ being arbitrarily small, if $m/(\alpha + 1) \geqq 1$, $p = 1$, if $m/(\alpha + 1) < 1$, and numbers c_1 and $r_1 < \infty$ independent of λ. Employing the estimates (3.5)–(3.7), we find that the quantity at the right-hand side of (3.26) has order $o(\lambda^{-m(2-\alpha)/2\alpha})$ as $\lambda \downarrow 0$, so that we come to (3.24).

In order to verify (3.25), apply Lemma 3.6 and write the kernel $\imath(\theta, \theta'; \lambda)$ of the operator $R_{1,2}(\lambda)$ in the form

$$\imath(\theta, \theta'; \lambda) = h(\theta; \lambda)h(\theta'; \lambda)\tilde{B}(\theta, \theta')|\theta - \theta'|^{-m+\alpha+2}$$

where the function $\tilde{B}$ is uniformly bounded on $\mathcal{O} \times \mathcal{O}$, and independent of λ.

Assume at first $\alpha + 2 < m/2$, and apply Lemma 2.4 with $\mathscr{D} = \mathcal{O}$, $a = b = h(\lambda)$, $t = \tilde{B}$ and $\delta = \alpha + 2$. Thus we get

$$\nu(\tau; R_{1,2}(\lambda)) \leqq c_2 \mathscr{I}_{p/2}(\lambda; 0, r_1)^{2m/p(\alpha+2)} \tag{3.27}$$

where $p > 2m/(\alpha + 2)$ and the numbers c_2 and $r_1 < \infty$ are independent of λ. Obviously, the right-hand side of (3.27) remains uniformly bounded as $\lambda \downarrow 0$, provided that the number $-1 + (\alpha + 2)p/2m > 0$ is small enough (see (3.7)).

If $\alpha + 2 = m/2$, choose $\varepsilon > 0$ small enough, and apply again Lemma 2.4 with $\mathscr{D} = \mathcal{O}$, $a = b = h(\lambda)$, $t(\theta, \theta') = \tilde{B}(\theta, \theta')|\theta - \theta'|^{\varepsilon}$ and $\delta = \alpha + 2 - \varepsilon$ in order to obtain the estimate

$$\nu(\tau; R_{1,2}(\lambda)) \leqq O(1),\ \lambda \downarrow 0,\ \forall \tau > 0\,. \tag{3.28}$$

Next, suppose $m/2 < \alpha + 2 < m$, and apply Lemma 2.5 with $\mathscr{D} = \mathcal{O}$, $a = b = h(\lambda)$, $t = \tilde{B}$ and $\delta = \alpha + 2$. Thus we get

$$\nu(\tau; R_{1,2}(\lambda)) \leqq c_2 \mathscr{I}_{m/(\alpha+2)}(\lambda; 0, r_1)^{2(\alpha+2)/m} = O(1),\ \lambda \downarrow 0\,, \tag{3.29}$$

(see (3.7)).

Finally, if $\alpha + 2 \geqq m$, we just use (2.4) with $p = 2$ which yields

$$\nu(\tau; R_{1,2}(\lambda)) \leqq \tau^{-2}\|R_{1,2}(\lambda)\|^2_{\mathfrak{S}_2} \leqq c_4\ \|h(\lambda)\|^4_{L^2} \leqq$$

$$\leqq c_5 \mathscr{I}_1(\lambda; O, r_1)^2 = \begin{cases} O(1), & m \geqq 3, \\ O(|\ln \lambda|^2), & m = 2, \end{cases}\quad \lambda \downarrow 0\,, \tag{3.30}$$

(see (3.5) and (3.6)) where the numbers c_4, c_5 and $r_1 < \infty$ do not depend on λ.

Putting together (3.27)–(3.30), we come to (3.25). □

Lemma 3.8 *The estimates*

$$N(\mu, \infty \mid \pm R_2(\lambda)) = o(\lambda^{-m(2-\alpha)/2\alpha}),\quad \lambda \downarrow 0\,, \tag{3.31$_\pm$}$$

hold for each $\mu > 0$.

