Spectral asymptotics for quantum Hamiltonians in strong magnetic fields

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1 Introduction

The aim of this article is to present some of the authors' recent results on the asymptotic behaviour of the discrete spectrum of the Schrödinger, Pauli, and Dirac operators in strong magnetic fields perturbed by electric potentials which decay at infinity. Since the article is oriented towards readers who are not necessarily specialists in the field, we allocate considerable attention to the physical prerequisites concerning the quantization scheme of classical Hamiltonians involving magnetic fields (see Subsections 2.1 and 2.2). Moreover, we describe some general, mainly well-known, spectral properties of the unperturbed quantum magnetic Hamiltonians: in Subsection 2.3 we deal with constant magnetic fields, while in Subsection 2.4 we treat special cases of non-constant ones. Although we have described those spectral properties of the unperturbed Hamiltonians which are needed to present our results in the spectral asymptotics of the perturbed Hamiltonians, we have tried to consider briefly also some other features of the unperturbed Hamiltonians, which we find especially important.

Sections 2 and 3 contain our main results. In the theorems of Section 2 we obtain only *the main asymptotic term* as the coupling constant of the magnetic field tends to infinity, of the appropriate spectral counting function for a given quantum Hamiltonian.

On the other hand, in Section 3 we aim at assumptions concerning the perturbation V which are as general as possible. The methods we use here are of variational nature. Moreover, we apply the approach due to Kac-Murdock-Szegö and developed by H.Widom to the spectral asymptotics of compact operators of Toeplitz type. As a matter of fact, one of our main purposes was to reveal the relation between the spectral theory of quantum Hamiltonians with the theory of holomorphic spaces of Segal-Bargmann type and the associated Toeplitz operators. It seems to us that this relation is not understood deeply enough yet.

Section 4 is devoted to precise spectral theorems containing a sharp remainder estimate or even complete asymptotic expansions. However, the assumptions in this case are by far more restrictive than those in the previous section. The methods applied in this chapter are of Tauberian nature. They are closely related to the microlocal analysis and, in particular, the functional calculus due to Helffer-Sjöstrand.

With very few exceptions, the variational and the Tauberian methods rarely co-exist in one and the same article. One of our purposes was to overcome the gap between these two types of methods. We have tried to give the reader the opportunity to compare the two types of results which can be obtained by them, dealing with the problem of the asymptotic behaviour of the discrete spectrum of quantum Hamiltonians in strong magnetic fields perturbed by decaying scalar potentials.

There exist many other interesting problems in the spectral theory of quantum Hamiltonians in strong magnetic fields which we have not considered just in order to avoid an unreasonable increase of the size of the article. Among these problems we would mention the asymptotics of the ground state of atomic Hamiltonians in strong magnetic fields (see [Ba.So.Y]), and the asymptotics of the integrated density of states for the Schrödinger operator with ergodic scalar potential and strong constant magnetic field (see [Br.Hu.Le], [Wa], [Ki.R]). We hope that we will discuss these, and other interesting problems in a future second part of this article.

2 Quantum magnetic Hamiltonians

2.1 Classical magnetic Hamiltonians

2.1.1. Let us begin with the equations of motion of a three-dimensional classical non-relativistic particle in an electromagnetic field $(\mathbf{E}, \mathbf{B})$. Here $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ is the electric component of the field, and $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ is its magnetic component. We assume that the field is stationary, i.e. independent of time t. Moreover, we suppose that the medium is uniform and isotropic, so that the electric permeability and the magnetic permittivity are scalar constants.

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the coordinates of the particle, and M be its mass. Then the equations of motion of the particle can be written in their Newtonian form

$$M\ddot{\mathbf{x}} = \mu\dot{\mathbf{x}} \wedge \mathbf{B} + g\mathbf{E} \tag{2.1.1}$$

(see [La.Li]). Here $\mu \in \mathbb{R}$ is the coupling constant of the magnetic field, and $g \in \mathbb{R}$ is the coupling constant of the electric field. Moreover, $\dot{\mathbf{x}} := \frac{\partial \mathbf{x}}{\partial t}$ is the velocity of the particle, and $\ddot{\mathbf{x}} := \frac{\partial^2 \mathbf{x}}{\partial t^2}$ is its acceleration. Finally, "$\wedge$" denotes the vector product. Notice that the vector appearing at the right-hand side of (2.1.1) is the Lorentz force acting on the particle.

In order to pass to the Lagrangian version of (2.1.1), we introduce the electromagnetic potential

$(\mathbf{A}, V)$ with $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$, $V : \mathbb{R}^3 \to \mathbb{R}$, satisfying $\mathbf{B} = \text{curl } \mathbf{A}$, $\mathbf{E} = -\nabla V$, and the Lagrangian function

$$L(\mathbf{x}, \dot{\mathbf{x}}) := \frac{M|\dot{\mathbf{x}}|^2}{2} + \mu \mathbf{A}(\mathbf{x}).\dot{\mathbf{x}} - gV(\mathbf{x}). \tag{2.1.2}$$

Evidently, (2.1.2) admits an immediate generalization to an arbitrary dimension m; in this case $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{A} : \mathbb{R}^m \to \mathbb{R}^m$, and $V : \mathbb{R}^m \to \mathbb{R}$. The equations of motion of the m-dimensional particle can be written in their Lagrangian form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0, \quad j = 1, \dots, m. \tag{2.1.3}$$

If $m = 3$, (2.1.3) is equivalent to (2.1.1).

2.1.2. In order to pass to the canonical Hamiltonian formulation of the equations of motion of our m-dimensional particle, let us introduce the generalized momentum $\mathbf{p} = (p_1, \dots, p_m)$, with $p_j := \frac{\partial L}{\partial \dot{x}_j}$, $j = 1, \dots, m$, write $\dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{x}, \mathbf{p})$ as a function of $\mathbf{x}$ and $\mathbf{p}$, and compose the Hamiltonian function

$$\mathcal{H}(\mathbf{x}, \mathbf{p}) := \mathbf{p}.\dot{\mathbf{x}}(\mathbf{x}, \mathbf{p}) - L(\mathbf{x}, \dot{\mathbf{x}}(\mathbf{x}, \mathbf{p})).$$

Taking into account (2.1.2), we get $\mathbf{p} = M\dot{\mathbf{x}} + \mu\mathbf{A}(\mathbf{x})$ or $\dot{\mathbf{x}} = (\mathbf{p} - \mu\mathbf{A}(\mathbf{x}))/M$. Therefore,

$$\mathcal{H}(\mathbf{x}, \mathbf{p}) = \frac{1}{2M}|\mathbf{p} - \mu\mathbf{A}(\mathbf{x})|^2 + gV(\mathbf{x}), \quad (\mathbf{x}, \mathbf{p}) \in T^*\mathbb{R}^m = \mathbb{R}^{2m}. \tag{2.1.4}$$

The Hamiltonian function $\mathcal{H}$ is related to the canonical system of Hamiltonian equations

$$\dot{x}_j = \frac{\partial \mathcal{H}}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial \mathcal{H}}{\partial x_j}, \quad j = 1, \dots, m, \tag{2.1.5}$$

which is equivalent to (2.1.3) (see [A, Section 15]). System (2.1.5) can be re-written in the form

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{p}} \end{pmatrix} = (J^T)^{-1}\nabla\mathcal{H} = J\nabla\mathcal{H}, \tag{2.1.6}$$

where $J = (J^T)^{-1} := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, I_m being the unit $m \times m$-matrix. Introduce the canonical symplectic form

$$\omega := \sum_{j=1}^{m} dx_j \wedge dp_j. \tag{2.1.7}$$

In the coordinates $(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2m}$ the matrix of ω is equal to J, and the co-tangent bundle $T^*\mathbb{R}^m = \mathbb{R}^{2m}$ equipped with the symplectic form ω is a symplectic manifold (see [A, Section 37]). We can re-write (2.1.6) using another Hamiltonian function

$$\tilde{\mathcal{H}}(\mathbf{x}, \mathbf{p}) := \frac{1}{2M}|\mathbf{p}|^2 + gV(\mathbf{x}), \tag{2.1.8}$$

which is independent of the magnetic field, and another symplectic form $\tilde{\omega}_\mu$. In order to define $\tilde{\omega}_\mu$, we introduce at first the 1-differential form of the magnetic potential $A := \sum_{j=1}^{m} A_j(\mathbf{x})dx_j$, and the 2-differential form of the magnetic field $B := dA \equiv \sum_{j<k} B_{j,k}(\mathbf{x})\, dx_j \wedge dx_k$ where

$$B_{j,k} := \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}, \quad j, k = 1, \dots, m. \tag{2.1.9}$$

We shall use the same notation for the antisymmetric matrix-valued function $B = B(\mathbf{x}) = \{B_{j,k}\}_{j,k=1}^m$. Notice that in the three-dimensional case there exists a standard way of identifying the components of the vector $\mathbf{B}$ with the coefficients of the 2-differential form B, namely $B_1 = B_{2,3}$, $B_2 = B_{3,1}$, $B_3 = B_{1,2}$. Introduce the magnetic symplectic form

$$\tilde{\omega}_\mu := \omega - \mu B \equiv \sum_{j=1}^m dx_j \wedge dp_j - \mu \sum_{j<k} B_{j,k}(\mathbf{x})\, dx_j \wedge dx_k, \quad \mu \in \mathbb{R}. \tag{2.1.10}$$

Notice that ω is equal to $\tilde{\omega}_\mu$ for $\mu = 0$. In the coordinates $(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2m}$ the matrix of $\tilde{\omega}_\mu$ is equal to $\tilde{J}_\mu(\mathbf{x}) := \begin{pmatrix} -\mu B(\mathbf{x}) & I_m \\ -I_m & 0 \end{pmatrix}$. Then (2.1.6) can be re-written as

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{p}} \end{pmatrix} = (\tilde{J}_\mu^T)^{-1} \nabla \tilde{\mathcal{H}} = K_\mu \nabla \tilde{\mathcal{H}}, \tag{2.1.11}$$

where $K_\mu := (\tilde{J}_\mu^T)^{-1} = \begin{pmatrix} 0 & I_m \\ -I_m & \mu B \end{pmatrix}$, and the Hamiltonian function $\tilde{\mathcal{H}}$ is defined in (2.1.8) (see [Ma, Subsection 2.10]). Notice that $\tilde{\mathcal{H}}$ is transformed into $\mathcal{H}$ under the change of the variables $\mathbf{x} \mapsto \mathbf{x}$, $\mathbf{p} \mapsto \mathbf{p} - \mu \mathbf{A}(\mathbf{x})$. In other words, we have $\tilde{\mathcal{H}}(\mathbf{x}, \mathbf{p} - \mu \mathbf{A}(\mathbf{x})) = \mathcal{H}(\mathbf{x}, \mathbf{p})$. Under the same change of the variables, the differential form $\tilde{\omega}_\mu$ is transformed into ω. Thus the passage from the couple $(\omega, \mathcal{H})$ to $(\tilde{\omega}_\mu, \tilde{\mathcal{H}})$ is related to a non-degenerate coordinate change in the underlying symplectic manifold.
Let $\mathcal{F}, \mathcal{G} \in C^1(T^*\mathbb{R}^m)$. Introduce the canonical and the magnetic Poisson bracket

$$\{\mathcal{F}, \mathcal{G}\} := \nabla \mathcal{F}.J \nabla \mathcal{G} = \sum_{j=1}^m \left(\frac{\partial \mathcal{F}}{\partial x_j} \frac{\partial \mathcal{G}}{\partial p_j} - \frac{\partial \mathcal{F}}{\partial p_j} \frac{\partial \mathcal{G}}{\partial x_j} \right), \tag{2.1.12}$$

$$\{\mathcal{F}, \mathcal{G}\}_\mu := \nabla \mathcal{F}.K_\mu \nabla \mathcal{G} = \{\mathcal{F}, \mathcal{G}\} + \mu \sum_{j,k=1}^m B_{j,k}(\mathbf{x}) \frac{\partial \mathcal{F}}{\partial p_j} \frac{\partial \mathcal{G}}{\partial p_k}. \tag{2.1.13}$$

In particular, if $j, k = 1, \ldots, m$, we have

$$\{x_j, x_k\} = 0, \{p_j, p_k\} = 0, \{x_j, p_k\} = \delta_{j,k}, \tag{2.1.14}$$

$$\{x_j, x_k\}_\mu = 0, \{p_j, p_k\}_\mu = \mu B_{j,k}(\mathbf{x}), \{x_j, p_k\}_\mu = \delta_{j,k}. \tag{2.1.15}$$

Denote by $(\mathbf{x}(t), \mathbf{p}(t))$ the solution of (2.1.6) related to fixed initial data, and by $(\tilde{\mathbf{x}}(t), \tilde{\mathbf{p}}(t))$ – the corresponding solution of (2.1.11). Then the dynamics of $\mathcal{F}(t) := \mathcal{F}(\mathbf{x}(t), \mathbf{p}(t))$ is governed by the equation $\frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\}$, while the dynamics of $\tilde{\mathcal{F}}(t) := \mathcal{F}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{p}}(t))$ – by $\frac{d\tilde{\mathcal{F}}}{dt} = \left\{\tilde{\mathcal{F}}, \tilde{\mathcal{H}}\right\}_\mu$.

2.1.3. Next, we solve explicitly the equations of motion of a classical particle in constant magnetic field, and zero electric potential (i.e. $V = 0$), and investigate the asymptotics of its trajectory as $\mu \to \infty$. We shall consider dimensions $m = 2, 3$, and assume $M = 1$.
Let at first $m = 2$. We suppose that $B_{1,2} = 1$. Then the coordinates $(x, y) \in \mathbb{R}^2$ of the particle satisfy the equations

$$\begin{cases} \ddot{x} = \mu \dot{y}, \\ \ddot{y} = -\mu \dot{x}, \end{cases} \tag{2.1.16}$$

and the initial conditions

$$\begin{cases} x(0) = x_0, \dot{x}(0) = x_1, \\ y(0) = y_0, \dot{y}(0) = y_1. \end{cases} \tag{2.1.17}$$

The solution of Cauchy problem (2.1.16)–(2.1.17) is given by

$$\begin{cases} x(t;\mu) = x_0 + \frac{x_1}{\mu}\sin\ \mu t + \frac{y_1}{\mu}(1-\cos\ \mu t), \\ y(t;\mu) = y_0 + \frac{y_1}{\mu}\sin\ \mu t - \frac{x_1}{\mu}(1-\cos\ \mu t), \end{cases} \tag{2.1.18}$$

i.e. unless $\dot{\mathbf{x}}(0) = 0$, the trajectory of the particle is a circle of radius $\frac{\sqrt{x_1^2+y_1^2}}{\mu}$ centered at the point $(x_0 + y_1/\mu, y_0 - x_1/\mu)$. Evidently, for each $t \in \mathbb{R}$ we have $\lim_{\mu\to\infty} x(t,\mu) = x_0$, $\lim_{\mu\to\infty} y(t,\mu) = y_0$. If $m = 3$, we suppose $B_{1,2} = 1$, $B_{2,3} = B_{3,1} = 0$, i.e. $\mathbf{B} = (0,0,1)$. Then the first two coordinates (x,y) in the plane perpendicular to the magnetic field $\mathbf{B}$ satisfy (2.1.16)–(2.1.17), and their dynamics is given by (2.1.18). The third coordinate z along $\mathbf{B}$ satisfies the free-motion equation $\ddot{z} = 0$, and the initial conditions $z(0) = z_0,\ \dot{z}(0) = z_1$. Therefore,

$$z(t) = z_0 + z_1 t. \tag{2.1.19}$$

Combining (2.1.18) and (2.1.19), we find that generically our three-dimensional particle moves along a helix whose axis is parallel to the magnetic field, and whose radius tends to zero as $\mu \to \infty$. Therefore, asymptotically as $\mu \to \infty$ the trajectory of the particle is a straight line (provided that $z_1 \neq 0$), parallel to $\mathbf{B}$ and passing through the initial point (x_0, y_0, z_0).
More examples of trajectories of classical particles moving in electromagnetic fields can be found in [Iv, Chapter 6, Appendix F].

2.2 The concept of quantization. Definitions and basic properties of quantum magnetic Hamiltonians

2.2.1. The physical observables are described by the classical mechanics as sufficiently regular functions defined over a given symplectic manifold. Here we shall consider only the case where the manifold coincides with $T^*\mathbb{R}^m = \mathbb{R}^{2m}$, $m > 1$. The symplectic 2-form will be chosen as the canonical one (see (2.1.7)), or the magnetic one (see (2.1.10)).
The same physical observables are described by the quantum mechanics as linear operators acting in a given Hilbert space $\mathbf{H}$. The quantization of a classical physical system could be understood as the construction of a mapping $\mathbf{Q} : \mathcal{F} \mapsto \mathbf{Q}(\mathcal{F})$, which puts into correspondence to the function $\mathcal{F}$ defined over $T^*\mathbb{R}^m$ the linear operator $\mathbf{Q}(\mathcal{F})$ acting in $\mathbf{H}$.
In the case where the symplectic manifold on which the classical observables are defined, coincides with $T^*\mathbb{R}^m$ equipped with the canonical symplectic form, the mapping $\mathbf{Q}$ should satisfy the following axioms (see e.g. [Ha, Subsection 3.7] and the references cited there):

1. $\mathbf{Q}$ is linear, and $\mathbf{Q}(1) = \mathrm{Id}$.

2. If $\mathcal{F}$ and $\mathcal{G}$ coincide with one of the components of the vectors $\mathbf{x} = (x_1, \ldots, x_m)$ or $\mathbf{p} = (p_1, \ldots, p_m)$, then $\mathbf{Q}(\{\mathcal{F}, \mathcal{G}\}) = \frac{1}{i\hbar}[\mathbf{Q}(\mathcal{F}), \mathbf{Q}(\mathcal{G})]$, where the canonical Poisson bracket $\{.,.\}$ is defined in (2.1.12), $\hbar > 0$ is the Planck constant, and $[S,T] := ST - TS$ denotes the commutator of the linear operators S and T. In other words, in agreement with (2.1.14) we have,

$$[\mathbf{Q}(x_j), \mathbf{Q}(x_k)] = 0,\ [\mathbf{Q}(p_j), \mathbf{Q}(p_k)] = 0,\ [\mathbf{Q}(x_j), \mathbf{Q}(p_k)] = i\hbar\mathbf{Q}(1) \equiv i\hbar\,\mathrm{Id}, \quad j,k = 1, \ldots, m. \tag{2.2.1}$$

3. $\mathbf{H}$ is irreducible under the action of $\mathbf{Q}(x_j)$ and $\mathbf{Q}(p_j)$, $j = 1, \ldots, m$.

4. If $\mathcal{F}$ is real-valued, then $\mathbf{Q}(\mathcal{F})$ is self-adjoint.

The Stone – von Neumann theorem (see e.g. [Re.S, vol.1, Theorem VIII.14]) implies that up to a unitary equivalence $\mathbf{H} = L^2(\mathbb{R}^m)$, $\mathbf{Q}(x_j)$ is equal to the multiplier by x_j, and $\mathbf{Q}(p_j) = -i\hbar\frac{\partial}{\partial x_j}$,

$j = 1, \ldots, m$. Define the unitary Fourier transform $\Phi : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)$ by

$$(\Phi u)(\mathbf{p}) := (2\pi\hbar)^{-m/2} \int_{\mathbb{R}^m} e^{-i\mathbf{x}.\mathbf{p}/\hbar} u(\mathbf{x}) d\mathbf{x}.$$

If $\mathcal{F}(\mathbf{x}, \mathbf{p})$ can be written as the sum

$$\mathcal{F}(\mathbf{x}, \mathbf{p}) = \mathcal{F}_1(\mathbf{x}) + \mathcal{F}_2(\mathbf{p}), \tag{2.2.2}$$

then at least formally the operator $\mathbf{Q}(\mathcal{F})$ can be defined as

$$\mathbf{Q}(\mathcal{F}) = \mathbf{Q}(\mathcal{F}_1) + \mathbf{Q}(\mathcal{F}_2) := \mathcal{F}_1 + \Phi^* \mathcal{F}_2 \Phi. \tag{2.2.3}$$

Here we use the same notation for the multipliers by $\mathcal{F}_j$, $j = 1, 2$, and the functions $\mathcal{F}_j$ themselves. The attempt to extend straightforwardly this quantization scheme to *products* $\mathcal{F}(\mathbf{x}, \mathbf{p}) = \mathcal{F}_1(\mathbf{x})\mathcal{F}_2(\mathbf{p})$ meets a difficulty since $\mathcal{F}_1(\mathbf{x})\mathcal{F}_2(\mathbf{p}) = \mathcal{F}_2(\mathbf{p})\mathcal{F}_1(\mathbf{x})$ while the operators $\mathbf{Q}(\mathcal{F}_1)$ and $\mathbf{Q}(\mathcal{F}_2)$ generically do not commute, i.e. $\mathcal{F}_1 \Phi^* \mathcal{F}_2 \Phi \neq \Phi^* \mathcal{F}_2 \Phi \mathcal{F}_1$. One of the possible ways to overcome this difficulty is to introduce the Weyl quantization scheme. According to it, if $\mathcal{F} \in C^\infty(T^*\mathbb{R}^m)$ possesses certain regularity properties, we define the operator $\mathbf{Q}(\mathcal{F})$ by

$$(\mathbf{Q}(\mathcal{F})u)(\mathbf{x}) := \frac{1}{(2\pi\hbar)^m} \int_{\mathbb{R}^m} \mathcal{F}\left(\frac{\mathbf{x} + \mathbf{x}'}{2}, \mathbf{p}\right) e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}/\hbar} u(\mathbf{x}') d\mathbf{x}' d\mathbf{p}, \tag{2.2.4}$$

the integral at the right-hand side being an oscillatory one. If $\mathcal{F}$ can be written as in (2.2.2), then the operators defined by (2.2.3) and (2.2.4) coincide, provided that $\mathcal{F}_1$ and $\mathcal{F}_2$ are regular enough. In this case we shall say that $\mathbf{Q}(\mathcal{F})$ is a *pseudo-differential operator* (ΨDO) with *Weyl symbol* $\mathcal{F}$ and shall use the notation $\mathbf{Q}(\mathcal{F}) := \mathcal{F}^w(x, \hbar D)$, $D = (-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_m})$. A detailed theory of the ΨDO acting in $L^2(\mathbb{R}^m)$, and, in particular, the ΨDO with Weyl symbols can be found in [Shu, Chapter IV] (see also [Hö, vol.3, Section 18.5], and [D.Sj, Chapter 7]).