Proof. Choose an integer $n > m/2$ and write the kernel of the operator $R_2(\lambda)$ in the form

$$(-1)^n \frac{h(\theta; \lambda)h(\theta'; \lambda)}{\operatorname{vol}\Omega} \sum_{\substack{\gamma' \in \Gamma^*,\ \gamma'' \in \Gamma^* \\ \gamma' \neq \gamma''}} \frac{\overline{\psi_{\gamma'}(\theta')\psi_{\gamma''}(\theta)}}{|\theta' - \theta + \gamma' - \gamma''|^{2n}} (\widehat{\Delta^n V})(\theta' - \theta + \gamma' - \gamma'')$$

where θ, $\theta' \in \mathcal{O}$. Here we assume that $\operatorname{diam} \mathcal{O}$ is so small that the vector $\theta' - \theta + \gamma' - \gamma''$ does not vanish if $\gamma' \in \Gamma^*$, $\gamma'' \in \Gamma^*$, $\gamma' \neq \gamma''$, $\theta \in \bar{\mathcal{O}}$ and $\theta' \in \bar{\mathcal{O}}$. Moreover, the estimates (0.2) combined with the inequality $2n + \alpha > m$ imply that $(\widehat{\Delta^n V})(y)$ is uniformly bounded with respect to $y \in \mathbb{R}^m$ (see Lemma 3.6). Therefore, we have

$$\| R_{1,2}(\lambda) \|_{\mathfrak{S}_2}^2 \leqq (\operatorname{vol} \Omega)^{-2} \sup_{y \in \mathbb{R}^m} |(\widehat{\Delta^n V})(y)|^2 \int_{\mathcal{O}} \int_{\mathcal{O}} h(\theta; \lambda)^2 h(\theta'; \lambda)^2$$
$$\cdot \left(\sum_{\substack{\gamma' \in \Gamma^*, \gamma'' \in \Gamma^* \\ \gamma' \neq \gamma''}} \frac{|\psi_{\gamma'}(\theta)| |\psi_{\gamma''}(\theta')|}{|\theta' - \theta + \gamma' - \gamma''|^{2n}} \right)^2 d\theta \, d\theta' . \tag{3.32}$$

Applying the Schwarz inequality and then changing the summation variables, we get

$$\sum_{\substack{\gamma' \in \Gamma^*, \gamma'' \in \Gamma^* \\ \gamma' \neq \gamma''}} |\theta' - \theta + \gamma' - \gamma''|^{-2n} |\psi_{\gamma'}(\theta)| |\psi_{\gamma''}(\theta')| \leqq$$
$$\left(\sum_{\beta' \in \Gamma^*} |\psi_{\beta'}(\theta)|^2 \sum_{\substack{\beta'' \in \Gamma^* \\ \beta'' \neq 0}} |\theta' - \theta + \beta''|^{-2n} \right)^{1/2} \left(\sum_{\beta' \in \Gamma^*} |\psi_{\beta'}(\theta')|^2 \sum_{\substack{\beta'' \in \Gamma^* \\ \beta'' \neq 0}} |\theta' - \theta + \beta''|^{-2n} \right)^{1/2}$$
$$= \operatorname{vol} \Omega \sum_{\substack{\beta'' \in \Gamma^* \\ \beta'' \neq 0}} |\theta' - \theta + \beta''|^{-2n} . \tag{3.33}$$

Since the $2n > m$, the last series in (3.33) is absolutely convergent with respect to $\theta \in \bar{\mathcal{O}}$ and $\theta' \in \bar{\mathcal{O}}$, provided that $\operatorname{diam} \mathcal{O}$ is small enough. Inserting (3.33) into (3.32), we get the estimate

$$\| R_2(\lambda) \|_{\mathfrak{S}_2}^2 \leqq c_6 \| h(\lambda) \|_{L^2}^4 \leqq c_7 \mathscr{I}_1(\lambda; O, r_1)^2 = \begin{cases} O(1), & m \geqq 3, \\ O(|\ln \lambda|^2), & m = 2, \end{cases} \quad \lambda \downarrow 0 , \tag{3.34}$$

(see (3.5) and (3.6)) with numbers c_6, c_7 and $r_1 < \infty$ independent of λ. Now, the estimates (2.4) with $p = 2$, and (3.34) imply $(3.31)_\pm$. □

The combination of $(2.3)_\pm$, $(3.23)_\pm$ and $(3.31)_\pm$ yields the following result.

Proposition 3.9 *The estimates*

$$\pm N(1 \mp \tau, \infty | \mathfrak{R}(\lambda)) \leqq \pm N(1 \mp \tau', \infty | R(V; \lambda, \mathcal{O})) + o(\lambda^{-m(2-\alpha)/2\alpha}), \quad \lambda \downarrow 0, \tag{3.35$_\pm$}$$

hold for each $\tau \in (0,1)$, *and* $\tau' \in (\tau, 1)$ *and sufficiently small* $\operatorname{diam} \mathcal{O}$.

3.6 **Proposition 3.10** *The estimates*

$$\pm N(1 \mp \tau', \infty | R(V; \lambda, \mathcal{O})) \leqq \pm N(1 \mp \tau', \infty | R(V; \lambda, \mathbb{R}^m)) + o(\lambda^{-m(2-\alpha)/2\alpha}), \quad \lambda \downarrow 0 , \tag{3.36$_\pm$}$$

hold for each $\tau \in (0, 1)$, $\tau' \in (\tau, 1)$ *and every ball* $\mathcal{O}$ *containing* θ_k.

Proof. Recall that ζ denotes the characteristic function of $\mathcal{O}$. Let $\eta := 1 - \zeta$ be the characteristic function of $\mathbb{R}^m \backslash \mathcal{O}$. Define the orthogonal projection $\mathscr{P} : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)$ by

$$(\mathscr{P}u)(\theta) := \zeta(\theta) u(\theta), \; u \in L^2(\mathbb{R}^m) ,$$

and set $\mathscr{Q} := \mathrm{Id} - \mathscr{P}$, so that we have

$$(\mathscr{Q}u)(\theta) := \eta(\theta)u(\theta),\ u \in L^2(\mathbb{R}^m)\,.$$

Further, write the identity

$$R(V; \lambda, \mathbb{R}^m) = \mathscr{P}R(V; \lambda, \mathbb{R}^m)\mathscr{P} + \mathscr{Q}R(V; \lambda, \mathbb{R}^m)\mathscr{Q} + \\ + \mathscr{P}R(V; \lambda, \mathbb{R}^m)\mathscr{Q} + \mathscr{Q}R(V; \lambda, \mathbb{R}^m)\mathscr{P} \tag{3.37}$$

and apply the elementary estimates

$$\pm 2\mathrm{Re} \int_{\mathbb{R}^m} V(\mathbf{x})(\widehat{\zeta u})(\mathbf{x})\overline{(\widehat{\eta u})(\mathbf{x})}\, d\mathbf{x} \leqq$$

$$\leqq \int_{\mathbb{R}^m} |V(\mathbf{x})|^{2s}|(\widehat{\zeta u})(\mathbf{x})|^2\, d\mathbf{x} + \int_{\mathbb{R}^m} |V(\mathbf{x})|^{2-2s}|(\widehat{\eta u})(\mathbf{x})|^2\, d\mathbf{x}, \quad s \in (\tfrac{1}{2}, 1),$$

in order to verify the operator inequalities

$$\pm(\mathscr{P}R(V; \lambda, \mathbb{R}^m)\mathscr{Q} + \mathscr{Q}R(V; \lambda, \mathbb{R}^m)\mathscr{P}) \leqq \\ \leqq \pm\mathscr{P}R(\mp|V|^{2s}; \lambda, \mathbb{R}^m)\mathscr{P} \pm \mathscr{Q}R(\mp|V|^{2-2s}; \lambda, \mathbb{R}^m)\mathscr{Q}\,. \tag{3.38$_\pm$}$$