In the case of general measurable $\mathcal{F}_1$ and $\mathcal{F}_2$ occurring in (2.2.2), the operator $\mathbf{Q}(\mathcal{F}_1) = \mathcal{F}_1$ appearing in (2.2.3) is closed on the domain $\left\{u \in L^2(\mathbb{R}^m) | \mathcal{F}_1 u \in L^2(\mathbb{R}^m)\right\}$ and is self-adjoint on this domain provided that $\mathcal{F}_1 = \bar{\mathcal{F}}_1$. Similarly, the operator $\mathbf{Q}(\mathcal{F}_2) = \Phi^* \mathcal{F}_2 \Phi$ is closed on the domain $\left\{u \in L^2(\mathbb{R}^m) | \mathcal{F}_2 \Phi u \in L^2(\mathbb{R}^m)\right\}$ and is self-adjoint on this domain provided that $\mathcal{F}_2 = \bar{\mathcal{F}}_2$. In order to define the sum (2.2.3) as a closed or self-adjoint operator, we need additional assumptions concerning $\mathcal{F}_j$, $j = 1, 2$. Let us consider the important example

$$\Pi_j(\hbar, \mu) = \mathbf{Q}(p_j - \mu A_j) := -i\hbar \frac{\partial}{\partial x_j} - \mu A_j, \quad j = 1, \ldots, m, \quad \mu \in \mathbb{R}.$$

We shall call the operator $\Pi(\hbar, \mu) := -i\hbar\nabla - \mu\mathbf{A} = (\Pi_1(\hbar, \mu), \ldots, \Pi_m(\hbar, \mu))$ the operator of the magnetic momentum. Notice the commutation relations

$$[\Pi_j(\hbar, \mu), \Pi_k(\hbar, \mu)] = i\hbar\mu B_{j,k}, \quad j, k = 1, \ldots, m, \tag{2.2.5}$$

where the components $B_{j,k}$ of the magnetic field B are defined in (2.1.9).

Assume $\mathbf{A} \in L^2_{\rm loc}(\mathbb{R}^m; \mathbb{R}^m)$. The operators $\Pi_j(\hbar, \mu)$, $j = 1, \ldots, m$, are symmetric on $C_0^\infty(\mathbb{R}^m)$. In the sequel we shall denote by $\Pi_j(\hbar, \mu)$, $j = 1, \ldots, m$, the closure of the operator $-i\hbar\frac{\partial}{\partial x_j} - \mu A_j$ defined originally on $C_0^\infty(\mathbb{R}^m)$. We could obtain the operator $\Pi(\hbar, \mu)$ using another closely related quantization scheme. We choose the underlying symplectic manifold on which the classical observables are defined as $T^*\mathbb{R}^m$ equipped with the *magnetic* symplectic form (see (2.1.10)). Further, we introduce the quantization mapping $\mathbf{Q}_\mu$ satisfying the same axioms as $\mathbf{Q}$ except that axiom 2 is replaced by:

2_μ. If $\mathcal{F}$ and $\mathcal{G}$ coincide with one of the components of the vectors $\mathbf{x} = (x_1, \ldots, x_m)$ or $\mathbf{p} = (p_1, \ldots, p_m)$, then $\mathbf{Q}_\mu(\{\mathcal{F}, \mathcal{G}\}_\mu) = \frac{1}{i\hbar}[\mathbf{Q}_\mu(\mathcal{F}), \mathbf{Q}_\mu(\mathcal{G})]$, where the magnetic Poisson bracket $\{.,.\}_\mu$ is defined in (2.1.13). In other words, in agreement with (2.1.15), we have

$$[\mathbf{Q}_\mu(x_j), \mathbf{Q}_\mu(x_k)] = 0, \ [\mathbf{Q}_\mu(p_j), \mathbf{Q}_\mu(p_k)] = i\hbar\mu \mathbf{Q}_\mu(B_{j,k}), \ [\mathbf{Q}(x_j), \mathbf{Q}(p_k)] = i\hbar \, \mathrm{Id}, \ j,k = 1, \ldots, m. \tag{2.2.6}$$

Notice that the only difference between (2.2.1) and (2.2.6) is the second relation, i.e. the commutation relation between $\mathbf{Q}(p_j)$ and $\mathbf{Q}(p_k)$, $j,k = 1, \ldots, m$. We can define again $\mathbf{Q}_\mu(x_j)$, $j = 1, \ldots, m$, as the multiplier by x_j, and consequently $\mathbf{Q}_\mu(\mathcal{F})$ with $\mathcal{F} = \mathcal{F}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^m$, as the multiplier by $\mathcal{F}$; in particular, $\mathbf{Q}_\mu(B_{j,k}) = B_{j,k}$, $j,k = 1, \ldots, m$. Further, we set $\mathbf{Q}_\mu(p_j) = -i\hbar\frac{\partial}{\partial x_j} - \mu A_j \equiv \Pi_j(\hbar, \mu)$, $j = 1, \ldots, m$. Then (2.2.5) coincides with second commutation relation in (2.2.6). In order to enumerate several spectral properties of the momentum operators, we shall recall some definitions.
Let S and T be two closed linear operators acting in a given Hilbert space $\mathbf{H}$. The operators S and T are said to be *unitarily equivalent* if there exists a unitary operator $U : \mathbf{H} \to \mathbf{H}$, $U : \mathrm{Dom}\ (T) \to \mathrm{Dom}\ (S)$, such that $U^*SU = T$. Assume in addition that $\mathbf{H} = L^2(\mathbb{R}^m)^s$, $s \geq 1$. Let Γ be the antilinear operator of the complex conjugation, i.e. $(\Gamma u)(\mathbf{x}) = \overline{u(\mathbf{x})}$, $u \in \mathbf{H}$, $\mathbf{x} \in \mathbb{R}^m$. The operators S and T are said to be *anti-unitarily equivalent* if the operators $\Gamma S \Gamma$ and T are unitarily equivalent.
If $S = S^*$ and $T = T^*$ are unitarily or anti-unitarily equivalent, their spectra, essential spectra, and discrete spectra coincide, i.e. $\sigma(S) = \sigma(T)$, $\sigma_{\mathrm{ess}}(S) = \sigma_{\mathrm{ess}}(T)$, $\sigma_{\mathrm{disc}}(S) = \sigma_{\mathrm{disc}}(T)$.
We shall say that two magnetic potentials $\mathbf{A}^{(l)} \in L^2_{\mathrm{loc}}(\mathbb{R}^m; \mathbb{R}^m)$, $l = 1, 2$, are *gauge equivalent* if there exists a scalar function φ such that

$$\mathbf{A}^{(1)} - \mathbf{A}^{(2)} = \nabla\varphi. \tag{2.2.7}$$

Moreover, we shall say that the linear operators S and T acting in $\mathbf{H}$ are *gauge unitarily equivalent* if there exists a measurable real-valued function ψ over $\mathbb{R}^m$ such that we have

$$S = e^{-i\psi} T e^{i\psi}. \tag{2.2.8}$$

The components of any two momentum operators corresponding to gauge equivalent magnetic potentials are gauge unitarily equivalent. More precisely, let the potentials $\mathbf{A}^{(l)} \in L^2_{\mathrm{loc}}(\mathbb{R}^m; \mathbb{R}^m)$, $l = 1, 2$, satisfy (2.2.7). Set $\Pi^{(l)}(\hbar, \mu) := -i\hbar\mu\nabla - \mu\mathbf{A}^{(l)}$, $l = 1, 2$. Then we have

$$\Pi_j^{(2)}(\hbar, \mu) = e^{-i\mu\varphi/\hbar} \Pi_j^{(1)}(\hbar, \mu) e^{i\mu\varphi/\hbar}, \quad j, \ldots, m, \tag{2.2.9}$$

i.e. (2.2.8) holds with $S = \Pi_j^{(2)}(\hbar, \mu)$, $T = \Pi_j^{(1)}(\hbar, \mu)$, and $\psi = \mu\varphi/\hbar$. Moreover,

$$\Gamma \Pi_j(\hbar, \mu) \Gamma = -\Pi_j(\hbar, -\mu), \quad j = 1, \ldots, m. \tag{2.2.10}$$

In particular, the operators $\Pi_j(\hbar, \mu)$ and $-\Pi_j(\hbar, -\mu)$, $j = 1, \ldots, m$ are anti-unitarily equivalent.

2.2.2. Our next goal is the construct the operator $\mathbb{H}$ corresponding to the Hamiltonian function $\mathcal{H}(\mathbf{x}, \mathbf{p})$ defined in (2.1.4); in other words, $\mathbb{H} = \mathbf{Q}(\mathcal{H})$. Assume at first $g = 0$. Then the operator $\mathbb{H}_0(\hbar, \mu)$ corresponding to the kinetic part $\mathcal{H}_0 := \frac{1}{2M}|\mathbf{p} - \mu\mathbf{A}(\mathbf{x})|^2$ of the Hamiltonian function can be written as $\mathbb{H}_0(\hbar, \mu) := \mathbf{Q}(\mathcal{H}_0) := \frac{1}{2M}\sum_{j=1}^m \Pi_j(\hbar, \mu)^2$. In order to define a self-adjoint realization of $\mathbb{H}_0(\hbar, \mu)$ in the general case $\mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^m; \mathbb{R}^m)$, let us introduce the quadratic form

$$\frac{1}{2M}\sum_{j=1}^m \int_{\mathbb{R}^m} |\Pi_j(\hbar, \mu)u|^2 \, d\mathbf{x}, \quad u \in C_0^\infty(\mathbb{R}^m), \tag{2.2.11}$$

and close it in $L^2(\mathbb{R}^m)$. Then $\mathbb{H}_0(\hbar, \mu)$ is the unique self-adjoint operator generated in $L^2(\mathbb{R}^m)$ by the closed quadratic form (2.2.11) (see [Av.H.S, Section 2])). In our further considerations of $\mathbb{H}_0$ we shall use such a system of units that $M = 1/2$ so that $1/2M = 1$.
The operator $\mathbb{H}_0$ satisfies *the diamagnetic inequality*, presented below in a convenient form.

Lemma 2.1 *Let* $\mathbf{A} \in L^2_{\rm loc}(\mathbb{R}^m, \mathbb{R}^m)$*,* $\hbar > 0$*,* $\lambda > 0$*,* $\mu \in \mathbb{R}$*, and* $\gamma > 0$*. Then for each* $u \in L^2(\mathbb{R}^m)$*, and for almost every* $\mathbf{x} \in \mathbb{R}^m$ *we have* $|(\mathbb{H}_0(\hbar, \mu) + \lambda)^{-\gamma} u(\mathbf{x})| \leq \left((-\hbar^2\Delta + \lambda)^{-\lambda}|u|\right)(\mathbf{x})$*.*

Before we proceed with some corollaries of Lemma 2.1, let us recall again some definitions.
Let S and T be two self-adjoint operators acting in Hilbert space. The operator T is called S-bounded (respectively, S-compact) if the operator $T(S+i)^{-1}$ is bounded (respectively, compact). Assume now that S is lower bounded so that for some $\lambda > 0$ the operator $S + \lambda$ is positive definite. Then the operator T is called S-form-bounded (resp. S-form-compact) if the operator $|T|^{1/2}(S + \lambda)^{-1/2}$ is bounded (resp. compact).
If T is S-bounded, then the limit $\lim_{\lambda\to\infty} \|T(S + i\lambda)^{-1}\|$ exists and is called the relative S-bound of T. If this relative bound is smaller than one, the operator $S + T$ is self-adjoint on Dom (S). Notice that if the operator S is lower bounded, then the S-bound of the operator T can be written as $\lim_{\lambda\to\infty} \|T(S + \lambda)^{-1}\|$.
Similarly, if T is S-form-bounded, then the limit $\lim_{\lambda\to\infty} \||T|^{1/2}(S + \lambda)^{-1/2}\|^2$ exists and is called the relative S-form-bound of T. If this bound is smaller than one, the quadratic form of the operator $S+T$ is closed on the domain of the quadratic form of the operator S, and hence the operator $S+T$ is well-defined and self-adjoint as a form sum.
Finally, if T is S-compact (respectively, S-form-compact), then the relative S-bound (respectively, S-form-bound) of T is equal to zero. Moreover, in this case we have $\sigma_{\rm ess}(S + T) = \sigma_{\rm ess}(S)$.

Corollary 2.1 *Let* $\mathbf{A} \in L^2_{\rm loc}(\mathbb{R}^m; \mathbb{R}^m)$*,* $\mu \in \mathbb{R}$*. Let* $V : \mathbb{R}^m \to \mathbb{R}$ *be measurable function.*
a) Assume that the multiplier by V *is* $\hbar^2\Delta$*-bounded with relative bound* α*. Then* V *is* $\mathbb{H}_0(\hbar, \mu)$*-bounded with relative bound at most* α*. If* $|g|\alpha < 1$*, then the operator*

$$\mathbb{H}(\hbar, \mu, g) := \mathbb{H}_0(\hbar, \mu) + gV \tag{2.2.12}$$

is self-adjoint on Dom $(\mathbb{H}_0)$*. Moreover, if* V *is* Δ*-compact, then it is also* $\mathbb{H}_0(\hbar, \mu)$*-compact, and*

$$\sigma_{\rm ess}(\mathbb{H}(\hbar, \mu, g)) = \sigma_{\rm ess}(\mathbb{H}_0(\hbar, \mu)). \tag{2.2.13}$$

b) Assume that V *is* $-\hbar^2\Delta$*-form-bounded with relative bound* α*. Then* V *is* $\mathbb{H}_0(\hbar, \mu)$*-form-bounded with relative bound at most* α*. If* $|g|\alpha < 1$*, then the quadratic form of the operator (2.2.12) is closed on the domain of the quadratic form of* $\mathbb{H}_0$ *and generates a unique self-adjoint operator* $\mathbb{H}$ *in* $L^2(\mathbb{R}^m)$*. Moreover, if* V *is* $-\Delta$*-form-compact, then it is also* $\mathbb{H}_0(\hbar, \mu)$*-form-compact, and (2.2.13) holds.*

Next, we give explicit examples of potentials V which are Δ-compact and $-\Delta$-form-compact. We shall say that the function V is in the class $\mathcal{L}_p = \mathcal{L}_p(\mathbb{R}^n)$, $p \geq 1$, $n \geq 1$, if for every $\varepsilon > 0$ we have $V = V_1 + V_2$ with $V_1 \in L^p(\mathbb{R}^n)$ and $\sup_{\mathbf{x}\in\mathbb{R}^n} |V_2(\mathbf{x})| \leq \varepsilon$.
In contrast to the usual Lebesgue spaces $L^p(\mathbb{R}^n)$, the classes $\mathcal{L}_p$ are embedded, i.e. $\mathcal{L}_p \subset \mathcal{L}_r$ if $p > r$.

Lemma 2.2 *a) Let* $V \in \mathcal{L}_p(\mathbb{R}^n)$ *with* $p = 2$ *if* $n = 1, 2, 3$*,* $p > 2$ *if* $n = 4$*, and* $p = n/2$ *if* $n \geq 5$*. Then* V *is* Δ*-compact.*
b) Let $V \in \mathcal{L}_p(\mathbb{R}^n)$ *with* $p = 1$ *if* $n = 1$*,* $p > 1$ *if* $n = 2$*, and* $p = n/2$ *if* $n \geq 3$*. Then* V *is* $-\Delta$*-form-compact.*

The operator $\mathbb{H}(\hbar, \mu, g)$ defined in (2.2.12) is called the Schrödinger operator. It is the Hamiltonian of a quantum non-relativistic particle of zero spin. Notice that (2.2.9) implies that two Schrödinger operators corresponding to gauge equivalent magnetic potentials, are gauge unitarily equivalent. Moreover, (2.2.10) entails $\Gamma\mathbb{H}(\hbar, \mu, g)\Gamma = \mathbb{H}(\hbar, -\mu, g)$. Hence, the operators $\mathbb{H}(\hbar, \mu, g)$ and $\mathbb{H}(\hbar, -\mu, g)$ are anti-unitarily equivalent.

2.2.3. The Schrödinger operator $\mathbb{H}$ does not take account of the spin effects. The appropriate Hamiltonian operator of a quantum non-relativistic particle of $\frac{1}{2}$-spin is the Pauli operator (see e.g.

[Mes, vol.2, Chapter XII, Section 18]). Although it is possible to define this operator in arbitrary dimension (see e.g. [Sh]), here we shall consider it only for $m = 2, 3$. Introduce the Pauli matrices $\hat{\sigma}_j$, $j = 1, 2, 3$, which are constant 2×2 Hermitian matrices satisfying

$$\hat{\sigma}_j \hat{\sigma}_k + \hat{\sigma}_k \hat{\sigma}_j = 2\delta_{j,k}, \quad j, k = 1, 2, 3. \tag{2.2.14}$$

In the standard representation which is to be used in the sequel, we have

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Introduce the unperturbed Pauli operator

$$\mathbb{P}_0(\hbar, \mu) := \frac{1}{2M} \left(\sum_{j=1}^{m} \hat{\sigma}_j \Pi_j(\hbar, \mu) \right)^2, \quad m = 2, 3,$$

acting in $L^2(\mathbb{R}^m)^2$. In our further considerations of $\mathbb{P}_0$ we assume $M = 1/2$, as in the case of the Schrödinger operator. Notice that if $m = 2$,

$$\mathbb{P}_0(\hbar, \mu) = \mathbb{H}_0(\hbar, \mu) I_2 - \hbar\mu \hat{\sigma}_3 B_{1,2}, \tag{2.2.15}$$

while in the case $m = 3$,

$$\mathbb{P}_0(\hbar, \mu) = \mathbb{H}_0(\hbar, \mu) I_2 - \hbar\mu \sum_{j=1}^{3} \hat{\sigma}_j B_j, \tag{2.2.16}$$

where we have used the standard identification of the entries of the matrix B with the components of the vector $\mathbf{B}$. In order to define a self-adjoint realization of $\mathbb{P}_0(\hbar, \mu)$ in the general case $\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^m; \mathbb{R}^m)$, let us introduce the operator $\Sigma(\hbar, \mu) := \sum_{j=1}^{m} \hat{\sigma}_j \Pi_j(\hbar, \mu)$ defined originally on $C_0^\infty(\mathbb{R}^m)^2$, and then closed in $L^2(\mathbb{R}^m)^2$. Then $\mathbb{P}_0(\hbar, \mu)$ is self-adjoint on the domain

$$\text{Dom}(\mathbb{P}_0(\hbar, \mu)) := \{u \in \text{Dom}(\Sigma(\hbar, \mu)) | \Sigma(\hbar, \mu)u \in \text{Dom}(\Sigma(\hbar, \mu)^*)\},$$

and $\mathbb{P}_0(\hbar, \mu) = \Sigma(\hbar, \mu)^* \Sigma(\hbar, \mu)$. Notice that if the magnetic field B is bounded, i.e. if $B_{j,k} \in L^\infty(\mathbb{R}^m)$, the operator $\mathbb{P}_0(\hbar, \mu) - \mathbb{H}_0(\hbar, \mu) I_2$ is bounded, and $\mathbb{P}_0(\hbar, \mu)$ is self-adjoint on Dom $(\mathbb{H}_0(\hbar, \mu))^2$. Let $V : \mathbb{R}^m \to \mathbb{R}$ be a measurable function over. Set

$$\mathbb{P}(\hbar, \mu, g) := \mathbb{P}_0(\hbar, \mu) + gVI_2, \quad g \in \mathbb{R}.$$

Tackling the self-adjointness of $\mathbb{P}(\hbar, \mu, g)$, one can make use of the following assertions.

Lemma 2.3 *Let $m = 2, 3$. Assume that $\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^m; \mathbb{R}^m)$, and the magnetic field B is bounded. Then the relative $\mathbb{H}_0$-bound (respectively, $\mathbb{H}_0$-form-bound) of the multiplier by V is equal to the $\mathbb{P}_0$-bound (respectively, $\mathbb{P}_0$-form-bound) of the operator VI_2. Moreover, V is $\mathbb{H}_0$-compact (respectively, $\mathbb{H}_0$-form-compact) if and only if VI_2 is $\mathbb{P}_0$-compact (respectively, $\mathbb{P}_0$-form-compact).*

Corollary 2.2 *Let $m = 2, 3$. Assume that $\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^m; \mathbb{R}^m)$, and the magnetic field B is bounded.*
a) Let V satisfy the assumptions of Lemma 2.2 a). Then the multiplier by V is $\mathbb{P}_0$-compact.
b) Let V satisfy the assumptions of Lemma 2.2 b). Then the multiplier by V is $\mathbb{P}_0$-form-compact.