Combining (3.37) and (3.38)$_\pm$, we get

$$\pm R(V; \lambda, \mathbb{R}^m) \leqq \pm\mathscr{P}R(V \mp |V|^{2s}; \lambda, \mathbb{R}^m)\mathscr{P} \pm \mathscr{Q}R(V \mp |V|^{2-2s}; \lambda, \mathbb{R}^m)\mathscr{Q}\,. \tag{3.39$_\pm$}$$

Now, note that for each f the operator $\mathscr{P}R(f; \lambda, \mathbb{R}^m)\mathscr{P}$ is unitarily equivalent to $R(f; \lambda, \mathcal{O})$ and the operator $\mathscr{Q}R(f; \lambda, \mathbb{R}^m)\mathscr{Q}$ is unitarily equivalent to $R(f; \lambda, \mathbb{R}^m \setminus \mathcal{O})$.

Then, by the minimax principle, the estimates (3.39)$_\pm$ imply the inequalities

$$\pm N(\mu, \infty|R(V; \lambda, \mathbb{R}^m)) \leqq \pm N(\mu, \infty|R(V \mp |V|^{2s}; \lambda, \mathcal{O})) \pm \\ \pm N(\mu, \infty|R(V \mp |V|^{2-2s}; \lambda, \mathbb{R}^m \setminus \mathcal{O})), \quad \forall\mu > 0, \forall s \in (\tfrac{1}{2}, 1)\,, \tag{3.40$_\pm$}$$

Applying Lemma 2.4 with $\mathscr{D} = \mathbb{R}^m \setminus \mathcal{O}$, $a = b = h(\lambda)$, $g(\mathbf{x}, \theta) = (2\pi)^{-m/2} \cdot \exp\{-i\mathbf{x}\cdot\theta\}$, and $f = V$ or $f = |V|^{2-2s}$, and bearing in mind the inequalities (2.3)$_\pm$, we get

$$\pm N(\mu, \infty|R(V \mp |V|^{2-2s}; \lambda, \mathbb{R}^m \setminus \mathcal{O})) \leqq c_8 \mathscr{I}_p(\lambda; r_0, \infty), \ \mu > 0\,, \tag{3.41$_\pm$}$$

where $p > m/\alpha(2 - 2s) > m/\alpha > m/2$, and the numbers c_8 and $r_0 > 0$ are independent of λ. By the estimate (3.8), we find that the second term at the right-hand side of (3.40)$_\pm$ remains uniformly bounded as $\lambda \downarrow 0$.

The estimates (2.3)$_\pm$ entail

$$\pm N(1 \mp \tau, \infty|R(V \mp |V|^{2s}; \lambda, \mathcal{O})) \leqq \pm N((1 \mp \tau)(1 \mp \varepsilon), \infty|R(V; \lambda, \mathcal{O})) + \\ + N((1 \mp \tau)\varepsilon, \infty|R(-|V|^{2s}; \lambda, \mathcal{O})), \quad \forall\tau \in (0, 1), \forall\varepsilon \in (0, 1)\,. \tag{3.42$_\pm$}$$

Applying Lemma 2.4 with $\mathscr{D} = \mathcal{O}$, $a = b = h(\lambda)$, $g(\mathbf{x}, \theta) = (2\pi)^{-m/2}\exp\{-i\mathbf{x}\cdot\theta\}$, and $f = -|V|^{2s}$, we get

$$N((1 \mp \tau)\varepsilon, \infty|R(-|V|^{2s}; \lambda, \mathcal{O})) \leqq c_9 \mathscr{I}_p(\lambda; 0, r_1) \tag{3.43}$$

with $p = m/2s\alpha + \delta$, $\delta > 0$ being small enough, if $m/2s\alpha \geqq 1$, and $p = 1$ if $m/2s\alpha < 1$, and numbers c_9 and r_1 independent of λ. The estimates (3.5)–(3.7) imply that the quantity at the right-hand side of (3.43) has order $o(\lambda^{-m(2-\alpha)/2\alpha})$ as $\lambda \downarrow 0$.