Results on the self-adjointness of $\mathbb{P}$ with unbounded magnetic field can be found in [Sob 1] and [Sob 2]. As in the case of the Schrödinger operator, (2.2.9) implies that two Pauli operators corresponding to gauge equivalent magnetic potentials, are gauge unitarily equivalent.
Put $U_{\text{Pauli}} := i\hat{\sigma}_2$. We have $U^*_{\text{Pauli}} \bar{\hat{\sigma}}_j U_{\text{Pauli}} = -\hat{\sigma}_j$, $j = 1, 2, 3$. By (2.2.10), $U^*_{\text{Pauli}} \Gamma \mathbb{P}(\hbar, \mu, g) \Gamma U_{\text{Pauli}} =$

$\mathbb{P}(\hbar, -\mu, g)$. Hence, the operators $\mathbb{P}(\hbar, \mu, g)$ and $\mathbb{P}(\hbar, -\mu, g)$ are antiunitarily equivalent.

2.2.4. The Pauli operator does not take into account the relativistic effects. The appropriate Hamiltonian operator of a quantum relativistic particle of $\frac{1}{2}$-spin is the Dirac operator. As in the case of the Pauli operator, we shall consider it only for $m = 2, 3$. First, we introduce the Dirac matrices $\hat{\alpha}_j$, $j = 1, \dots, m$, and $\hat{\beta}$. For $l_2 = 2$, $l_3 = 4$, they are constant Hermitian $l_m \times l_m$ matrices satisfying

$$\hat{\alpha}_j\hat{\alpha}_k + \hat{\alpha}_k\hat{\alpha}_j = 2\delta_{j,k},\ j,k = 1,\dots,m,\ \hat{\alpha}_j\hat{\beta} + \hat{\beta}\hat{\alpha}_j = 0,\ j = 1,\dots,m,\ \hat{\beta}^2 = I_{l_m}. \qquad (2.2.17)$$

In what follows we shall use the standard representation of the Dirac matrices: if $m = 2$, then $\hat{\alpha}_j = \hat{\sigma}_j$, $j = 1, 2$, $\hat{\beta} = \hat{\sigma}_3$, and if $m = 3$, then

$$\hat{\alpha}_j = \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix}, \quad j = 1, 2, 3, \quad \hat{\beta} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

Introduce the unperturbed Dirac operator

$$\mathbb{D}_0(\hbar, \mu) := c\sum_{j=1}^{m} \hat{\alpha}_j \Pi_j(\hbar, \mu) + Mc^2\hat{\beta}$$

acting in $L^2(\mathbb{R}^m)^{l_m}$. Here c denotes the speed of light in vacuum, and M – as usually, the mass of the particle. In our further considerations of the Dirac operator, we use such a system of units that $c = 1$ and $M = 1$. For simplicity, we shall consider the operator $\mathbb{D}_0(\hbar, \mu)$ only for $\mathbf{A} \in C^\infty(\mathbb{R}^m; \mathbb{R}^m)$; in this case $\mathbb{D}_0(\hbar, \mu)$ is essentially self-adjoint on the domain $C_0^\infty(\mathbb{R}^m)^{l_m}$. Moreover, $\mathbb{D}_0(\hbar, \mu)$ satisfies the identity

$$\mathbb{D}_0(\hbar, \mu)^2 = \begin{cases} \mathbb{P}_0(\hbar, \mu) + I_2 & \text{if} \quad m = 2, \\ \begin{pmatrix} \mathbb{P}_0(\hbar, \mu) + I_2 & 0 \\ 0 & \mathbb{P}_0(\hbar, \mu) + I_2 \end{pmatrix} & \text{if} \quad m = 3. \end{cases} \qquad (2.2.18)$$

Let $V : \mathbb{R}^m \to \mathbb{R}$ be a measurable function, $m = 2, 3$. Set

$$\mathbb{D}(\hbar, \mu, g) := \mathbb{D}_0(\hbar, \mu) + gVI_{l_m}.$$

Approaching the problem of the self-adjointness of $\mathbb{D}$, one should not forget that $\mathbb{D}_0$ is not semi-bounded so that the self-adjoint realization of $\mathbb{D}$ as *a form sum* is possible only after essential modifications of the method. Hence, the self-adjoint realization of $\mathbb{D}$ as an operator sum is more common. Making use of (2.2.18), we could easily prove the following

Lemma 2.4 *Let $m = 2, 3$, $\mathbf{A} \in L^2_{\rm loc}(\mathbb{R}^m; \mathbb{R}^m)$. Then the relative $\mathbb{D}_0$-bound of the operator VI_{l_m} does not exceed the relative $\mathbb{P}_0$-form-bound of the operator V^2I_2. Moreover, if the operator V^2I_2 is $\mathbb{P}_0$-form-compact, then the operator VI_{l_m} is $\mathbb{D}_0$-compact.*

Taking into account Corollary 2.1, we obtain the following

Corollary 2.3 *Let $m = 2, 3$, $\mathbf{A} \in L^2_{\rm loc}(\mathbb{R}^m; \mathbb{R}^m)$. Assume that the magnetic field B is bounded. Let $V \in \mathcal{L}_p$ with $p > 2$ if $m = 2$, and $p = 3$ if $m = 3$. Then the operator VI_{l_m} is $\mathbb{D}_0$-compact.*

As in the case of the Schrödinger and Pauli operators, (2.2.9) implies that two Dirac operators corresponding to gauge equivalent magnetic potentials, are gauge unitarily equivalent. Put

$$U_{\rm Dirac} := \begin{pmatrix} 0 & i\hat{\sigma}_2 \\ -i\hat{\sigma}_2 & 0 \end{pmatrix}.$$

We have $U^*_{\rm Dirac}\bar{\hat{\alpha}}_j U_{\rm Dirac} = \hat{\alpha}_j$, $j = 1, 2, 3$, $U^*_{\rm Dirac}\bar{\hat{\beta}} U_{\rm Dirac} = -\hat{\beta}$ (cf. [Th, Section 1.4.6]). Therefore, by (2.2.10), $U^*_{\rm Dirac}\Gamma\mathbb{D}(\hbar, \mu, g)\Gamma U_{\rm Dirac} = -\mathbb{D}(\hbar, -\mu, -g)$. Hence, the operators $\mathbb{D}(\hbar, \mu, g)$ and

$-\mathbb{D}(\hbar, -\mu, -g)$ are anti-unitarily equivalent.

In what follows we will be mainly interested in the case $\hbar = g = 1$; in this case we shall skip $\hbar$ and g in the notations of the Schrödinger, Pauli, and Dirac operators.

2.3 Quantum Hamiltonians with constant magnetic fields

2.3.1. In this subsection we shall discuss the case where the magnetic field B is constant, i.e. its entries $B_{j,k}$, $j,k = 1,\dots,m$, are independent of $\mathbf{x} \in \mathbb{R}^m$. In this case B can be regarded as a antisymmetric linear mapping $\mathbb{R}^m \to \mathbb{R}^m$. Set $2d := \dim \operatorname{Ran} B$, and $k := m - 2d = \dim \operatorname{Ker} B$. Throughout the subsection we assume that $B \neq 0$, and hence $d \geq 1$. The spectral theory of the operators $\mathbb{H}_0(\mu)$, $\mathbb{P}_0(\mu)$, and $\mathbb{D}_0(\mu)$ is quite different in the case $k = 0$ and $k \geq 1$; that is why we shall consider them separately. The two leading examples illustrating these two cases, are respectively $m = 2$, i.e. $d = 1$ and $k = 0$, and $m = 3$, i.e. $d = 1$ and $k = 1$.
Let $b_1 \geq \dots \geq b_d > 0$ be such numbers that the non-zero eigenvalues of B coincide with $-ib_j$ and ib_j, $j = 1,\dots,d$. Then there exist Cartesian coordinates $X = (x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^{2d} = \mathbb{R}^m = \operatorname{Ran} B$ if $k = 0$ (respectively, $(X, z) = (x_1, y_1, \dots, x_d, y_d, z_1, \dots, z_k)$ with $X = (x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^{2d} = \operatorname{Ran} B$, and $z = (z_1, \dots, z_k) \in \mathbb{R}^k = \operatorname{Ker} B$ if $k \geq 1$), in which the operator $\mathbb{H}_0(\mu)$ can be written as

$$\mathbb{H}_0(\mu) := \sum_{j=1}^{d} \left\{ \left(-i\frac{\partial}{\partial x_j} + \mu\frac{b_j y_j}{2}\right)^2 + \left(-i\frac{\partial}{\partial y_j} - \mu\frac{b_j x_j}{2}\right)^2 \right\} \quad \text{if} \quad k = 0, \tag{2.3.1}$$

$$\mathbb{H}_0(\mu) := \sum_{j=1}^{d} \left\{ \left(-i\frac{\partial}{\partial x_j} + \mu\frac{b_j y_j}{2}\right)^2 + \left(-i\frac{\partial}{\partial y_j} - \mu\frac{b_j x_j}{2}\right)^2 \right\} - \sum_{l=1}^{k} \frac{\partial^2}{\partial z_l^2} \quad \text{if} \quad k \geq 1. \tag{2.3.2}$$

In both cases $k = 0$ and $k \geq 1$ we have $B = \sum_{j=1}^{d} b_j dx_j \wedge dy_j$.

2.3.2. First, we consider the case $k = 0$; then (2.3.1) is valid.
Let us start with the leading example $m = 2$, i.e. $d = 1$. In this case we shall assume without any loss of generality that $b := b_1 = 1$; we can always restore the general formulae with arbitrary $b > 0$ by substituting μ for μb. Hence, (2.3.1) reduces to $\mathbb{H}_0(\mu) = \Pi_1(\mu)^2 + \Pi_2(\mu)^2$ where

$$\Pi_1(\mu) = -i\frac{\partial}{\partial x} + \frac{\mu y}{2},\ \Pi_2(\mu) = -i\frac{\partial}{\partial y} - \frac{\mu x}{2},$$

with $(x, y) := (x_1, y_1)$. Set

$$a(\mu) := \Pi_1(\mu) - i\Pi_2(\mu), a(\mu)^* := \Pi_1(\mu) + i\Pi_2(\mu). \tag{2.3.3}$$

We have $[a(\mu)^*, a(\mu)] = 2\mu \operatorname{Id}$. The operators $a(\mu)$ and $a(\mu)^*$ play the role respectively of creation and annihilation operators in the spectral theory of $\mathbb{H}_0(\mu)$ since

$$\mathbb{H}_0(\mu) = a(\mu)a(\mu)^* + \mu \operatorname{Id} = a(\mu)^* a(\mu) - \mu \operatorname{Id}. \tag{2.3.4}$$

Introduce the operators

$$\hat{\Pi}_1 := -i\frac{\partial}{\partial x},\ \hat{\Pi}_2 := -x,\ \hat{\mathbb{H}}_0 := \hat{\Pi}_1^2 + \hat{\Pi}_2^2 = -\frac{\partial^2}{\partial x^2} + x^2,$$

acting in $L^2(\mathbb{R}^2_{x,y})$. Further, define the unitary operator $\mathcal{W}_\mu : L^2(\mathbb{R}^2_{x,y}) \to L^2(\mathbb{R}^2_{x,y})$ by

$$(\mathcal{W}_\mu u)(x, y) := \frac{\sqrt{\mu}}{2\pi} \int_{\mathbb{R}^2} e^{i\varphi_\mu(x,y;x',y')} u(x', y') dx' dy' \tag{2.3.5}$$

where $\varphi_\mu(x,y;x',y') := \mu\frac{xy}{2} - \mu^{1/2}(xy'+yx') + x'y'$, $\mu > 0$. Evidently,

$$\mathcal{W}_\mu^* \Pi_j(\mu) \mathcal{W}_\mu = \mu^{1/2}\hat{\Pi}_j, \quad j = 1,2, \tag{2.3.6}$$

and, hence,

$$\mathcal{W}_\mu^* \mathbb{H}_0(\mu) \mathcal{W}_\mu = \mu \hat{\mathbb{H}}_0. \tag{2.3.7}$$

The unitarity of $\mathcal{W}_\mu$ as well as the validity of (2.3.6) can be easily checked by direct calculations but there are exist a deeper reason for these facts which we are going to discuss now.
On $T^*\mathbb{R}^d \oplus T^*\mathbb{R}^d = \mathbb{R}^{2d} \oplus \mathbb{R}^{2d}$ introduce the symplectic bilinear form $w(\mathbf{x},\mathbf{p};\mathbf{x}',\mathbf{p}') := \sum_{j=1}^d (x_j p_j' - p_j x_j')$, $(\mathbf{x},\mathbf{p}) \in \mathbb{R}^{2d}$, $(\mathbf{x}',\mathbf{p}') \in \mathbb{R}^{2d}$, which corresponds to the canonical symplectic differential 2-form (2.1.7). We shall say that the linear mapping $S : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is symplectic if $w(S(\mathbf{x},\mathbf{p}); S(\mathbf{x}',\mathbf{p}')) = w(\mathbf{x},\mathbf{p};\mathbf{x}',\mathbf{p}')$ for each $(\mathbf{x},\mathbf{p}) \in \mathbb{R}^{2d}$, $(\mathbf{x}',\mathbf{p}') \in \mathbb{R}^{2d}$.

Lemma 2.5 [D.Sj, Chapter 7, Theorem A2] *Let T_1 and T_2 be two pseudo-differential operators acting in $L^2(\mathbb{R}^d)$, and having the Weyl symbols a_1 and a_2, respectively. Assume that there exists a linear symplectic mapping $S : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ such that $a_2(\mathbf{x},\mathbf{p}) = a_1(S(\mathbf{x},\mathbf{p}))$, $(\mathbf{x},\mathbf{p}) \in \mathbb{R}^{2d}$. Then there exists a unitary operator $\mathcal{U} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that*

$$T_2 = \mathcal{U}^* T_1 \mathcal{U}. \tag{2.3.8}$$

Remark: We do not claim that the unitary operator $\mathcal{U}$ satisfying the assertion of Lemma 2.5 is unique. For example, if there exists another unitary operator $\mathcal{U}'$ commuting with T_1, then (2.3.8) is valid again if we replace $\mathcal{U}$ by $\widetilde{\mathcal{U}} := \mathcal{U}'\mathcal{U}$.
Let us consider the linear symplectic mapping $S_\mu : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$S_\mu(\mathbf{x},\mathbf{p}) := \left(\frac{1}{\sqrt{\mu}}(x-\eta), \frac{1}{\sqrt{\mu}}(\xi - y), \frac{\sqrt{\mu}}{2}(\xi + y), -\frac{\sqrt{\mu}}{2}(\eta + x)\right), \ \mathbf{x} = (x,y), \ \mathbf{p} = (\xi,\eta). \tag{2.3.9}$$

The Weyl symbols of the operators $\Pi_1(\mu)$, $\Pi_2(\mu)$, and $\mathbb{H}_0(\mu)$ are transformed under S_μ into the Weyl symbols of the operators $\sqrt{\mu}\ \hat{\Pi}_1$, $\sqrt{\mu}\ \hat{\Pi}_2$, and $\mu\hat{\mathbb{H}}_0$, respectively, while $\mathcal{W}_\mu$ defined by (2.3.5) is a unitary operator generated by S_μ according to Lemma 2.5.
Since the operators $\mathbb{H}_0(\mu)$ and $\mu\hat{\mathbb{H}}_0$ are unitarily equivalent (see (2.3.7)), they have identical spectral properties. On the other hand,

$$\hat{\mathbb{H}}_0 = \int_{\mathbb{R}} \oplus h dy \tag{2.3.10}$$

where $h := -\frac{d^2}{dx^2} + x^2$ is the standard harmonic oscillator acting in $L^2(\mathbb{R}_x)$. Let us recall its spectral properties. Introduce the creation and annihilation operators $\alpha := -i\frac{d}{dx} + ix$, $\alpha^* = -i\frac{d}{dx} - ix$. We have $h = \alpha\alpha^* + 1$, $[\alpha^*, \alpha] = 2$. Moreover, Ker $\alpha^* = \left\{c\tilde{f}_1, c \in \mathbb{C}\right\}$ where $\tilde{f}_1(x) := e^{-x^2/2}$, $x \in \mathbb{R}$. Hence, $\sigma(h)$ is discrete, it consists of simple eigenvalues $2q-1$, $q \in \mathbb{N}_* := \{1,2,\ldots\}$, and the corresponding eigenfunctions can be written as $\tilde{f}_q := \alpha^{q-1}\tilde{f}_1$, $q \in \mathbb{N}_*$. Evidently, the system $\{\tilde{f}_q\}_{q=1}^\infty$ is an orthogonal basis in $L^2(\mathbb{R})$. Introduce the orthonormalized basis

$$f_q := \tilde{f}_q/\|\tilde{f}_q\| = \frac{H_{q-1}(x)e^{-x^2/2}}{(\sqrt{\pi}2^{q-1}(q-1)!)^{1/2}}, \quad x \in \mathbb{R}, \quad q \in \mathbb{N}_*, \tag{2.3.11}$$

where $H_s(x) := (-1)^s e^{x^2} \frac{d^s}{dx^s} e^{-x^2}$, $x \in \mathbb{R}$, $s \in \mathbb{N} := \{0,1,2,\ldots\}$ is the Hermite polynomial of order s (see e.g. [Mes, vol.1, Appendix B, Section III]). Summarizing, we get

$$u(x) = \sum_{q=1}^\infty f_q(x) \int_{\mathbb{R}} u(x') f_q(x') dx', \ u \in L^2(\mathbb{R}),$$

$$(hu)(x) = \sum_{q=1}^{\infty}(2q-1)f_q(x)\int_{\mathbb{R}} u(x')f_q(x')dx', \; u \in \mathrm{Dom}(h).$$

By (2.3.10) we find that $\hat{\mathbb{H}}_0$ has purely point spectrum, its eigenvalues are equal to $2q-1$, $q \in \mathbb{N}_*$, each of them being of infinite multiplicity. The corresponding eigenfunctions have the form $v_q(x,y) = f_q(x)w(y)$, $q \in \mathbb{N}_*$, with arbitrary non-zero function $w \in L^2(\mathbb{R}_y)$. In other words,

$$(\hat{\mathbb{H}}_0 u)(x,y) = \sum_{q=1}^{\infty}(2q-1)(\hat{p}_q u)(x,y), \quad u \in \mathrm{Dom}(\hat{\mathbb{H}}_0), \tag{2.3.12}$$

where

$$(\hat{p}_q u)(x,y) := f_q(x)\int_{\mathbb{R}} u(x',y)f_q(x')dx', \quad q \in \mathbb{N}_*, \quad u \in L^2(\mathbb{R}^2_{x,y}). \tag{2.3.13}$$

In view of the unitary equivalence of $\mu\hat{\mathbb{H}}_0$ and $\mathbb{H}_0(\mu)$ (see (2.3.7)), we get

$$\sigma(\mathbb{H}_0(\mu)) = \sigma_{\mathrm{ess}}(\mathbb{H}_0(\mu)) = \sigma_{\mathrm{pp}}(\mathbb{H}_0(\mu)) = \bigcup_{q=1}^{\infty}\{\mu(2q-1)\}, \tag{2.3.14}$$

and all the eigenvalues $\mu(2q-1)$, $q \in \mathbb{N}_*$, are of infinite multiplicity. Moreover, (2.3.12) implies

$$(\mathbb{H}_0(\mu)u)(x,y) = \mu\sum_{q=1}^{\infty}(2q-1)(p_q u)(x,y), \quad u \in \mathrm{Dom}(\mathbb{H}_0(\mu)), \tag{2.3.15}$$

where $p_q := \mathcal{W}_\mu \hat{p}_q \mathcal{W}_\mu^*$, $q \in \mathbb{N}_*$. Denote by $K_{q,\mu}$ the integral kernel of $\hat{p}_q$, $q \in \mathbb{N}_*$, i.e.

$$(p_q u)(x,y) = \int_{\mathbb{R}^2} K_{q,\mu}(x,y;x',y')u(x',y')dx'dy', \quad u \in L^2(\mathbb{R}^2).$$

Taking into account (2.3.5) and (2.3.13), we get

$$K_{q,\mu}(x,y;x',y') = \frac{\mu}{(2\pi)^2}\int_{\mathbb{R}^3} \exp\{i[\varphi_\mu(x,y;\xi_2,\eta) - \varphi_\mu(x',y';\xi_1,\eta)]\}f_q(\xi_1)f_q(\xi_2)d\xi_1 d\xi_2 d\eta =$$

$$\frac{\mu}{2\pi}\exp\{-\frac{\mu}{4}[(x-x')^2+(y-y')^2+2i(xy'-yx')]\}L_{q-1}\left(\frac{\mu}{2}\left((x-x')^2+(y-y')^2\right)\right), \; s \in \mathbb{N}_*, \tag{2.3.16}$$

where $L_s(\xi) := \frac{e^\xi}{s!}\frac{d^s}{d\xi^s}\left(\xi^s e^{-\xi}\right)$, $\xi \in \mathbb{R}$, $s \in \mathbb{N}$, is the Laguerre polynomial of order s.
There exists yet another interpretation of the orthogonal projection p_1. Since $p_1 u = u$ is equivalent to $\mathbb{H}_0(\mu)u = \mu u$, (2.3.4) implies that $u \in \mathrm{Ran}\; p_1$ is equivalent to $a(\mu)a(\mu)^* u = 0$, or to $a(\mu)^* u = 0$. Therefore, p_1 projects onto $\mathrm{Ker}\; a(\mu)^*$. Introduce the complex variable $\zeta = x+iy$. Then $a(\mu)^* u = 0$ if and only if $\partial u/\partial\bar{\zeta} + \mu\zeta u/4 = 0$. The general solution of the last equation is $u = ve^{-\mu|\zeta|^2/4}$, where v is an arbitrary holomorphic function, i.e. $\partial v/\partial\bar{\zeta} = 0$. Hence, the system $\{u_s\}_{s=0}^{\infty}$ with $u_s(x,y) := (x+iy)^s e^{-\mu(x^2+y^2)/4}$, $s \in \mathbb{N}$, forms a basis of $\mathrm{Ran}\; p_1 = \mathrm{Ker}\, a(\mu)^*$. Moreover, the basis $\{u_s\}_{s=0}^{\infty}$ is orthogonal, and the basis $\{w_s\}_{s=0}^{\infty}$ with $w_s := \sqrt{\frac{\mu}{2\pi s!}}\left(\frac{\mu}{2}\right)^{s/2} u_s$, $s \in \mathbb{N}$, is orthonormalized. Therefore, the integral kernel $K_{1,\mu}$ of $p_1(\mu)$ can be written as

$$K_{1,\mu}(x,y;x',y') = \sum_{s=0}^{\infty} w_s(x,y)\overline{w_s(x',y')} = \frac{\mu}{2\pi}\exp\{-\frac{\mu}{4}[(x-x')^2+(y-y')^2+2i(xy'-yx')]\}$$

which coincides with (2.3.16) for $q = 1$.
Let $\Omega \subseteq \mathbb{R}^{2d}$, $d \geq 1$, be an open non-empty set. Assume that $\mathbf{w} \in C(\Omega)$ is a strictly positive function

on Ω. Let $L^2(\Omega;\mathbf{w})$ be the standard weighted space over Ω with weight $\mathbf{w}$. Denote by $\mathcal{H}(\Omega;\mathbf{w})$ be the weighted holomorphic space

$$\mathcal{H}(\Omega;\mathbf{w}) := \left\{f \in L^2(\Omega;\mathbf{w}) | \partial f/\partial\bar{\zeta}_j = 0,\ \zeta_j = x_j + iy_j,\ j = 1,\ldots,d\right\} \tag{2.3.17}$$

(see [Ha, Definition 2.1]). Therefore, Ker $a(\mu)^*$ coincides with the weighted holomorphic space $\mathcal{H}\left(\mathbb{R}^2; e^{-\mu(x^2+y^2)/2}\right)$ called, up to minor variations, the Segal-Bargmann space (see [Ha, Section 3.2]), or the Fock space (see [B.Sol 2]).

Let us go back now to the general case $k = 0$ and $d \geq 1$ where $\mathbb{H}_0(\mu)$ is a sum of d pairwise commuting two-dimensional operators (see (2.3.1)). Hence, in this case (2.3.14) should be replaced by

$$\sigma(\mathbb{H}_0(\mu)) = \bigcup_{s_1=1}^{\infty} \ldots \bigcup_{s_d=1}^{\infty} \left\{\mu\left((2s_1-1)b_1 + \ldots + (2s_d-1)b_d\right)\right\}. \tag{2.3.18}$$

Let us re-write (2.3.18) using the strictly increasing sequence of the *Landau levels* $\{\Lambda_q\}$, $q \geq 1$:

$$\begin{cases} \Lambda_1 := b_1 + \ldots + b_d = \frac{1}{2}\mathrm{Tr}\,\sqrt{B^*B}, \\ \Lambda_q := \inf\left\{\lambda \in \mathbb{R} | \lambda > \Lambda_{q-1}, \lambda = \sum_{j=1}^{d}(2s_j-1)b_j,\ (s_1,\ldots,s_d) \in \mathbb{N}_*^d\right\},\ q > 1. \end{cases} \tag{2.3.19}$$

Lemma 2.6 *Assume that B is constant, and $B \neq 0$. Let $k = 0$. Then we have*

$$\sigma(\mathbb{H}_0(\mu)) = \sigma_{\rm ess}(\mathbb{H}_0(\mu)) = \sigma_{\rm pp}(\mathbb{H}_0(\mu)) = \cup_{q=1}^{\infty}\{\mu\Lambda_q\}. \tag{2.3.20}$$

For further references, we introduce here the orthogonal projections $P_q = P_q(\mu)$, $q \geq 1$, onto the qth Landau level. In other words, $P_q u = u$ implies $u \in$ Dom $(\mathbb{H}_0(\mu))$, and $\mathbb{H}_0(\mu)u = \mu\Lambda_q u$. Let $\mathbf{K}_{q,\mu}$ be the integral kernel of the operator P_q, $q \geq 1$. For $X = (X_1,\ldots,X_d) = (x_1,y_1,\ldots,x_d,y_d) \in \mathbb{R}^{2d}$, $X' = (X_1',\ldots,X_d') = (x_1',y_1',\ldots,x_d',y_d') \in \mathbb{R}^{2d}$, we have

$$\mathbf{K}_{q,\mu}(X;X') = \sqrt{\det\ B} \times \sum_{\substack{(s_1,\ldots,s_d) \in \mathbb{N}_*^d : \\ (2s_1-1)b_1 + \ldots (2s_d-1)b_d = \Lambda_q}} \prod_{j=1}^{d} K_{s_j,\mu}(b_j^{1/2}x_j, b_j^{1/2}y_j; b_j^{1/2}x_j', b_j^{1/2}y_j')$$

(see (2.3.16)). In particular,

$$\mathbf{K}_{q,\mu}(X;X) = \mu^d \mathcal{C}_q(B),\ X \in \mathbb{R}^{2d}, \tag{2.3.21}$$

where

$$\mathcal{C}_q(B) := (2\pi)^{-d}\kappa_q\sqrt{\det\ B}, \tag{2.3.22}$$

and

$$\kappa_q := \#\left\{\mathbf{s} = (s_1,\ldots,s_d) \in \mathbb{N}_*^d | \sum_{j=1}^{d}(2s_j-1)b_j = \Lambda_q\right\},\ q \geq 1, \tag{2.3.23}$$

is the *multiplicity* of the Landau level Λ_q.

2.3.3. Next, we describe the spectra of the two-dimensional Pauli and Dirac operators in constant magnetic field. At first we consider Pauli operator $\mathbb{P}_0(\mu)$. Without any loss of generality we assume $B_{1,2} > 0$ so that we have $b \equiv b_1 = B_{1,2}$. Hence, (2.2.15) reads

$$\mathbb{P}_0(\mu) = \begin{pmatrix} \mathbb{H}_0(\mu) - \mu b & 0 \\ 0 & \mathbb{H}_0(\mu) + \mu b \end{pmatrix}. \tag{2.3.24}$$

Therefore, (2.3.14) implies $\sigma(\mathbb{P}_0(\mu)) = \sigma(\mathbb{H}_0(\mu) - \mu b)\bigcup\sigma(\mathbb{H}_0(\mu) + \mu b) = \bigcup_{q=0}^{\infty}\{2\mu b q\}$. We have established the following

Lemma 2.7 *Let $m = 2$. Assume that B is constant, and $B \neq 0$. Then we have*

$$\sigma(\mathbb{P}_0(\mu)) = \sigma_{\rm ess}(\mathbb{P}_0(\mu)) = \sigma_{\rm pp}(\mathbb{P}_0(\mu)) = \bigcup_{q=0}^{\infty} \{2\mu b q\}\,. \tag{2.3.25}$$

In order to describe the spectrum of the two-dimensional Dirac operator with constant magnetic field, let us introduce the Dirac-Landau levels

$$\Lambda_q^-(\mu) := \begin{cases} -\sqrt{-2q\mu B_{1,2}+1} & q \in \mathbb{N} \quad \text{if} \quad \mu B_{1,2} < 0, \\ -\sqrt{2q\mu B_{1,2}+1} & q \in \mathbb{N}_* := \mathbb{N} \setminus \{0\} \quad \text{if} \quad \mu B_{1,2} > 0, \end{cases} \tag{2.3.26}$$

$$\Lambda_q^+(\mu) := \begin{cases} \sqrt{-2q\mu B_{1,2}+1} & q \in \mathbb{N}_* \quad \text{if} \quad \mu B_{1,2} < 0, \\ \sqrt{2q\mu B_{1,2}+1} & q \in \mathbb{N} \quad \text{if} \quad \mu B_{1,2} > 0. \end{cases} \tag{2.3.27}$$

Lemma 2.8 *Let $m = 2$. Assume that B is constant, and $B \neq 0$. Then we have*

$$\sigma(\mathbb{D}_0(\mu)) = \sigma_{\rm ess}(\mathbb{D}_0(\mu)) = \sigma_{\rm pp}(\mathbb{D}_0(\mu)) = \left\{\cup_q \left\{\Lambda_q^-(\mu)\right\}\right\} \bigcup \left\{\cup_q \left\{\Lambda_q^+(\mu)\right\}\right\}. \tag{2.3.28}$$

Sketch of the proof. By the antiunitary equivalence of $\mathbb{D}_0(\mu)$ and $-\mathbb{D}_0(-\mu)$, it suffices to consider the case $\mu > 0$ and $0 < B_{1,2} = b \equiv b_1$. In this case the operator $\mathbb{D}_0(\mu)$ is unitary equivalent under the abstract Foldy-Wouthuysen transformation to the operator

$$\begin{pmatrix} \sqrt{a(\mu)a(\mu)^*+1} & 0 \\ 0 & -\sqrt{a(\mu)^*a(\mu)+1} \end{pmatrix} = \begin{pmatrix} \sqrt{\mathbb{H}_0(\mu)-\mu b+1} & 0 \\ 0 & -\sqrt{\mathbb{H}_0(\mu)+\mu b+1} \end{pmatrix}$$

(see [Th, Theorem 5.13]). Therefore, $\sigma(\mathbb{D}_0(\mu)) = \sigma(\sqrt{\mathbb{H}_0(\mu)-\mu b+1}) \bigcup \sigma(-\sqrt{\mathbb{H}_0(\mu)+\mu b+1})$ which combined with (2.3.14) implies (2.3.28). $\diamondsuit$

2.3.4. Further, we discuss $\sigma(\mathbb{H}_0(\mu))$ in the case where the constant magnetic field B has a non-trivial kernel, i.e. $k = \dim \operatorname{Ker} B \geq 1$. In this case we have $\mathbb{H}_0(\mu) = \mathbb{H}_{0,1}(\mu) + \mathbb{H}_{0,2}(\mu)$ where

$$\mathbb{H}_{0,1}(\mu) := \sum_{j=1}^{d} \left\{ \left(-i\frac{\partial}{\partial x_j} + \mu\frac{b_j y_j}{2}\right)^2 + \left(-i\frac{\partial}{\partial y_j} - \mu\frac{b_j x_j}{2}\right)^2 \right\},\ \mathbb{H}_{0,2}(\mu) := -\sum_{l=1}^{k} \frac{\partial^2}{\partial z_l^2}$$

(see (2.3.2)). Moreover, the operators $\mathbb{H}_{0,1}(\mu)$ and $\mathbb{H}_{0,2}(\mu)$ commute. Hence it is easy to see that

$$\sigma(\mathbb{H}_0(\mu)) = \sigma(\mathbb{H}_{0,1}(\mu)) + \sigma(\mathbb{H}_{0,2}(\mu)) :=$$

$$\{\lambda \in \mathbb{R} | \lambda = \lambda_1 + \lambda_2,\ \lambda_1 \in \sigma(\mathbb{H}_{0,1}(\mu)),\ \lambda_2 \in \sigma(\mathbb{H}_{0,2}(\mu))\}\,. \tag{2.3.29}$$

Lemma 2.6 implies that

$$\sigma(\mathbb{H}_{0,1}(\mu)) = \bigcup_{q=1}^{\infty} \{\mu\Lambda_q\}. \tag{2.3.30}$$

The operator $\mathbb{H}_{0,2}(\mu)$ is unitarily equivalent under the partial Fourier transform with respect to the variable $z \in \mathbb{R}^k$, to the multiplier by $|\zeta|^2$, $\zeta \in \mathbb{R}^k$, acting in $L^2(\mathbb{R}^3_{x,y,\zeta})$. Therefore,

$$\sigma(\mathbb{H}_{0,2}(\mu)) = \sigma_{\rm ac}(\mathbb{H}_{0,2}(\mu)) = [0, \infty). \tag{2.3.31}$$

Combining (2.3.29), (2.3.30), and (2.3.31), we get

Lemma 2.9 *Assume that B is constant, and $B \neq 0$. Let $k \geq 1$. Then we have*

$$\sigma(\mathbb{H}_0(\mu)) = \sigma_{\rm ess}(\mathbb{H}_0(\mu)) = \sigma_{\rm ac}(\mathbb{H}_0(\mu)) = [\mu\Lambda_1, +\infty). \tag{2.3.32}$$

2.3.5. Finally, we describe the spectra of the three-dimensional Pauli and Dirac operators in constant magnetic fields. If $m = 3$, and $\mathbf{B} = (0,0,b)$, $b > 0$, (2.3.24) holds again (see (2.2.16)), and we have

$$\sigma(\mathbb{P}_0(\mu)) = \sigma_{\rm ess}(\mathbb{P}_0(\mu)) = \sigma(\mathbb{P}_0(\mu) - \mu b) \bigcup \sigma(\mathbb{H}_0(\mu) + \mu b) = [0, +\infty). \tag{2.3.33}$$

The operator $\mathbb{D}_0(\mu)$ is unitarily equivalent under the abstract Foldy-Wouthuysen transformation to the operator $\begin{pmatrix} \sqrt{\mathbb{P}_0(\mu)+1} & 0 \\ 0 & -\sqrt{\mathbb{P}_0(\mu)+1} \end{pmatrix}$ (see [Th, Theorem 5.13]). Therefore, we have $\sigma(\mathbb{D}_0(\mu))$ $= \sigma\left(\sqrt{\mathbb{P}_0(\mu)+1}\right) \bigcup \sigma\left(-\sqrt{\mathbb{P}_0(\mu)+1}\right)$ which combined with (2.3.33) entails

$$\sigma(\mathbb{D}_0(\mu)) = \sigma_{\rm ess}(\mathbb{D}_0(\mu)) = \sigma_{\rm ac}(\mathbb{D}_0(\mu)) = (-\infty, -1] \cup [1, +\infty). \tag{2.3.34}$$

2.4 Quantum Hamiltonians with non-constant magnetic fields

2.4.1 Let $m = 2$. We shall use the short-hand notation $b(X) = B_{1,2}(X) = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$, $X = (x,y) \in \mathbb{R}^2$, (cf. (2.1.9)).

Lemma 2.10 *Let $b \in L^2_{\rm loc}(\mathbb{R}^2)$. Then there exists a function $\varphi \in H^2_{\rm loc}(\mathbb{R}^2)$ where $H^2_{\rm loc}(\mathbb{R}^2)$ denotes the second local Sobolev space, such that $\Delta\varphi = b$ in the distributional sense.*

Remarks. (i) $H^2_{\rm loc}(\mathbb{R}^2)$ is embedded in $C(\mathbb{R}^2)$.
(ii) The function φ introduced in Lemma 2.10 is not unique.

By analogy with (2.3.3) introduce the creation and the annihilation operators $a(\mu)$ and $a(\mu)^*$. As in the case of constant magnetic field we have

$$\mathbb{P}_0(\mu) = \begin{pmatrix} \mathbb{P}_0^-(\mu) & 0 \\ 0 & \mathbb{P}_0^+(\mu) \end{pmatrix} := \begin{pmatrix} \mathbb{H}_0(\mu) - \mu b & 0 \\ 0 & \mathbb{H}_0(\mu) + \mu b \end{pmatrix} = \begin{pmatrix} a(\mu)a(\mu)^* & 0 \\ 0 & a(\mu)^*a(\mu) \end{pmatrix} \tag{2.4.1}$$

(cf. (2.3.4)). Hence, Ker $(\mathbb{P}_0(\mu)) = \{\mathbf{u} = (u_1, u_2) \in {\rm Dom}(\mathbb{P}_0(\mu)) | u_1 \in {\rm Ker}\ a(\mu)^*,\ u_2 \in {\rm Ker}\ a(\mu)\}$.

Lemma 2.11 *Under the assumptions of Lemma 2.10 we have*

$$\dim {\rm Ker}\ a(\mu) = \dim \left\{f \in L^2(\mathbb{R}^2) | f = ge^{\mu\varphi}, \partial g/\partial\zeta = 0\right\}, \tag{2.4.2}$$

$$\dim {\rm Ker}\ a(\mu)^* = \dim \left\{f \in L^2(\mathbb{R}^2) | f = ge^{-\mu\varphi}, \partial g/\partial\bar{\zeta} = 0\right\}, \tag{2.4.3}$$

where φ is any function introduced in Lemma 2.10, and $\zeta := x + iy$. Moreover, the dimensions at the right-hand side of (2.4.2)–(2.4.3) are independent of the choice of φ.

Proof: At first we prove the independence of the dimensions appearing at the right-hand sides of (2.4.2) and (2.4.3) of the choice of φ. Let $\varphi_j = \bar{\varphi}_j$, $j = 1,2$, satisfy $\Delta\varphi_j = b$. Set $\varphi_0 = \varphi_1 - \varphi_2$. Then the Green formula and $\Delta\phi_0 = 0$ imply that the function $\psi_0(X) := \int_\gamma \left(-\frac{\partial\varphi_0}{\partial y}dx + \frac{\partial\varphi_0}{\partial x}dy\right)$ where γ is an arbitrary piecewise smooth path connecting a fixed arbitrary point $X_0 \in \mathbb{R}^2$ with $X \in \mathbb{R}^2$, is well-defined and, in particular, independent of the choice of γ. Moreover, we have

$$\frac{\partial\varphi_0}{\partial x} = \frac{\partial\psi_0}{\partial y}, \quad \frac{\partial\varphi_0}{\partial y} = -\frac{\partial\psi_0}{\partial x}.$$

Hence, the function $\varphi_0 + i\psi_0$ is holomorphic with respect to to ζ. Assume that $f \in L^2(\mathbb{R}^2)$, $f = ge^{\mu\varphi_1}$, $\partial g/\partial\bar{\zeta} = 0$. Set $\tilde{f} := e^{-i\mu\psi_0}f$, $\tilde{g} := e^{-\mu(\varphi_0 + i\psi_0)}g$. Evidently, $\tilde{f} \in L^2(\mathbb{R}^2)$, $\tilde{f} = \tilde{g}e^{\mu\varphi_2}$, and $\partial\tilde{g}/\partial\bar{\zeta} = 0$. Therefore, the sets $\left\{f \in L^2(\mathbb{R}^2) | f = ge^{-\mu\varphi_1}, \partial g/\partial\bar{\zeta} = 0\right\}$ and $\left\{\tilde{f} \in L^2(\mathbb{R}^2) | f = ge^{-\mu\varphi_2}, \partial g/\partial\bar{\zeta} = 0\right\}$

are isomorphic under the mapping $f \mapsto \tilde{f} = e^{-i\mu\psi_0} f$, and hence their dimensions coincide. Analogously, the sets $\left\{f \in L^2(\mathbb{R}^2) | f = ge^{\mu\varphi_1}, \partial g/\partial\zeta = 0\right\}$ and $\left\{\tilde{f} \in L^2(\mathbb{R}^2) | f = ge^{\mu\varphi_2}, \partial g/\partial\zeta = 0\right\}$ are isomorphic under the same mapping and their dimensions coincide.
Now fix the function φ introduced in Lemma 2.10, and choose $\mathbf{A} = (A_1, A_2) = (-\partial\varphi/\partial y, \partial\varphi/\partial x)$. Up to gauge unitary equivalence we have

$$a(\mu) = -2i\left(\frac{\partial}{\partial\zeta} - \mu\frac{\partial\varphi}{\partial\zeta}\right) = -2ie^{\mu\varphi}\frac{\partial}{\partial\zeta}e^{-\mu\varphi}, \quad a(\mu)^* = -2i\left(\frac{\partial}{\partial\overline{\zeta}} + \mu\frac{\partial\varphi}{\partial\overline{\zeta}}\right) = -2ie^{-\mu\varphi}\frac{\partial}{\partial\overline{\zeta}}e^{\mu\varphi}.$$

Hence, in the particular gauge chosen we have

$$\text{Ker } a(\mu) = \left\{f \in L^2(\mathbb{R}^2) | f = ge^{\mu\varphi}, \frac{\partial g}{\partial\zeta} = 0\right\}, \text{ Ker } a(\mu)^* = \left\{f \in L^2(\mathbb{R}^2) | f = ge^{-\mu\varphi}, \frac{\partial g}{\partial\overline{\zeta}} = 0\right\},$$

which entails (2.4.2) and (2.4.3). $\diamondsuit$

Remarks: (i) The assumptions of Lemma 2.11 do not exclude the cases Ker $a(\mu) = \{0\}$ and/or Ker $a(\mu)^* = \{0\}$.
(ii) Up to gauge equivalence, Ker $a(\mu) = \mathcal{H}(\mathbb{R}^2; e^{2\mu\varphi})$, Ker $a(\mu)^* = \mathcal{H}(\mathbb{R}^2; e^{-2\mu\varphi})$ (see (2.3.17)).