Fix the pair $\tau \in (0, 1)$, $\tau' \in (\tau, 1)$ and choose $\varepsilon \in (0, 1)$ so that we have $\pm (1 \mp \tau)(1 \mp \varepsilon) \leqq \pm (1 \mp \tau')$. Now the relations $(3.40)_\pm$–(3.43) entail $(3.36)_\pm$. □

3.7 The Birman–Schwinger principle implies

$$N(1 - \tau, \infty \,|\, R(V; \lambda, \mathbb{R}^m)) =$$
$$= N(-\infty, -\lambda \,|\, -\tfrac{1}{2}(M_k^+(\theta_k)\nabla, \nabla) + (1 - \tau)^{-1} V), \;\forall \tau \in (-1, 1) . \quad (3.44)$$

For $\tau \in (-1, 1)$ and $\lambda > 0$ set

$$\Xi(\lambda; \tau) := (2\pi)^{-m} \mathrm{vol}\{(\mathbf{x}, \theta) \in \mathbb{R}^{2m} : \tfrac{1}{2}(M_k^+(\theta_k)\theta, \theta) + (1 - \tau)^{-1} V(\mathbf{x}) < -\lambda\} .$$

Using Proposition 1.1 with $\mathscr{M} = \frac{1}{2} M_k^+(\theta_k)$, we establish the following asymptotics

$$N(-\infty, -\lambda \,|\, -\tfrac{1}{2}(M_k^+(\theta_k)\nabla, \nabla) + (1 - \tau)^{-1} V) = \Xi(\lambda; \tau) + o(\lambda^{-m(2-\alpha)/2\alpha}),$$
$$\lambda \downarrow 0, \;\forall \tau \in (-1, 1) . \quad (3.45)$$

Set

$$\mathscr{D}_V(t) = \{\mathbf{x} \in \mathbb{R}^m : V(\mathbf{x}) + t < 0\}, t \in \mathbb{R} .$$

Writing the identity

$$\Xi(\lambda; \tau) - \Xi(\lambda; 0) = \mathscr{C}_k^+ \Big\{ [(1 - \tau)^{-m/2} - 1] \int_{\mathscr{D}_V((1-\tau)\lambda)} (-V(\mathbf{x}) - (1 - \tau)\lambda))^{m/2} d\mathbf{x} +$$
$$+ \frac{m}{2} \int_{(1-\tau)\lambda}^{\lambda} dt \int_{\mathscr{D}_V(t)} (-V(\mathbf{x}) - t))^{m/2 - 1} d\mathbf{x} \Big\}, \lambda > 0, \tau \in (-1, 1) ,$$

where the quantity $\mathscr{C}_k^+$ is introduced in $(1.7)_+$, we find that under the hypotheses of the Main Theorem, the estimate

$$|\Xi(\lambda; \tau) - \Xi(\lambda; 0)| \leqq c(\tau) \lambda^{-m(2-\alpha)/2\alpha}$$

holds for $\lambda > 0$ small enough and a quantity $c(\tau)$ which tends to zero as $\tau \to 0$. Hence we have

$$\limsup_{\tau \downarrow 0} \limsup_{\lambda \downarrow 0} \pm \Xi(\lambda; \pm \tau)/\Xi(\lambda; 0) \leqq \pm 1 . \quad (3.46)_\pm$$

Now the asymptotics $(1.6)_\pm$ follow from the sequence of estimates (3.1), $(3.9)_\pm$, $(3.20)_\pm$, $(3.35)_\pm$, $(3.36)_\pm$ and (3.44)–$(3.46)_\pm$.

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