The following result is an important example of the application of Lemma 2.11.

Proposition 2.1 [Ah.C] *Assume that* $0 \leq b(X) \leq C(1 + |X|)^{-2-\varepsilon}$, $X \in \mathbb{R}^2$, $C > 0$, $\varepsilon > 0$. *Then*

$$\dim \text{Ker } a(\mu) = 0, \ \dim \text{Ker } a(\mu)^* = \left[\frac{\mu}{2\pi}\int_{\mathbb{R}^2} b(X)dX\right],$$

where $[t]$ *denotes the largest integer* strictly smaller *than* t *if* $0 < t \in \mathbb{R}$, *and* $[0] = 0$.

Proof: We can choose $\varphi = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|X - X'|b(X')dX'$. Then we have $\varphi(X) = \frac{\ln|X|}{2\pi}\int_{\mathbb{R}^2} b(X')dX' + \mathcal{O}(|X|^{-\varepsilon_1})$ as $|X| \to \infty$ with $\varepsilon_1 = \min\{1, \varepsilon\}$. Therefore, by (2.4.2), dim Ker $a(\mu) = 0$. Similarly, if $\frac{\mu}{2\pi}\int_{\mathbb{R}^2} b(X)dX \leq 1$, then Ker $a(\mu)^* = \{0\}$. However, if $\frac{\mu}{2\pi}\int_{\mathbb{R}^2} b(X)dX > 1$, the linearly independent functions $(x + iy)^l e^{-\mu\varphi(x,y)}$ with $l = 0, \ldots, \left[\frac{\mu}{2\pi}\int_{\mathbb{R}^2} b(X)dX\right] - 1$, form a basis of Ker $a(\mu)^*$. $\diamondsuit$

Lemma 2.12 [Sh, Lemma 3.4] *Assume that* $b \in C^1(\mathbb{R}^2)$, *and* $|X|^2 b(X) \to +\infty$ *as* $|X| \to \infty$. *Then* dim Ker $a(\mu)^* = \infty$.

Remark: The dimension of dim Ker $a(\mu)^*$ can be infinite, even if the hypotheses of Lemma 2.12 do not hold. Assume, for example, that b is a periodic function. Denote by Γ its lattice of periods. Set $\mathbb{T} := \mathbb{R}^2/\Gamma$. If $\frac{\mu}{2\pi}\int_{\mathbb{T}} b(X)dX$ is a positive integer, then dim Ker $a(\mu)^* = \infty$ (see [Du.No] and [Sob 3, Theorem 7.1]), although $|X|^2 b(X)$ may not tend to infinity as $|X| \to \infty$. On the other hand, if $\int_{\mathbb{T}} b(X)dX = 0$, then $\sigma(\mathbb{P}_0(\mu))$ is purely absolutely continuous (see [B.Sus]), and, in particular, Ker $\mathbb{P}_0(\mu) = \{0\}$.

Lemma 2.13 *Assume that there exist constants* $c_j > 0$, $j = 1, 2, 3$, *such that*

$$c_1 \leq b(X) \leq c_2, \ |\nabla b(X)| \leq c_3, \ X \in \mathbb{R}^2. \tag{2.4.4}$$

Then

$$0 \in \sigma_{\text{ess}}(\mathbb{P}_0(\mu)), \tag{2.4.5}$$

$$\sigma(\mathbb{P}_0(\mu)) \setminus \{0\} \subseteq [2\mu c_1, \infty). \tag{2.4.6}$$

Proof: By Lemma 2.12, dim Ker $a(\mu)^* = \infty$. Hence, $0 \in \sigma_{\rm ess}(a(\mu)a(\mu)^*) \subseteq \sigma_{\rm ess}(\mathbb{P}_0(\mu))$ which entails (2.4.5). On the other hand, $a(\mu)^*a(\mu) = a(\mu)a(\mu)^* + 2\mu b$, $a(\mu)a(\mu)^* \geq 0$, and $b \geq c_1 > 0$ imply

$$\sigma(a(\mu)^*a(\mu)) \subseteq [2\mu c_1, \infty). \qquad (2.4.7)$$

An elementary supersymmetric argument yields

$$\sigma(a(\mu)^*a(\mu)) = \sigma(a(\mu)a(\mu)^*) \setminus \{0\} = \sigma(\mathbb{P}_0(\mu)) \setminus \{0\}. \qquad (2.4.8)$$

Combining (2.4.7) and (2.4.8), we get (2.4.6). $\diamondsuit$

Let $P(\mu) = P_{\rm Pauli}(\mu)$ be the spectral projection onto Ker $a(\mu)^* =$ Ker $\mathbb{P}_0^-(\mu)$. Denote by $K_\mu = K_{\mu,{\rm Pauli}}$ the integral kernel of $P(\mu)$, i.e.

$$(P(\mu)u)(X) = \int_{\mathbb{R}^2} K_\mu(X,X')u(X')dX'. \qquad (2.4.9)$$

The function $K_\mu(X,X)$, $X \in \mathbb{R}^2$ is continuous since $K_\mu(X,X')e^{2\mu\varphi(X')}$ is the reproducing kernel of the weighted holomorphic space $\mathcal{H}(\mathbb{R}^2; e^{-2\mu\varphi})$ (see [BdM.Sj]) and [Ha, Section 1, Theorem 3]).

Proposition 2.2 [E, Main Theorem] *Let $W \in C_0^\infty(\mathbb{R}^2)$. Assume that (2.4.4) holds. Then we have*

$$\lim_{\mu\to\infty} \mu^{-1} \int_{\mathbb{R}^2} W(X)K_{\mu,{\rm Pauli}}(X,X)dX = \frac{1}{2\pi}\int_{\mathbb{R}^2} W(X)b(X)dX. \qquad (2.4.10)$$

In the case where $\mathbb{R}^2$ is replaced by a compact manifold, an analogue of (2.4.10) can be deduced from the general theory of Toeplitz operators developed in [BdM.Gu]. In the case where $\mathbb{R}^2$ is replaced by a bounded pseudoconvex domain, an analogue of (2.4.10) is obtained in [Eng].

2.4.2. Let $m = 3$. Assume that the magnetic filed $\mathbf{b}$ has a constant direction. i.e.

$$\mathbf{B} = (0, 0, b). \qquad (2.4.11)$$

Since div $\mathbf{B} = 0$, we have $\frac{\partial b}{\partial z} = 0$, i.e. $b = b(X)$. Performing, if necessary, a gauge transformation, we can find a magnetic potential $\mathbf{A}$ such that curl $\mathbf{A} = \mathbf{B}$, and $\mathbf{A} = (A_1, A_2, 0)$ with $A_j = A_j(X)$, $j = 1, 2$; for example, we can choose $A_1 = -\partial\varphi/\partial y$ and $A_2 = \partial\varphi/\partial x$ where φ satisfies the assertion of Lemma 2.10.

Lemma 2.14 *Let $m = 3$. Assume that (2.4.11) is valid, and b satisfies (2.4.4). Then*

$$\sigma(\mathbb{P}_0(\mu)) = [0, \infty). \qquad (2.4.12)$$

Proof: Evidently, $\mathbb{P}_0(\mu) \geq 0$, and hence

$$\sigma(\mathbb{P}_0(\mu)) \subseteq [0, \infty). \qquad (2.4.13)$$

On the other hand, $\mathbb{P}_0(\mu) = \begin{pmatrix} \tilde{\mathbb{P}}_0^-(\mu) - \frac{\partial^2}{\partial z^2} & 0 \\ 0 & \tilde{\mathbb{P}}_0^+(\mu) - \frac{\partial^2}{\partial z^2} \end{pmatrix}$ where $\tilde{\mathbb{P}}_0^\pm(\mu) := \int_{\mathbb{R}} \oplus\, \mathbb{P}_0^\pm dz$. Hence, $\sigma(\mathbb{P}_0(\mu)) = \sigma\left(\tilde{\mathbb{P}}_0^-(\mu) - \frac{\partial^2}{\partial z^2}\right) \bigcup \sigma\left(\tilde{\mathbb{P}}_0^+(\mu) - \frac{\partial^2}{\partial z^2}\right)$. The operators $\tilde{\mathbb{P}}_0^\pm(\mu)$ and $-\frac{\partial^2}{\partial z^2}$ commute. By Lemma 2.13, $0 = \inf \sigma(\tilde{\mathbb{P}}_0^-(\mu))$. Moreover, $\sigma\left(-\frac{\partial^2}{\partial z^2}\right) = [0, \infty)$. Therefore,

$$\sigma\left(\tilde{\mathbb{P}}_0^-(\mu) - \frac{\partial^2}{\partial z^2}\right) = [0, \infty) \subseteq \sigma(\mathbb{P}_0(\mu)). \qquad (2.4.14)$$

Now (2.4.13) and (2.4.14) entail (2.4.12). $\diamondsuit$

Remark: Analyzing the proof of Lemma 2.4.12, we find that if $\mathbf{B}$ has a constant direction, then Ker $\mathbb{P}_0^-(\mu) = \{0\}$. In the case of non-constant direction of $\mathbf{B}$ this is not always true; counter-examples in this respect can be found in [E.So 3].

3 Spectral asymptotics for quantum Hamiltonians in strong magnetic fields: rough theorems

3.1 Formulations of main results

In this section we shall study the asymptotic behaviour as $\mu \to \infty$ of the discrete spectrum of the Hamiltonians introduced in Subsection 1.2, i.e. the Schrödinger, the Pauli, and the Dirac operator, with electric potentials V which decay at infinity.
Let T be a self-adjoint operator in a Hilbert space. Denote by $\mathcal{P}_I(T)$ the spectral projection of T related to the interval $I \subset \mathbb{R}$. Set

$$\mathcal{N}(I; T) := \text{rank}\ \mathcal{P}_I(T) \equiv \text{Tr}\ \mathcal{P}_I(T).$$

If the spectrum of the operator T situated on I is discrete, i.e. if $I \cap \sigma_{\text{ess}}(T) = \emptyset$, then $\mathcal{N}(I; T)$ is equal to the number of the eigenvalues of T lying on I, and counted with the multiplicities. If $\lambda_1, \lambda_2 \in \mathbb{R}$, and $\lambda_1 < \lambda_2$, we shall write $\mathcal{N}(\lambda_1, \lambda_2; T)$ instead of $\mathcal{N}((\lambda_1, \lambda_2); T)$. Similarly, if $\lambda \in \mathbb{R}$, we shall write $N(\lambda; T)$ instead of $\mathcal{N}((-\infty, \lambda); T)$.
Let T be a linear compact operator which is not necessarily self-adjoint. For $s > 0$ set

$$n_*(s; T) := \text{rank}\ \mathcal{P}_{(s^2, \infty)}(T^*T). \tag{3.1.1}$$

If, moreover, $T = T^*$, put

$$n_\pm(s; T) := \text{rank}\ \mathcal{P}_{(s,\infty)}(\pm T), \quad s > 0. \tag{3.1.2}$$

We shall choose intervals $I = I(\mu)$ which satisfy $I(\mu) \cap \sigma_{\text{ess}}(\mathbb{H}(\mu)) = \emptyset$ for μ large enough, and shall investigate the asymptotic behaviour as $\mu \to \infty$ of $\mathcal{N}(I(\mu); \mathbb{H}(\mu))$. The phrase "rough theorems" appearing in the title of the section, means that we shall derive asymptotic formulae containing only the main term and no remainder estimate; on the other hand, we shall consider maximally general perturbations V. On the contrary, in the next section we shall aim at asymptotic formulae involving a sharp remainder estimates, or even at expansions containing several asymptotic terms, imposing more restrictive assumptions on V.

3.1.1. Let us begin with the Schrödinger operator in constant magnetic fields. In this case we shall consider arbitrary dimensions $m \geq 2$. We refer the reader to (2.3.20) and (2.3.32) in order to recall the structure of the spectrum of $\mathbb{H}_0(\mu)$, and, in particular, the fact that this structure is essentially different for $k = \dim \text{Ker}\ B = 0$, and for $k \geq 1$.
First, we state our results concerning the case $k = 0$. Fix the Landau level Λ_q, $q \geq 1$ (see (2.3.19)). Choose the real numbers λ_1 and λ_2 such that $\lambda_1 < \lambda_2$, and $\lambda_1\lambda_2 > 0$. Since (2.3.20) with $k = 0$ implies $(\mu\Lambda_q + \lambda_1, \mu\Lambda_q + \lambda_2) \cap \sigma(\mathbb{H}_0(\mu)) = \emptyset$ for sufficiently large μ, we shall study the asymptotic behaviour as $\mu \to \infty$ of $\mathcal{N}(\mu\Lambda_q + \lambda_1, \mu\Lambda_q + \lambda_2; \mathbb{H}(\mu))$.
Assume now that $V \in \mathcal{L}_r$, $r \geq 1$ (see the definition before Lemma 2.1). For $\lambda \neq 0$ set

$$\text{k}_{\text{Schr}}(\lambda) = \text{k}_{\text{Schr}}(\lambda; V) := \begin{cases} \frac{\sqrt{\det B}}{(2\pi)^d} \text{vol}\ \{\mathbf{x} \in \mathbb{R}^m | V(\mathbf{x}) < \lambda\} & \text{if} \quad \lambda < 0, \\ -\frac{\sqrt{\det B}}{(2\pi)^d} \text{vol}\ \{\mathbf{x} \in \mathbb{R}^m | V(\mathbf{x}) > \lambda\} & \text{if} \quad \lambda > 0, \end{cases}$$

where vol $\mathcal{E}$ denotes the Lebesgue measure of the measurable set $\mathcal{E} \subset \mathbb{R}^m$. Evidently the function k_{Schr} is non-decreasing on $(-\infty, 0)$ and $(0, \infty)$.

Theorem 3.1 (cf. [R 2, Theorem 2.1]) *Assume that B is constant, and $k \equiv \dim \text{Ker}\ B = 0$. Let $V \in \mathcal{L}_r$ with $r = 2$ if $m = 2$, $r > 2$ if $m = 4$ and $r = d$ if $m \equiv 2d > 4$. Suppose that the real numbers*

λ_1 and λ_2 such that $\lambda_1 < \lambda_2$, and $\lambda_1\lambda_2 > 0$, are continuity points of $\mathrm{k_{Schr}}$. *Fix the Landau level Λ_q, $q \geq 1$. Then we have*

$$\lim_{\mu\to\infty} \mu^{-d}\mathcal{N}(\mu\Lambda_q + \lambda_1, \mu\Lambda_q + \lambda_2; \mathbb{H}(\mu)) = \kappa_q \left(\mathrm{k_{Schr}}(\lambda_2) - \mathrm{k_{Schr}}(\lambda_1)\right), \tag{3.1.3}$$

κ_q being defined in (2.3.23).

Notice that $\lambda \neq 0$ is a continuity point of $\mathrm{k_{Schr}}$ if and only if $\mathrm{vol}\,\{\mathbf{x} \in \mathbb{R}^m | V(\mathbf{x}) = \lambda\} = 0$. Hence, under the hypotheses of Theorem 3.1, $\kappa_q\left(\mathrm{k_{Schr}}(\lambda_2) - \mathrm{k_{Schr}}(\lambda_1)\right) = \mathcal{C}_q(B)\mathrm{vol}\ \{\mathbf{x} \in \mathbb{R}^m | \lambda_1 < V(\mathbf{x}) < \lambda_2\}$, the quantity $\mathcal{C}_q(B)$ being defined in (2.3.22).
Furthermore, if we consider the first Landau level Λ_1 and deal with the case $\lambda_1 < \lambda_2 < 0$, i.e. if we study the asymptotics of the discrete spectrum of $\mathbb{H}(\mu)$ *below* the bottom of its essential spectrum then we could replace the bounded interval (λ_1, λ_2) by $(-\infty, \lambda)$ with $\lambda < 0$, and could slightly relax the assumptions on V if $m = 2$. More precisely, the following assertion is true.

Theorem 3.2 *Assume that B is constant, and $k = 0$. Let $V \in \mathcal{L}_r$ with $r > 1$ if $m = 2$, and $r = d$ if $m \equiv 2d > 2$. Suppose that $\lambda < 0$ is a continuity point of* $\mathrm{k_{Schr}}$. *Then we have*

$$\lim_{\mu\to\infty} \mu^{-d}N(\mu\Lambda_1 + \lambda; \mathbb{H}(\mu)) = \mathrm{k_{Schr}}(\lambda). \tag{3.1.4}$$

Writing (3.1.4), we have taken into account that always $\kappa_1 = 1$.

Next, we pass to the formulation of our results concerning the Schrödinger operator $\mathbb{H}(\mu)$ in constant magnetic field B with non-trivial kernel, i.e. $k \geq 1$. If $\mathbf{x} \in \mathbb{R}^m$, we shall write $\mathbf{x} = (X_\perp, z)$, where $X_\perp \in \mathbb{R}^{2d} = \mathrm{Ran}\ B$, and $z \in \mathbb{R}^k = \mathrm{Ker}\ B$. Notice that $k \geq 1$ implies $m \geq 3$, so that the inclusion $V \in \mathcal{L}_{m/2}$ guarantees the self-adjointness of the operator $\mathbb{H}(\mu) = \mathbb{H}_0(\mu) + V$ defined as a sum in the sense of the quadratic forms.
Assume $V \in \mathcal{L}_{m/2}$, fix $X_\perp \in \mathbb{R}^{2d}$, and write

$$\chi(X_\perp) = -\Delta_z + V(X_\perp, z), \tag{3.1.5}$$

where $\Delta_z = \sum_{l=1}^k \frac{\partial^2}{\partial z_l^2}$ is the Laplacian defined on the Sobolev space $\mathrm{H}^2(\mathbb{R}^k)$, self-adjoint in $L^2(\mathbb{R}^k)$.

Proposition 3.1 *Let $k \geq 1$, and $V \in \mathcal{L}_{m/2}$. Then for almost every $X_\perp \in \mathbb{R}^{2d}$ the operator $|V(X_\perp, .)|^{1/2}(-\Delta_z + 1)^{-1/2}$ is compact in $L^2(\mathbb{R}^k)$.*

The proof of the proposition can be found in Subsection 3.6.

Assume $V \in \mathcal{L}_{m/2}$, and set $\Omega_{\mathrm{Schr}} := \left\{X_\perp \in \mathbb{R}^{2d} | |V(X_\perp, .)|^{1/2}(-\Delta_z + 1)^{-1/2} \text{ is compact}\right\}$. If, eventually, $\mathbb{R}^{2d} \neq \Omega_{\mathrm{Schr}}$, for definiteness we set $\chi(X_\perp) := -\Delta_z$ if $X_\perp \in \mathbb{R}^{2d} \setminus \Omega_{\mathrm{Schr}}$.
Since for $X_\perp \in \Omega_{\mathrm{Schr}}$ the multiplier by $V(X_\perp, .)$ is $-\Delta_z$-form-compact, we obtain the following

Corollary 3.1 *Let $k \geq 1$, and $V \in \mathcal{L}_{m/2}$. Then for every $X_\perp \in \Omega_{\mathrm{Schr}}$ the operator $\chi(X_\perp)$ is well-defined by (3.1.5) as a self-adjoint sum in the sense of the quadratic forms. Moreover, for every $X_\perp \in \mathbb{R}^{2d}$ we have*

$$\sigma_{\mathrm{ess}}(\chi(X_\perp)) = \sigma_{\mathrm{ess}}(-\Delta_z) = [0, \infty). \tag{3.1.6}$$

Let $\lambda < 0$. Then (3.1.6) implies that for every $X_\perp \in \mathbb{R}^{2d}$ the quantity $N(\lambda; \chi(X_\perp))$ is well-defined and finite. If, eventually, $\Omega_{\mathrm{Schr}} \neq \mathbb{R}^{2d}$, we have $N(\lambda; \chi(X_\perp)) = 0$ for $X_\perp \in \mathbb{R}^{2d} \setminus \Omega_{\mathrm{Schr}}$, $\lambda < 0$.

Proposition 3.2 *Let $k \geq 1$, $V \in \mathcal{L}_{m/2}$, and $\lambda < 0$. Then the function is $N(\lambda; \chi(.))$ is measurable on $\mathbb{R}^{2d}$.*

The proof of this proposition is contained in Subsection 3.6.

Fix $X_\perp \in \Omega_{\rm Schr}$. Assume at first that the discrete spectrum of $\chi(X_\perp)$ is non-empty, and denote by $\beta_j(X_\perp)$, $j \geq 1$, its eigenvalues enumerated in a non-decreasing order, i.e. $\beta_1(X_\perp) < \beta_2(X_\perp) \leq \beta_3(X_\perp) \leq \ldots < 0$. In the case where $\chi(X_\perp)$ has exactly J negative eigenvalues, $0 \leq J < \infty$, we put $\beta_j(X_\perp) = 0$ for $j \geq J+1$. Thus the non-decreasing infinite sequence $\{\beta_j(X_\perp)\}_{j=1}^\infty$ is well-defined for every $X_\perp \in \Omega_{\rm Schr}$. If, eventually, $\Omega_{\rm Schr} \neq \mathbb{R}^{2d}$, we have $\beta_j(X_\perp) = 0$ for $X_\perp \in \mathbb{R}^{2d} \setminus \Omega_{\rm Schr}$, $j \geq 1$. For each $\lambda < 0$ we have $N(\lambda; \chi(.)) : \mathbb{R}^{2d} \to \mathbb{N}$, and

$$\left\{X_\perp \in \mathbb{R}^{2d} | N(\lambda; \chi(X_\perp)) = 0\right\} = \left\{X_\perp \in \mathbb{R}^{2d} | \beta_1(X_\perp) \geq \lambda\right\},$$

$$\left\{X_\perp \in \mathbb{R}^{2d} | N(\lambda; \chi(X_\perp)) = j\right\} = \left\{X_\perp \in \mathbb{R}^{2d} | \beta_j(X_\perp) < \lambda\right\} \bigcap \left\{X_\perp \in \mathbb{R}^{2d} | \beta_{j+1}(X_\perp) \geq \lambda\right\}, j \geq 1.$$

Therefore,

$$\left\{X_\perp \in \mathbb{R}^{2d} | \beta_j(X_\perp) < \lambda\right\} = \left\{X_\perp \in \mathbb{R}^{2d} | N(\lambda; \chi(X_\perp)) \geq j\right\} =$$

$$\bigcup_{l=j}^{\infty} \left\{X_\perp \in \mathbb{R}^{2d} | N(\lambda; \chi(X_\perp)) = l\right\}, \ \lambda < 0, \ j \geq 1.$$

Hence the measurability of $N(\lambda; \chi(X_\perp))$ for all $\lambda < 0$ is equivalent to the measurability of the functions β_j, $j \geq 1$. In other words, Proposition 3.2 implies

Corollary 3.2 *Let $k \geq 1$, $V \in \mathcal{L}_{m/2}$. Then the functions β_j, $j \geq 1$, are measurable on $\mathbb{R}^{2d}$.*

Introduce the magnetic integrated density of states

$$\mathcal{K}_{\rm Schr}(\lambda) := \frac{\sqrt{\det B_+}}{(2\pi)^d} \int_{\mathbb{R}^{2d}} N(\lambda; \chi(X_\perp))\, dX_\perp, \quad \lambda < 0,$$

where B_+ is the restriction of the matrix $(B^*B)^{1/2}$ on Ran B. Notice that we have

$$\mathcal{K}_{\rm Schr}(\lambda) = \frac{\sqrt{\det B_+}}{(2\pi)^d} \sum_{j=1}^\infty {\rm vol}\, \left\{X_\perp \in \mathbb{R}^{2d} | \beta_j(X_\perp) < \lambda\right\}, \quad \lambda < 0.$$

Proposition 3.3 *Let $k \geq 1$, $V \in \mathcal{L}_{m/2}$, and $\lambda < 0$. Then we have $\mathcal{K}_{\rm Schr}(\lambda) < \infty$.*

The proof of this proposition can be found also in Subsection 3.6.

Theorem 3.3 [R 2, Theorem 2.2] *Assume that B is constant, and $k =$ dim Ker $B \geq 1$. Let $V \in \mathcal{L}_{m/2}$. Suppose that $\lambda < 0$ is a continuity point of $\mathcal{K}_{\rm Schr}$. Then we have*

$$\lim_{\mu \to \infty} \mu^{-d} N(\mu\Lambda_1 + \lambda; \mathbb{H}(\mu)) = \mathcal{K}_{\rm Schr}(\lambda). \tag{3.1.7}$$

The assumption that λ be a continuity point of $\mathcal{K}_{\rm Schr}$ is equivalent to vol $\mathcal{E}_\lambda = 0$, where $\mathcal{E}_\lambda := \left\{X_\perp \in \mathbb{R}^{2d} | \text{dim Ker } (\chi(X_\perp) - \lambda) \geq 1\right\}$. Notice that the measurability of $\mathcal{E}_\lambda$ follows from Corollary 3.2 since we have $\mathcal{E}_\lambda = \cup_{j=1}^\infty \left\{X_\perp \in \mathbb{R}^{2d} | \beta_j(X_\perp) = \lambda\right\}$.

Let us comment the term *magnetic integrated density of states* chosen by analogy with the usual (spatial) integrated density of states (IDOS) (see [R 3, p.3]). Let us recall the definition and the basic property of this spatial IDOS for the operator $\tilde{\mathbb{H}} = -\Delta + W$ where W is a periodic function over $\mathbb{R}^m$. Denote by Γ (respectively, by Γ^*) the lattice (respectively, the dual lattice) of the periods of

W, and set $\mathbb{T} := \mathbb{R}^m/\Gamma$, $\mathbb{T}^* := \mathbb{R}^m/\Gamma^*$. Define the auxiliary operator $\tilde{\chi}(\xi) := (i\nabla - \xi)^2 + W$, $\xi \in \mathbb{T}^*$, on the Sobolev space $\mathrm{H}^2(\mathbb{T})$. Then the spatial IDOS for the operator $\mathbb{H}$ can be written as

$$\mathcal{D}(\lambda) := (2\pi)^{-m} \int_{\mathbb{T}^*} N(\lambda; \tilde{\chi}(\xi))\, d\xi, \quad \lambda \in \mathbb{R}. \tag{3.1.8}$$

Set $\mathbb{T}_R := \mathbb{R}^m/R\Gamma$ with an integer $R \geq 1$. Evidently, vol $\mathbb{T}_R = R^m$vol $\mathbb{T}$.
Define the operator $\tilde{\mathbb{H}}_R = -\Delta + W$ on the Sobolev space $\mathrm{H}^2(\mathbb{T}_R)$. Then we have

$$\lim_{R\to\infty} \frac{N(\lambda; \tilde{\mathbb{H}}_R)}{\mathrm{vol}\, \mathbb{T}_R} = \mathcal{D}(\lambda) \tag{3.1.9}$$

(see e.g. [Re.S, vol.4, Theorem XIII.101]). Obviously, (3.1.9) is equivalent to

$$\lim_{R\to\infty} R^{-m} N(\lambda; \tilde{\mathbb{H}}_R) = (2\pi)^{-m} \mathrm{vol}\, \mathbb{T} \int_{\mathbb{T}^*} N(\lambda; \tilde{\chi}(\xi))\, d\xi, \quad \lambda \in \mathbb{R}, \tag{3.1.10}$$

while (3.1.7) can be written as

$$\lim_{\mu\to\infty} \mu^{-d} N(\lambda; \mathbb{H}(\mu) - \mu\Lambda_1) = (2\pi)^{-d} \sqrt{\det B_+} \int_{\mathbb{R}^{2d}} N(\lambda; \chi(X_\perp))\, dX_\perp, \quad \lambda < 0. \tag{3.1.11}$$

The formal resemblance between (3.1.10) and (3.1.11) was at the origin of the choice of the term "magnetic IDOS" used perhaps for the first time in [R 2].

3.1.2. Our next goal is to formulate the analogues of Theorems 3.1 – 3.3 for the Pauli operator $\mathbb{P}(\mu)$ in strong non-constant magnetic fields.
Let at first $m = 2$. Assume that b satisfies (2.4.4), and recall the structure of the spectrum of $\mathbb{P}_0(\mu)$ in this case (see (2.4.5) - (2.4.6)). Assume that $V \in \mathcal{L}_r$, $r \geq 1$. For $\lambda \neq 0$ set

$$\mathrm{k}_{\mathrm{Pauli}}(\lambda) := \begin{cases} \frac{1}{2\pi} \int_{\mathbb{R}^2} \theta(\lambda - V(X)) b(X) dX & \text{if} \quad \lambda < 0, \\ -\frac{1}{2\pi} \int_{\mathbb{R}^2} \theta(V(X) - \lambda) b(X) dX & \text{if} \quad \lambda > 0, \end{cases}$$

where θ is the Heaviside function

$$\theta(t) := \begin{cases} 0 & \text{if} \quad t < 0, \\ 1 & \text{if} \quad t \geq 0. \end{cases}$$

Evidently, $\mathrm{k}_{\mathrm{Pauli}}$ is non-decreasing on $(-\infty, 0)$ and $(0, \infty)$.
Notice that $\lambda \neq 0$ is a continuity point of $\mathrm{k}_{\mathrm{Pauli}}$ if and only if it is a continuity point of $\mathrm{k}_{\mathrm{Schr}}$, and recall that the last condition is equivalent to vol $\{X \in \mathbb{R}^2 | V(X) = \lambda\} = 0$.

Theorem 3.4 [R 5, Theorem 2.2] *Let $m = 2$. Assume that b satisfies (2.4.4). Let $V \in \mathcal{L}_2$. Suppose that the real numbers λ_1 and λ_2 such that $0 < \lambda_1 < \lambda_2$, are continuity points of* $\mathrm{k}_{\mathrm{Pauli}}$. *Then we have*

$$\lim_{\mu\to\infty} \mu^{-1} \mathcal{N}(\lambda_1, \lambda_2; \mathbb{P}(\mu)) = \mathrm{k}_{\mathrm{Pauli}}(\lambda_2) - \mathrm{k}_{\mathrm{Pauli}}(\lambda_1). \tag{3.1.12}$$

Theorem 3.5 [R 5, Theorem 2.1] *Let $m = 2$. Assume (2.4.4) is valid. Let $V \in \mathcal{L}_r$ with $r > 1$. Suppose that $\lambda < 0$ is a continuity point of* $\mathrm{k}_{\mathrm{Pauli}}$. *Then we have*

$$\lim_{\mu\to\infty} \mu^{-1} N(\lambda; \mathbb{P}(\mu)) = \mathrm{k}_{\mathrm{Pauli}}(\lambda). \tag{3.1.13}$$

Let now $m = 3$. Assume that the magnetic field has a constant direction, i.e. $\mathbf{B} = (0, 0, b(X_\perp))$, and b satisfies (2.4.4). Recall that in this case we have $\sigma(\mathbb{P}_0(\mu)) = [0, \infty)$ (see (2.4.12)).
Since $m = 3$ (i.e. $k = 1$), the family $\chi(X_\perp)$ (see (3.1.5)) consists of *ordinary* differential operators

$$\chi(X_\perp) = -\frac{d^2}{dz^2} + V(X_\perp, z). \tag{3.1.14}$$

Moreover, the magnetic field can be identified with the vector field $\mathbf{B} = \text{curl } A = (0,0,b)$ whose integral curves are straight lines. Hence, the operator $\chi(X_\perp)$ carries some information about the magnetic field: it is an ordinary differential operator with respect to the variable z along the magnetic field $\mathbf{B}$, and depends on the parameters $X_\perp$ on the plane perpendicular to $\mathbf{B}$.
Assume $V \in \mathcal{L}_{3/2}$. Then Proposition 3.1 implies that for almost every $X_\perp \in \mathbb{R}^2 = \text{Ran } B$ the operator $\chi(X_\perp)$ defined in (3.1.14) is self-adjoint, and for almost every $X_\perp \in \mathbb{R}^2$ we have $\sigma_{\text{ess}}(\chi(X_\perp)) = [0,\infty)$. Moreover, Proposition 3.2 combined with the continuity of b on $\mathbb{R}^2$ guarantees the measurability of $N(\lambda;\chi(.))b(.)$ on $\mathbb{R}^2$ for each $\lambda < 0$. For $\lambda < 0$ set

$$\mathcal{K}_{\text{Pauli}}(\lambda) := \frac{1}{2\pi}\int_{\mathbb{R}^2} N(\lambda;\chi(X_\perp))b(X_\perp)\,dX_\perp.$$

Proposition 3.3 combined with the estimate $b(X_\perp) \leq c_2$, $X_\perp \in \mathbb{R}^2$ (see (2.4.4)), implies $\mathcal{K}_{\text{Pauli}}(\lambda) < \infty$ if $V \in \mathcal{L}_{3/2}$ and $\lambda < 0$. Moreover, $\lambda < 0$ is a continuity point of $\mathcal{K}_{\text{Pauli}}$ if and only if $\text{vol}\left\{X_\perp \in \mathbb{R}^2 | \dim \text{Ker } (\chi(X_\perp) - \lambda) \geq 1\right\} = 0$.

Theorem 3.6 [R 5, Theorem 2.3] *Let $m = 3$. Suppose that $\mathbf{B} = (0,0,b(X_\perp))$, $X_\perp \in \mathbb{R}^2$, and b satisfies (2.4.4). Let $\lambda < 0$ be a continuity point of $\mathcal{K}_{\text{Pauli}}$. Then we have*

$$\lim_{\mu\to\infty} \mu^{-1} N(\lambda;\mathbb{P}(\mu)) = \mathcal{K}_{\text{Pauli}}(\lambda). \tag{3.1.15}$$

3.1.3. Finally, we shall formulate our results concerning the Dirac operator. We shall consider again only the cases $m = 2, 3$, and shall discuss just the case of a constant magnetic field.
Let at first $m = 2$. Recall the definition of the Dirac-Landau levels $\Lambda_q^\pm(\mu)$, and recall that $\sigma(\mathbb{D}_0(\mu)) = \left(\cup_q\{\Lambda_q^-(\mu)\}\right) \bigcup \left(\cup_q\{\Lambda_q^+(\mu)\}\right)$ (see (2.3.28)).

Theorem 3.7 *Assume that $m = 2$, B is constant, and $B \neq 0$. Let $V \in \mathcal{L}_r$, $r > 2$. Suppose that the real numbers λ_1 and λ_2 such that $\lambda_1 < \lambda_2$, and $\lambda_1\lambda_2 > 0$, are continuity points of the function k_{Schr}. Fix the Dirac-Landau Level $\Lambda_q^\pm(\mu)$. Then we have*

$$\lim_{\mu\to\infty} \mu^{-1}\mathcal{N}(\Lambda_q^\pm(\mu) + \lambda_1, \Lambda_q^\pm(\mu) + \lambda_2; \mathbb{P}(\mu)) = \text{k}_{\text{Schr}}(\lambda_2) - \text{k}_{\text{Schr}}(\lambda_1). \tag{3.1.16}$$

Let now $m = 3$. Recall that we have $\sigma_{\text{ess}}(\mathbb{D}_0(\mu)) = (-\infty,-1] \cup [1,\infty)$ (see (2.3.34)). Introduce the auxiliary family of ordinary differential operators

$$\partial(X_\perp) := \partial_0 + V(X_\perp, z)I_2, \tag{3.1.17}$$

with

$$\partial_0 := \begin{pmatrix} 1 & -i\frac{d}{dz} \\ -i\frac{d}{dz} & -1 \end{pmatrix},$$

acting in $L^2(\mathbb{R}_z)^2$, $\mathbb{R}_z = \text{Ker } B$, and depending on the parameters $X_\perp \in \mathbb{R}^2 = \text{Ran } B$.

Proposition 3.4 [R 4, Lemma 7.1] *Let $m = 3$, and $V \in \mathcal{L}_3$. For almost every $X_\perp \in \mathbb{R}^2$ the operator $V(X_\perp,.)\partial_0^{-1}$ is compact in $L^2(\mathbb{R}_z)^2$.*

Set $\Omega_{\text{Dirac}}\colon = \left\{X_\perp \in \mathbb{R}^2 | V(X_\perp,.)\partial_0^{-1} \text{ is compact}\right\}$. If eventually, $\Omega_{\text{Dirac}} \neq \mathbb{R}^2$, for definiteness set $\partial(X_\perp) = \partial_0$ if $X_\perp \in \mathbb{R}^2 \setminus \Omega_{\text{Dirac}}$.

Corollary 3.3 *Let $m = 3$, and $V \in \mathcal{L}_3$. Then for each $X_\perp \in \Omega_{\text{Dirac}}$ the operator $\partial(X_\perp)$ defined as an operator sum on the domain $\text{H}^1(\mathbb{R})^2$ of ∂_0, is self-adjoint in $L^2(\mathbb{R})^2$. Moreover, for each $X_\perp \in \mathbb{R}^2$*

$$\sigma_{\text{ess}}(\partial(X_\perp)) = \sigma_{\text{ess}}(\partial_0) = \sigma(\partial_0) = (-\infty,-1] \cup [1,\infty).$$

The proof of the corollary is quite similar to that of Corollary 3.1, and that is why we omit it.

Fix the real numbers λ_1 and λ_2 such that $-1 < \lambda_1 < \lambda_2 < 1$. Then Corollary 3.3 implies that for every $X_\perp \in \Omega_{\rm Dirac}$ the quantity $\mathcal{N}(\lambda_1, \lambda_2; \partial(X_\perp))$ is well-defined and finite. If, eventually, $\Omega_{\rm Dirac} \neq \mathbb{R}^2$, we have $\mathcal{N}(\lambda_1, \lambda_2; \partial(X_\perp)) = 0$ for $X_\perp \in \mathbb{R}^2 \setminus \Omega_{\rm Dirac}$.

Proposition 3.5 *Let $m = 3$, and $V \in \mathcal{L}_3$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $-1 < \lambda_1 < \lambda_2 < 1$. Then the function $\mathcal{N}(\lambda_1, \lambda_2; \partial(.))$ is measurable on $\mathbb{R}^2$.*

The proof of the proposition is contained in Subsection 2.6.

Corollary 3.4 *Let $V \in \mathcal{L}_3$, $\lambda \in (-1, 1)$. Then* dim Ker $(\partial(.) - \lambda)$ *is measurable on* $\mathbb{R}^2$.

For $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $-1 < \lambda_1 < \lambda_2 < 1$ set

$$\mathcal{K}_{\rm Dirac}(\lambda_1, \lambda_2) := \frac{b}{2\pi} \int_{\mathbb{R}^2} \mathcal{N}(\lambda_1, \lambda_2; \partial(X_\perp))\, dX_\perp,$$

where $b = b_1 \equiv \sqrt{\det B_+}$.

Proposition 3.6 [R 4, Corollary 7.2] *Under the hypotheses of Proposition 3.5* $\mathcal{K}_{\rm Dirac}(\lambda_1, \lambda_2) < \infty$.

Theorem 3.8 [R 4, Theorem 2.1] *Let $m = 3$. Assume that B is constant, and $B \neq 0$. Let $V \in \mathcal{L}_3$. Suppose that the real numbers λ_1 and λ_2 such that $-1 < \lambda_1 < \lambda_2 < 1$ satisfy*

$$\text{vol } \left\{X_\perp \in \mathbb{R}^2 | \text{dim Ker } (\partial(X_\perp) - \lambda_j) \geq 1\right\} = 0, \quad j = 1, 2.$$

Then we have

$$\lim_{\mu \to \infty} \mu^{-1} \mathcal{N}(\lambda_1, \lambda_2; \mathbb{D}(\mu)) = \mathcal{K}_{\rm Dirac}(\lambda_1, \lambda_2). \tag{3.1.18}$$

3.2 Comments

3.2.1. The exposition concerning the Schrödinger operator with *constant* magnetic fields follows closely [R 2]. The results concerning the Pauli operator with *non-constant* magnetic fields are borrowed from [R 5]. Theorem 3.8 dealing with three-dimensional Dirac operator was first proved in [R 4]. Formally, Theorem 3.7 related to the two-dimensional Dirac operator is new. However, combining the methods applied in the proof of the results concerning the two-dimensional Schrödinger operator with constant magnetic field (see Theorem 3.1 or [R 5, Theorem 2.1]) and the ones related to the three-dimensional Dirac operator (see [R 5, Theorem 2.1]), one can easily deduce the proof of Theorem 3.7.
In this subsection we give alternative formulations of some of the main results, and offer certain generalizations of them. Moreover, we compare these results with existing ones on uniform semiclassical spectral asymptotics of quantum Hamiltonians in strong magnetic fields. Subsection 3.3 contains a summary of variational methods which play an important role in the proofs of the results of this section. In Subsection 3.4 we give a detailed proof of Theorem 3.1 which is essentially different and much simpler than the original proof of [R 2, Theorem 2.1] whose assumptions are slightly modified. The new proof which we present here is based on the generalized Birman-Schwinger principle (see Lemma 3.7 below). Subsection 3.5 contains a sketch of the proof of Theorem 3.4. We follow mainly the original proof of [R 5, Theorem 2.2] but offer some minor improvements. Subsection 3.5 is devoted to the proofs of Propositions 3.1 – 3.3 and 3.5. While the results of Propositions 3.1 and 3.2 and their analogues for the Pauli operator with non-constant magnetic fields have been discussed in more or less detail in [R 2] and [R 5], the measurability of $N(\lambda; \chi(X_\perp))$ and $\mathcal{N}(\lambda_1, \lambda_2; \partial(X_\perp))$ treated in

Propositions 3.2 and 3.5 respectively, has not been considered in the original papers [R 2] and [R 4]. The reason of this seeming neglect is the fact that the measurability of the rank of spectral projections of quantum Hamiltonians H_ω depending on a parameter $\omega \in \Omega$, Ω being a measure space, has been studied within the general abstract framework of the theory of measurable self-adjoint operators (see, e.g. [Car.Lac]), and could be considered as a standard fact. In order to avoid the impression that important technical details have been swept under the rug, we offer here an independent proof of the measurability of $N(\lambda;\chi(X_\perp))$ and $\mathcal{N}(\lambda_1,\lambda_2;\partial(X_\perp))$, based on the Birman-Schwinger principle, and simple facts from the theory of measurable functions (see [Ko.Fo, Chapter V]). We refer the reader to the original paper [R 4] for the proof of Propositions 3.4 and 3.6. Finally, Subsection 3.7 contains a brief summary of the proof of Theorem 3.6, following quite closely the exposition of the original paper [R 5].

3.2.2. Let us comment the assumptions of Theorems 3.1 – 3.8 that the fixed energy levels λ_1 and λ_2, or λ be continuity points of the limiting functions appearing at the right-hand sides of (3.1.3), (3.1.4), (3.1.7), (3.1.12), (3.1.13), (3.1.16), and (3.1.18). This assumption prompts that all these asymptotic formulae should be interpreted as limiting relations of measures. In order to discuss thus aspect in more detail, let us recall some basic facts from the theory of measure.
Let g_μ, $\mu > 0$, and g be non-decreasing real-valued functions such that $\mathrm{Dom}(g_\mu) \subseteq \mathbb{R}$, $\mathrm{Dom}(g) \subseteq \mathbb{R}$. Assume that the open interval $I \subseteq \mathbb{R}$, independent of μ, is contained in $\mathrm{Dom}(g)$, and $\mathrm{Dom}(g_\mu)$ for μ large enough. We shall say that the function g is the *vague limit* as $\mu \to \infty$ of the family g_μ on I if $\lim_{\mu\to\infty} g_\mu(\lambda) = g(\lambda)$ for $\lambda \in I$ which are continuity points of g.
Evidently, Theorems 3.1 – 3.8 can all be re-formulated in the terms of vague limits.

Lemma 3.1 *The function g is a vague limit as $\mu \to \infty$ of the family g_μ on the open interval $I \subseteq \mathbb{R}$ if and only if* $\lim_{\mu\to\infty} \int_I \phi(\lambda)dg_\mu(\lambda) = \int_I \phi(\lambda)dg(\lambda)$ *for each* $\phi \in C_0^\infty(I)$.

As an illustration of Lemma 3.1, we shall give an alternative formulation of Theorems 3.5 and 3.6.

Theorem 3.9 *Fix* $\phi \in C_0^\infty(-\infty,0)$.
i) Under the hypotheses of Theorem 3.5 we have

$$\lim_{\mu\to\infty} \mu^{-1}\int_{-\infty}^0 \phi(\lambda)dN(\lambda;\mathbb{P}(\mu)) = \int_{-\infty}^0 \phi(\lambda)d\mathrm{k}_{\mathrm{Pauli}}(\lambda),$$

or, equivalently,

$$\lim_{\mu\to\infty} \mu^{-1}\mathrm{Tr}\ \phi(\mathbb{P}(\mu)) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \phi(V(X))\, b(X)dX. \tag{3.2.1}$$

ii) Under the hypotheses of Theorem 3.6 we have

$$\lim_{\mu\to\infty} \mu^{-1}\int_{-\infty}^0 \phi(\lambda)dN(\lambda;\mathbb{P}(\mu)) = \int_{-\infty}^0 \phi(\lambda)d\mathcal{K}_{\mathrm{Pauli}}(\lambda),$$

or, equivalently,

$$\lim_{\mu\to\infty} \mu^{-1}\mathrm{Tr}\ \phi(\mathbb{P}(\mu)) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}\ \phi(\chi(X_\perp))\, b(X_\perp)dX_\perp. \tag{3.2.2}$$

Limiting relations (3.2.1) – (3.2.2) admit some extensions to the case where the support of the functions ϕ is not compact. In this case, however, we need some *a priori* estimates of $\mathrm{Tr}\ \phi(\mathbb{P}(\mu))$. We shall not discuss this aspect of the theory in full generality, but will concentrate to the particular case where $\phi(\lambda) = \lambda_-^\gamma$, $\gamma > 0$; the quantity $\mathrm{Tr}\ \mathbb{P}(\mu)_-^\gamma$ is called the Riesz means of order $\gamma > 0$ for the negative eigenvalues of the Pauli operator . Although this case might seem rather too special, it is quite important from the point of view of the applications to the problems concerning the stability of matter. The necessary *a priori* estimates are provided by the following lemmas due to A.V.Sobolev.

Lemma 3.2 *Let $m = 2$. Assume that the scalar magnetic field b satisfies (2.4.4). Let $\gamma \geq 1$. Suppose that $V \in L^{\gamma+1}(\mathbb{R}^2) \cap L^\gamma(\mathbb{R}^2)$. Then we have*

$$\text{Tr}\, \mathbb{P}(\mu)_-^\gamma \leq c_4 \int_{\mathbb{R}^2} V(X)_-^{\gamma+1}\, dX + c_4' \mu \int_{\mathbb{R}^2} V(X)_-^\gamma\, b(X) dX, \tag{3.2.3}$$

where c_4 and c_4' are independent of μ, b, and V.

Lemma 3.3 *Let $m = 3$. Assume that $\mathbf{B} = (0, 0, b)$ and $b = b(X_\perp)$ satisfies (2.4.4). Let $\gamma > 1/2$. Suppose that $V \in L^{\gamma+3/2}(\mathbb{R}^3) \cap L^{\gamma+1/2}(\mathbb{R}^3)$. Then we have*

$$\text{Tr}\, \mathbb{P}(\mu)_-^\gamma \leq c_5 \int_{\mathbb{R}} \int_{\mathbb{R}^2} V(X_\perp, z)_-^{\gamma+3/2}\, dX_\perp dz + c_5' \mu \int_{\mathbb{R}} \int_{\mathbb{R}^2} V(X_\perp, z)_-^{\gamma+1/2}\, b(X_\perp) dX_\perp dz, \tag{3.2.4}$$

where c_5 and c_5' are independent of μ, b, and V.

Estimates (3.2.3) – (3.2.4) have been established in [Sob 1] – [Sob 2] (see also [Sob 3, Proposition 2.3]), under less restrictive assumptions; we reproduce them here in a form convenient for our purposes.

Corollary 3.5 *Under the hypotheses of Lemma 3.2 we have*

$$\lim_{\mu \to \infty} \mu^{-1} \text{Tr}\, \mathbb{P}(\mu)_-^\gamma = \frac{1}{2\pi} \int_{\mathbb{R}^2} V(X)_-^\gamma\, b(X) dX. \tag{3.2.5}$$

Corollary 3.6 *Under the hypotheses of Lemma 3.3 we have*

$$\lim_{\mu \to \infty} \mu^{-1} \text{Tr}\, \mathbb{P}(\mu)_-^\gamma = \frac{1}{2\pi} \int_{\mathbb{R}^2} \text{Tr}\, \chi(X_\perp)_-^\gamma\, b(X_\perp) dX_\perp. \tag{3.2.6}$$

Suppose that the hypotheses of Lemma 3.3 are fulfilled. Then for almost every $X_\perp \in \mathbb{R}^2$, the *Lieb-Thirring estimate*

$$\text{Tr}\, \chi(X_\perp)_-^\gamma \leq c_6 \int_{\mathbb{R}} V(X_\perp, z)_-^{\gamma+1/2} dz \tag{3.2.7}$$

holds with c_6 which is independent of V. Therefore, (3.2.6) agrees with (3.2.4).

Proof of Corollaries 3.5–3.6. Under the hypotheses of any of these corollaries we have

$$\text{Tr}\, \mathbb{P}(\mu)_-^\gamma = \int_{-\infty}^0 (-\lambda)^\gamma dN(\lambda; \mathbb{P}(\mu)) = \gamma \int_{-\infty}^0 (-\lambda)^{\gamma-1} N(\lambda; \mathbb{P}(\mu)) d\lambda. \tag{3.2.8}$$

Notice that the assumptions of Corollary 3.5 imply $V \in \mathcal{L}_p$ with $p > 1$, while those of Corollary 3.6 entail $V \in \mathcal{L}_{3/2}$. Hence, (3.1.13) or (3.1.15) hold for almost every $\lambda \in (-\infty, 0)$.
Therefore, the dominated convergence theorem guarantees that (3.1.13), (3.2.3), and (3.2.8), yield

$$\lim_{\mu \to \infty} \mu^{-1} \text{Tr}\, \mathbb{P}(\mu)_-^\gamma = \gamma \lim_{\mu \to \infty} \mu^{-1} \int_{-\infty}^0 (-\lambda)^{\gamma-1} N(\lambda; \mathbb{P}(\mu)) d\lambda =$$

$$\gamma \int_{-\infty}^0 (-\lambda)^{\gamma-1} \mathrm{k}_{\text{Pauli}}(\lambda)\, d\lambda = \int_{-\infty}^0 \lambda_-^\gamma\, d\mathrm{k}_{\text{Pauli}}(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}^2} V(X)_-^\gamma\, b(X) dX,$$

which is equivalent to (3.2.5). Similarly, (3.1.15), (3.2.4), and (3.2.8), imply

$$\lim_{\mu \to \infty} \mu^{-1} \text{Tr}\, \mathbb{P}(\mu)_-^\gamma = \gamma \lim_{\mu \to \infty} \mu^{-1} \int_{-\infty}^0 (-\lambda)^{\gamma-1} N(\lambda; \mathbb{P}(\mu)) d\lambda =$$

$$\gamma \int_{-\infty}^{0} (-\lambda)^{\gamma-1} \mathcal{K}_{\rm Pauli}(\lambda)\, d\lambda = \int_{-\infty}^{0} \lambda_-^{\gamma}\, d\mathcal{K}_{\rm Pauli}(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}^2} {\rm Tr}\ \chi(X_\perp)_-^{\gamma}\, b(X_\perp) dX_\perp,$$

which is equivalent to (3.2.6). $\diamondsuit$

3.2.3. Let us discuss now the correlation between Corollaries 3.5 – 3.6, and the *semiclassical* asymptotics of the Riesz means of the negative eigenvalues of the Pauli operator in dimensions two and three. Set $\mathbb{P}(\hbar, \mu) := \mathbb{P}(\hbar, \mu, 1)$, $\hbar > 0$, $\mu > 0$. Obviously, for an arbitrary $\hbar > 0$ estimates (3.2.3)–(3.2.4) can be re-written as

$$\mathrm{Tr}\ \mathbb{P}(\hbar, \mu)_-^{\gamma} \le c_4 \hbar^{-2} \int_{\mathbb{R}^2} V(X)_-^{\gamma+1}\, dX + c_4' \mu \hbar^{-1} \int_{\mathbb{R}^2} V(X)_-^{\gamma}\, b(X) dX,\ m = 2,$$

$$\mathrm{Tr}\ \mathbb{P}(\hbar, \mu)_-^{\gamma} \le c_5 \hbar^{-3} \int_{\mathbb{R}} \int_{\mathbb{R}^2} V(X_\perp, z)_-^{\gamma+3/2}\, dX_\perp dz +$$
$$c_5' \mu \hbar^{-2} \int_{\mathbb{R}} \int_{\mathbb{R}^2} V(X_\perp, z)_-^{\gamma+1/2}\, b(X_\perp) dX_\perp dz,\ m = 3.$$

For $\lambda < 0$ put

$$\mathcal{B}(\lambda) \equiv \mathcal{B}_0(\lambda; \hbar, \mu) :=$$
$$\begin{cases} \frac{\mu\hbar}{2\pi} \sum_{q=0}^{\infty} \varrho_q \int_{\mathbb{R}^2} \theta(\lambda - V(X) - 2q\mu\hbar b(X)) b(X) dX \text{ if } m = 2, \\ \frac{\mu\hbar}{(2\pi)^2} \sum_{q=0}^{\infty} \varrho_q \int_{T^*\mathbb{R}} \int_{\mathbb{R}^2} \theta(\lambda - |\zeta|^2 - V(X_\perp, z) - 2q\mu\hbar b(X_\perp)) b(X_\perp) dX_\perp dz\, d\zeta \text{ if } m = 3, \end{cases}$$

where $\varrho_0 = 1$, and $\varrho_q = 2$ if $q \ge 1$. Set $\mathcal{B}_\gamma(\hbar, \mu) := \int_{-\infty}^{0} \lambda_-^{\gamma}\, d\mathcal{B}(\lambda)$, $\gamma > 0$, $m = 2, 3$. The quantities $\mathcal{B}_\gamma(\hbar, \mu)$ are the moments of order γ of the measure $\mathcal{B}$ defined on $(-\infty, 0)$. Evidently,

$$\mathcal{B}_\gamma(\hbar, \mu) =$$
$$\begin{cases} \frac{\mu\hbar}{2\pi} \sum_{q=0}^{\infty} \varrho_q \int_{\mathbb{R}^2} (V(X) + 2q\mu\hbar b(X))_-^{\gamma} b(X) dX \text{ if } m = 2, \\ \frac{\mu\hbar}{(2\pi)^2} \sum_{q=0}^{\infty} \varrho_q \int_{\mathbb{R}^2} \int_{T^*\mathbb{R}} \left(|\zeta|^2 + V(X_\perp, z) + 2q\mu\hbar b(X_\perp)\right)_-^{\gamma} b(X_\perp)\, dz\, d\zeta\, dX_\perp \text{ if } m = 3, \end{cases} \quad \gamma > 0. \tag{3.2.9}$$

Theorem 3.10 *Let $m = 2$ (respectively, $m = 3$). Under the assumptions of Lemma 3.2 (respectively, Lemma 3.3), we have*

$$\lim_{\hbar \downarrow 0} \frac{1}{\mu\hbar + 1} \left\{ \hbar^m\ \mathrm{Tr}\ \mathbb{P}(\hbar, \mu)_-^{\gamma} - \mathcal{B}_\gamma(\hbar, \mu) \right\} = 0 \tag{3.2.10}$$

uniformly with respect to $\mu > 0$.

Asymptotic formulae of the type of (3.2.10) appeared first in [L.So.Y 2] (see also [L.So.Y 1]) for the case of a constant magnetic field. Various versions of (3.2.10) concerning non-constant fields can be found in [E.So 2] (see also [E.So 1]), and later in [Sob 3]. Asymptotic formulae concerning the Schrödinger and the Dirac operators, which are similar to (3.2.10) but contain a sharp remainder estimates can be found in [Iv, Chapters 6–7]; these formulae have been obtained under numerous supplementary assumptions.

Our Theorem 3.10 is closest to [Sob 3, Theorem 2.2]. There the assumptions are less restrictive than ours, but the formulation is somewhat lengthy.

The principal merit of (3.2.10) is that it is uniform with respect to $\mu > 0$; in particular, it is allowed that $\mu \to \infty$ as $\hbar \downarrow 0$. Notice that if $\mu\hbar \le$ const, then (3.2.10) can be re-written as

$$\lim_{\hbar \downarrow 0} \left\{ \hbar^m\ \mathrm{Tr}\ \mathbb{P}(\hbar, \mu)_-^{\gamma} - \mathcal{B}(\hbar, \mu) \right\} = 0, \tag{3.2.11}$$

i.e. we can omit the factor $(\mu\hbar+1)^{-1}$ in front of the braces in (3.2.10). Moreover, if $\mu>0$ is a constant independent of $\hbar>0$, then $\mathcal{B}_\gamma(\hbar,\mu)$ (see (3.2.9)), can be interpreted as the Darboux sums for the Riemann integrals with respect to the variable t occurring in the following quantities:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_0^\infty (V(X)+t)_-^\gamma\, dt\, dX = \frac{1}{2\pi^2}\int_{T^*\mathbb{R}^2}\left(|\mathbf{p}|^2+V(\mathbf{x})\right)_-^\gamma\, d\mathbf{x}\, d\mathbf{p} \quad \text{if} \quad m=2,$$

and

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{T^*\mathbb{R}}\int_0^\infty (|\zeta|^2+V(X_\perp,z)+t)_-^\gamma\, dt\, dz\, d\zeta\, dX_\perp = \frac{1}{4\pi^3}\int_{T^*\mathbb{R}^3}\left(|\mathbf{p}|^2+V(\mathbf{x})\right)_-^\gamma\, d\mathbf{x}\, d\mathbf{p} \text{ if } m=3.$$

Therefore, in this case we have

$$\lim_{\hbar\downarrow 0}\mathcal{B}_\gamma(\hbar,\mu) = \frac{2}{(2\pi)^m}\int_{T^*\mathbb{R}^m}\left(|\mathbf{p}|^2+V(\mathbf{x})\right)_-^\gamma\, d\mathbf{x}\, d\mathbf{p}. \tag{3.2.12}$$

The standard Weyl coefficient $(2\pi)^{-m}\int_{T^*\mathbb{R}^m}\left(|\mathbf{p}|^2+V(\mathbf{x})\right)_-^\gamma\, d\mathbf{x}\, d\mathbf{p}$ appearing at the right-hand side of (3.2.12) is the moment of order γ related to the negative part of the Hamiltonian function $|\mathbf{p}|^2+V(\mathbf{x})$ of a m-dimensional classical particle. Notice that the Weyl coefficient is independent of μ and b. Moreover, the additional factor 2 occurring in (3.2.12) is due to the fact that the Pauli operator acts upon *two*-component functions). Combining (3.2.11) and (3.2.12), we get

$$\lim_{\hbar\downarrow 0}\hbar^m \text{Tr}\ \mathbb{P}(\hbar,\mu)_-^\gamma = \frac{2}{(2\pi)^m}\int_{T^*\mathbb{R}^m}\left(|\mathbf{p}|^2+V(\mathbf{x})\right)_-^\gamma\, d\mathbf{x}\, d\mathbf{p}. \tag{3.2.13}$$

Asymptotic relations of this type are well-known; see e.g. [R 1, Theorem 2.1] where the semiclassical asymptotics of the negative spectrum of the Schrödinger operator have been considered, but the passage to the asymptotics as $\hbar\downarrow 0$ of Tr $\mathbb{P}(\hbar,\mu)_-^\gamma$, $\mu>0$, $\gamma\geq 0$, is trivial. However, (3.2.12), and, hence, (3.2.13) are not uniform with respect to μ.
Let us compare the main asymptotic terms as $\mu\to\infty$ of the functionals $\mathcal{B}_\gamma(\hbar,\mu)$ appearing in (3.2.10), and the quantities at the right-hand sides of (3.2.5) and (3.2.6). Notice that if $\mu\to\infty$ only the term corresponding to $q=0$ in (3.2.9) survives, while the terms corresponding to $q\geq 1$ vanish. In other words, if $m=2$, we have

$$\lim_{\mu\to\infty}\mu^{-1}\,\mathcal{B}_\gamma(\hbar,\mu) = \frac{\hbar}{2\pi}\int_{\mathbb{R}^2} V(X)_-^\gamma\, b(X) dX, \quad \hbar>0, \tag{3.2.14}$$

and the right-hand side of (3.2.14) with $\hbar=1$ coincides with that of (3.2.5). In other words, in the two-dimensional case Tr $\mathbb{P}(\mu)_-^\gamma$ is asymptotically equivalent to $\mathcal{B}_\gamma(\hbar,\mu)$ as $\mu\to\infty$. In the three-dimensional case, however, we have

$$\lim_{\mu\to\infty}\mu^{-1}\,\mathcal{B}_\gamma(\hbar,\mu) = \frac{\hbar}{(2\pi)^3}\int_{\mathbb{R}^2}\int_{T^*\mathbb{R}}\left(|\zeta|^2+V(X_\perp,z)\right)_-^\gamma\, dz\, d\zeta\, b(X_\perp) dX_\perp, \quad \hbar>0. \tag{3.2.15}$$

Hence, the quantities at the right-hand side of (3.2.6), and (3.2.15) with $\hbar=1$, are of essentially different nature. The right-hand side of (3.2.6) is equal to the integral with respect to $X_\perp\in\mathbb{R}^2$ of the Riesz means of order γ of the bound states of one-dimensional *quantum* particle described by the Hamiltonian $\chi(X_\perp) = -\frac{d^2}{dz^2}+V(X_\perp,z)$, while the right-hand side of (3.2.15) with $\hbar=1$ coincides with the integral with respect to $X_\perp$ of the moment of order γ related to the negative part of the energy of a one-dimensional *classical* particle described by a Hamilton function $|\zeta|^2+V(X_\perp,z)$, $(z,\zeta)\in T^*\mathbb{R}$. Evidently, the Hamiltonian operator $-\frac{d^2}{dz^2}+V(X_\perp,z)$ is obtained from the Hamiltonian function $|\zeta|^2+V(X_\perp,z)$ by the standard quantization scheme (see part 1.2.1).

3.3 Auxiliary material

This subsection contains some useful tools of variational character which are used systematically in the proofs of the main results of the section. First, we formulate a suitable version of the minimax principle for compact operators.

Lemma 3.4 *Let $T = T^*$ be a compact operator in the Hilbert space $\mathbf{H}$. For $s > 0$ denote by $\mathcal{F}_s^{\pm}(T)$ the linear subsets of $\mathbf{H}$ whose nonzero elements u satisfy $\pm(Tu, u) > s\|u\|^2$, where $(.,.)$ and $\|.\|$ denote respectively the scalar product and the norm of $\mathbf{H}$. Then we have $n_{\pm}(s;T) = \sup\dim \mathcal{F}_s^{\pm}(T)$, the quantities $n_{\pm}(T)$ being defined in (3.1.2).*

The next lemma summarizes the well-known Weyl inequalities for the singular numbers and eigenvalues for compact operators.

Lemma 3.5 *Let T_j, $j = 1, 2$, be linear compact operators. Then for each $s > 0$ and $t \in (0, s)$*

$$n_*(s; T_1 + T_2) \le n_*(s - t; T_1) + n_*(t; T_2),$$

the quantity $n_*(s;T)$ being defined in (3.1.1). If, moreover, $T_j = T_j^*$, $j = 1, 2$, then for each $s > 0$ and $t \in (0, s)$

$$n_{\pm}(s; T_1 + T_2) \le n_{\pm}(s - t; T_1) + n_{\pm}(t; T_2), \quad n_{\pm}(s; T_1 + T_2) \ge n_{\pm}(s + t; T_1) - n_{\mp}(t; T_2).$$

In the sequel we shall denote by $\mathcal{S}_\infty$ the class of linear compact operators acting in a given Hilbert space, and by $\mathcal{S}_p$, $p \ge 1$, - the Schatten–von Neumann class of compact linear operators for which the norm $\|T\|_p := (\mathrm{Tr}\,|T|^p)^{1/p}$ is finite. Evidently, if $T \in \mathcal{S}_p$, $p \ge 1$, we have

$$n_*(s;T) \le s^{-p}\|T\|_p^p, \quad s > 0. \tag{3.3.1}$$

If $T = T^* \in S_p$, $p \ge 1$, then the estimates $n_{\pm}(s;T) \le n_*(s;t)$ combined with (3.3.1) imply

$$n_{\pm}(s;T) \le s^{-p}\|T\|_p^p, \quad s > 0. \tag{3.3.2}$$

The following two lemmas are devoted to the Birman-Schwinger principle, and a generalization of its.

Lemma 3.6 [B, Lemma 1.1] *Let $\mathcal{H}_0 \ge 0$ and $\mathcal{V}$ be two self-adjoint operators such that $|\mathcal{V}|^{1/2}(\mathcal{H}_0 + 1)^{-1/2} \in \mathcal{S}_\infty$. For $\lambda < 0$ set*

$$\mathcal{R}(\lambda; \mathcal{H}_0) := (\mathcal{H}_0 - \lambda)^{-1/2}, \tag{3.3.3}$$

$$\mathcal{T}(\lambda; \mathcal{H}_0, \mathcal{V}) := \mathcal{R}(\lambda; \mathcal{H}_0)\mathcal{V}\mathcal{R}(\lambda; \mathcal{H}_0). \tag{3.3.4}$$

Then we have $N(\lambda; \mathcal{H}_0 + \mathcal{V}) = n_-(1; \mathcal{T}(\lambda; \mathcal{H}_0, \mathcal{V}))$ where the sum $\mathcal{H}_0 + \mathcal{V}$ should be understood in the sense of the quadratic forms.

Lemma 3.7 [R 4, Lemma 4.1] *Let $\mathcal{H}_0$ be a self-adjoint operator, and let λ_1 and λ_2 be real numbers such that $\lambda_1 < \lambda_2$, and $[\lambda_1, \lambda_2] \subset \rho(\mathcal{H}_0) := \mathbb{C} \setminus \sigma(\mathcal{H}_0)$. Set*

$$\tilde{\mathcal{R}}(\lambda_1, \lambda_2; \mathcal{H}_0) := \left((\mathcal{H}_0 - \lambda_1)(\mathcal{H}_0 - \lambda_2)\right)^{-1/2}, \tag{3.3.5}$$

$$\mathcal{G}(\lambda_1, \lambda_2; \mathcal{H}_0) := \left(\mathcal{H}_0 - \frac{1}{2}(\lambda_1 + \lambda_2)\right)\tilde{\mathcal{R}}(\lambda_1, \lambda_2; \mathcal{H}_0). \tag{3.3.6}$$

Let $\mathcal{V}$ be a symmetric operator on $\mathrm{Dom}(\mathcal{H}_0)$ such that $\mathcal{V}(\mathcal{H}_0 + i)^{-1} \in \mathcal{S}_\infty$. Put

$$\tilde{\mathcal{T}}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V}) := \tilde{\mathcal{R}}(\lambda_1, \lambda_2; \mathcal{H}_0)\mathcal{V}^2\tilde{\mathcal{R}}(\lambda_1, \lambda_2; \mathcal{H}_0) + 2\mathrm{Re}\,\mathcal{G}(\lambda_1, \lambda_2; \mathcal{H}_0)\mathcal{V}\tilde{\mathcal{R}}(\lambda_1, \lambda_2; \mathcal{H}_0) \tag{3.3.7}$$

Then we have $\mathcal{N}(\lambda_1, \lambda_2; \mathcal{H}_0 + \mathcal{V}) = n_-(1; \tilde{\mathcal{T}}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V}))$ where the sum $\mathcal{H}_0 + \mathcal{V}$ should be understood in the operator sense.

Finally, we state a suitable version of the Kac-Murdock-Szegö theorem.

Lemma 3.8 [R 5, Lemma 3.2] *Let $\{T(\mu)\}_{\mu>0}$ be self-adjoint compact operators satisfying the estimate $\|T(\mu)\| \leq t_0$ with $t_0 > 0$ independent of μ. Assume that the function $\nu : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is non-decreasing on $(-\infty, 0)$ and $(0, \infty)$, non-negative on $(-\infty, 0)$, and non-positive on $(0, \infty)$. Let $\nu(t) = 0$ for $|t| > t_0$. Suppose that for some $p \geq 1$:*
(i) $T(\mu) \in S_p$ for each $\mu > 0$;
(ii) the quantity $\int_{\mathbb{R}\setminus\{0\}} |t|^p d\nu(t)$ is finite;
(iii) There exists a non-increasing function $\phi : (0,\infty) \to (0,\infty)$ such that $\lim_{\mu\to\infty} \phi(\mu) \mathrm{Tr}\, T(\mu)^l = \int_{\mathbb{R}\setminus\{0\}} t^l d\nu(t)$ for each integer $l \geq p$.
Let $t \neq 0$ be a continuity point of ν. Then we have

$$\lim_{\mu\to\infty} \phi(\mu) n_+(t; T(\mu)) = -\nu(t),\ t > 0, \quad \lim_{\mu\to\infty} \phi(\mu) n_+(-t; T(\mu)) = \nu(t),\ t < 0.$$

3.4 Proof of Theorem 3.1

Throughout the subsection we assume that $m = 2d$ with $d \geq 1$.

Lemma 3.9 *Let $W \in L^r(\mathbb{R}^m)$, $r \geq 2$, and $q \geq 1$. Then we have*

$$\|WP_q(\mu)\|_r^r \leq \mathcal{C}_q(B)\mu^d \int_{\mathbb{R}^m} |W|^r d\mathbf{x}, \tag{3.4.1}$$

$\mathcal{C}_q(B)$ being defined in (2.3.22).

Proof. Assume at first $W \in L^2(\mathbb{R}^m)$. Then we have

$$\|WP_q(\mu)\|_2^2 = \int_{\mathbb{R}^m} |W(\mathbf{x})|^2\, \mathbf{K}_{q,\mu}(\mathbf{x};\mathbf{x}) d\mathbf{x} = \mathcal{C}_q(B)\mu^d \int_{\mathbb{R}^m} |W|^2 d\mathbf{x} \tag{3.4.2}$$

(see (2.3.21)). Assume now $W \in L^\infty(\mathbb{R}^m)$. Evidently,

$$\|WP_q(\mu)\| \leq \|W\|_{L^\infty(\mathbb{R}^m)}. \tag{3.4.3}$$

Interpolating between (3.4.2) and (3.4.3), we get (3.4.1). ◇

Corollary 3.7 *Under the hypotheses of Lemma 3.9 we have*

$$n_*(\varepsilon; WP_q(\mu)) \leq \mathcal{C}_q(B)\varepsilon^{-r}\mu^d \int_{\mathbb{R}^m} |W|^r d\mathbf{x},\ \forall \varepsilon > 0. \tag{3.4.4}$$

Proof: It suffices to apply (3.3.1) and (3.4.1). ◇
Let λ_1 and λ_2 satisfy the hypotheses of Theorem 3.1. For $\mu > 0$ large enough set $r_q(\mu) = r_q(\mu; \lambda_1, \Lambda_2)$: $= \tilde{\mathcal{R}}(\lambda_1 + \mu\Lambda_q, \lambda_2 + \mu\Lambda_q; \mathbb{H}_0(\mu))$ (see (3.3.5)).

Corollary 3.8 *Under the hypotheses of Lemma 3.9 we have*

$$n_*(\varepsilon; Wr_q(\mu)P_q(\mu)) \leq \mathcal{C}_q(B) \left(\varepsilon\sqrt{\lambda_1\lambda_2}\right)^{-r} \mu^d \int_{\mathbb{R}^m} |W|^r d\mathbf{x},\ \forall \varepsilon > 0. \tag{3.4.5}$$

Proof: It suffices to notice that $n_*(\varepsilon; Wr_q(\mu)P_q(\mu)) = n_*(\varepsilon\sqrt{\lambda_1\lambda_2}; WP_q(\mu))$, and to apply (3.4.4). ◇

Corollary 3.9 *Under the hypotheses of Lemma 3.9 we have*

$$\limsup_{\mu\to\infty} \mu^{-d} n_*(\varepsilon; Wr_q(\mu)) \leq \mathcal{C}_q(B)(\varepsilon\sqrt{\lambda_1\lambda_2})^{-r} \int_{\mathbb{R}^m} |W|^r d\mathbf{x},\ \forall \varepsilon > 0. \tag{3.4.6}$$

Proof: Fix $\delta \in (0,1)$. By Lemma 3.5,

$$n_*(\varepsilon; Wr_q(\mu)) \leq n_*(\varepsilon(1-\delta); Wr_q(\mu)P_q(\mu)) + n_*(\varepsilon\delta; Wr_q(\mu)(I - P_q(\mu)). \tag{3.4.7}$$

Applying (3.4.4), we get

$$\limsup_{\mu\to\infty} \mu^{-d} n_*(\varepsilon; Wr_q(\mu)) \leq \mathcal{C}_q(B) \left(\varepsilon(1-\delta)\sqrt{\lambda_1\lambda_2}\right)^{-r} \int_{\mathbb{R}^m} |W|^r d\mathbf{x},\ \forall\varepsilon > 0. \tag{3.4.8}$$

On the other hand, $\|(\mathbb{H}_0(\mu)+\mu)r_q(\mu)(I-P_q(\mu))\| \leq C$ with $C > 0$ independent of $\mu > 0$. Therefore,

$$n_*(\varepsilon\delta; Wr_q(\mu)(I - P_q(\mu)) \leq n_*(\varepsilon\delta; CW(\mathbb{H}_0(\mu)+\mu)^{-1}). \tag{3.4.9}$$

The diamagnetic inequality (see Lemma 2.1) implies $\|W(\mathbb{H}_0(\mu)+\mu)^{-1}\| \leq \||W|(-\Delta+\mu)^{-1}\|$. Since the multiplier by W is Δ-compact, (see Lemma 2.2), we find that $\lim_{\mu\to\infty}\|W(-\Delta+\mu)^{-1}\| = 0$. Therefore, (3.4.9) implies

$$n_*(\varepsilon\delta; Wr_q(\mu)(I - P_q(\mu)) = 0 \tag{3.4.10}$$

provided that μ is large enough.
Combining (3.4.7), (3.4.8), and (3.4.10), and letting $\delta \downarrow 0$ in (3.4.8), we get (3.4.6). $\diamondsuit$

Lemma 3.10 *Let $W = \overline{W} \in C_0^\infty(\mathbb{R}^m)$. Fix $q \geq 1$. Then $P_q(\mu)WP_q(\mu) \in \mathcal{S}_1$, and we have*

$$\lim_{\mu\to\infty} \mu^{-d}\mathrm{Tr}\,(P_q(\mu)WP_q(\mu))^l = \mathcal{C}_q(B)\int_{\mathbb{R}^m} W(\mathbf{x})^l d\mathbf{x}, \quad l \in \mathbb{N}_*. \tag{3.4.11}$$

Proof: In the proof we follow the strategy of [K.M.Sz] (see also [Gr.Sz]). Let $l = 1$. Obviously,

$$\mathrm{Tr}\ P_q(\mu)WP_q(\mu) = \int_{\mathbb{R}^m} W(\mathbf{x})\mathbf{K}_{q,\mu}(\mathbf{x};\mathbf{x})d\mathbf{x} = \mathcal{C}_q(B)\mu^d \int_{\mathbb{R}^m} W(\mathbf{x})d\mathbf{x}$$

which implies (3.4.11) with $l = 1$. Let now $l \geq 2$. We have

$$\mathrm{Tr}\,(P_q(\mu)WP_q(\mu))^l = \mathrm{Tr}\,(WP_q(\mu))^l = \int_{\mathbb{R}^{ml}} W(\mathbf{x}_1)\mathbf{K}_{q,\mu}(\mathbf{x}_1;\mathbf{x}_2)\ldots W(\mathbf{x}_l)\mathbf{K}_{q,\mu}(\mathbf{x}_l;\mathbf{x}_1)d\mathbf{x}_1\ldots d\mathbf{x}_l.$$

Change the variables $\mathbf{x}_1 = \mathbf{x}_1'$, $\mathbf{x}_j = \mu^{-1/2}\mathbf{x}_j' + \mathbf{x}_1'$, $2,\ldots,l$. Then we have

$$\mathrm{Tr}\,(P_q(\mu)WP_q(\mu))^l = \mu^d \int_{\mathbb{R}^{ml}} W(\mathbf{x}_1')W(\mathbf{x}_1'+\mu^{-1/2}\mathbf{x}_2')\ldots W(\mathbf{x}_1'+\mu^{-1/2}\mathbf{x}_l')$$

$$\mathbf{K}_{q,1}(0;\mathbf{x}_2')\ldots\mathbf{K}_{q,1}(\mathbf{x}_l';0)d\mathbf{x}_1'\ldots d\mathbf{x}_l'.$$

Therefore,

$$\lim_{\mu\to\infty}\mu^{-d}\mathrm{Tr}\,(P_q(\mu)WP_q(\mu))^l = \int_{\mathbb{R}^m} W(\mathbf{x}_1)^l d\mathbf{x}_1 \int_{\mathbb{R}^{m(l-1)}} \mathbf{K}_{q,1}(0;\mathbf{x}_2)\ldots\mathbf{K}_{q,1}(\mathbf{x}_l;0)d\mathbf{x}_2\ldots d\mathbf{x}_l =$$

$$\mathbf{K}_{q,1}(0;0)\int_{\mathbb{R}^m} W(\mathbf{x})^l d\mathbf{x} = \mathcal{C}_q(B)\int_{\mathbb{R}^m} W(\mathbf{x})^l d\mathbf{x}$$

(see (2.3.21)), which is equivalent to (3.4.11) with $l \geq 2$. $\diamondsuit$
Let $W = \overline{W}$ be a measurable function. For $s > 0$ set

$$\tilde{\nu}_\pm(s) = \tilde{\nu}_\pm(s;W) := \mathcal{C}_q(B)\mathrm{vol}\,\{\mathbf{x}\in\mathbb{R}^m \mid \pm W(\mathbf{x}) > s\}.$$

Corollary 3.10 *Under the hypotheses of Lemma 3.10 we have*

$$\lim_{\mu\to\infty} \mu^{-d} n_\pm(s; P_q(\mu) W P_q(\mu)) = \tilde{\nu}_\pm(s; W),$$

provided that $s > 0$ *is a continuity point of* $\tilde{\nu}_\pm(.; W)$.

Proof: It suffices to apply Lemma 3.8 with $T(\mu) = P_q(\mu)WP_q(\mu)$, $t_0 = \|W\|_{L^\infty(\mathbb{R}^m)}$, $\phi(\mu) = \mu^{-d}$, and $\nu(s) = \begin{cases} \tilde{\nu}_-(-s; W) & \text{if} \quad s < 0, \\ -\tilde{\nu}_+(s; W) & \text{if} \quad s > 0, \end{cases}$ and take into account Lemma 3.10. ♢

Remark: Since $P_1(\mu)$ is the orthogonal projection onto (a version of) the Segal-Bargmann space, $P_1(\mu)WP_1(\mu)$ is a classical Toeplitz operator (see [BdM.Gu], [Ha, Section 8]).
Let $V = \bar{V}$ be a measurable function. Assume that λ_1 and λ_2 satisfy the hypotheses of Theorem 3.1. For $s > 0$ set

$$\nu_{\lambda_1,\lambda_2}(s; V) := \tilde{\nu}_-(s; \frac{1}{\lambda_1\lambda_2} V^2 - \frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2} V) = \mathcal{C}_q(B)\text{vol}\{\mathbf{x} \in \mathbb{R}^m | V(\mathbf{x})^2 - (\lambda_1+\lambda_2)V(\mathbf{x}) + \lambda_1\lambda_2 s < 0\}.$$

Evidently, $\nu_{\lambda_1,\lambda_2}(1; V) = \mathcal{C}_q(B)\text{vol}\ \{\mathbf{x} \in \mathbb{R}^m | (V(\mathbf{x}) - \lambda_1)(V(\mathbf{x}) - \lambda_2) < 0\}$. Hence, the assumption that the point $s = 1$ is a continuity point of $\nu_{\lambda_1,\lambda_2}(.; V)$ is equivalent to the assumption that λ_1 and λ_2 are continuity points of the function k_{Schr}. Under this assumption we have $\nu_{\lambda_1,\lambda_2}(1; V) = \kappa_q\left(\text{k}_{\text{Schr}}(\lambda_2) - \text{k}_{\text{Schr}}(\lambda_1)\right)$.
Introduce the operator $t(\mu) = t(\mu; V) = t_{\lambda_1,\lambda_2}(\mu) := \tilde{\mathcal{T}}(\lambda_1 + \mu\Lambda_q, \lambda_2 + \mu\Lambda_q; \mathbb{H}_0(\mu), V)$ (see (3.3.7)).

Proposition 3.7 *Assume that the hypotheses of Theorem 3.1 hold. Suppose in addition that* $V \in C_0^\infty(\mathbb{R}^m)$. *Then we have*

$$\lim_{\mu\to\infty} \mu^{-d} n_-(s; t_{\lambda_1,\lambda_2}(\mu; V)) = \nu_{\lambda_1,\lambda_2}(s; V), \tag{3.4.12}$$

provided that $s > 0$ *is a continuity point of* $\nu_{\lambda_1,\lambda_2}(.; V)$.

By the minimax principle (see Lemma 3.4), $n_-(s; P_q(\mu)t(\mu)P_q(\mu)) \leq n_-(s; t(\mu))$ for each $s > 0$. Obviously, $P_q(\mu)t(\mu)P_q(\mu) = P_q(\mu)WP_q(\mu)$ with $W = (V^2 - (\lambda_1+\lambda_2)V)/\lambda_1\lambda_2$. By Corollary 3.10,

$$\liminf_{\mu\to\infty} \mu^{-d} n_-(s; t(\mu)) \geq \lim_{\mu\to\infty} \mu^{-d} n_-(s; P_q(\mu)WP_q(\mu)) = \tilde{\nu}(s; W) = \nu_{\lambda_1,\lambda_2}(s; V). \tag{3.4.13}$$

On the other hand, for each $\varepsilon \in (0, s)$

$$n_-(s; t(\mu)) \leq n_-(s-\varepsilon; P_q(\mu)t(\mu)P_q(\mu)) +$$

$$n_-(s-\varepsilon; (I - P_q(\mu))t(\mu)(I - P_q(\mu))) + n_-(\varepsilon; 2\text{Re}\ P_q(\mu)t(\mu)(I - P_q(\mu))). \tag{3.4.14}$$

We have

$$\|(I-P_q(\mu))t(\mu)(I-P_q(\mu))\| \leq \|V\|^2_{L^\infty(\mathbb{R}^m)}\|r(\mu)(I-P_q(\mu))\|^2 + 2\|g(\mu)\|\|V\|_{L^\infty(\mathbb{R}^m)}\|r(\mu)(I-P_q(\mu))\|$$

where $g(\mu) := \mathcal{G}(\lambda_1 + \mu\Lambda_q, \lambda_2 + \mu\Lambda_q; \mathbb{H}_0(\mu))$ (see (3.3.6)). Since $\|r(\mu)(I - P_q(\mu))\| = \mathcal{O}(\mu^{-1})$ and $\|g(\mu)\| = \mathcal{O}(1)$ as $\mu \to \infty$, we have

$$\|(I - P_q(\mu))t(\mu)(I - P_q(\mu))\| = \mathcal{O}(\mu^{-1}), \quad \mu \to \infty. \tag{3.4.15}$$

Further,

$$\text{Re}\ P_q(\mu)t(\mu)(I - P_q(\mu)) = \text{Re}\ P_q(\mu)r(\mu)V^2 r(\mu)(I - P_q(\mu)) + 2\text{Re}\ P_q(\mu)g(\mu)Vr(\mu)(I - P_q(\mu)) +$$

$$\text{Re } P_q(\mu)r(\mu)[V,a]a(\mu)^*r(\mu)(I-P_q(\mu)) + \text{Re } P_q(\mu)r(\mu)a(\mu)[V,a^*]r(\mu)(I-P_q(\mu)).$$

Since $\|[V,a]\| = \|[V,a^*]\| = \sup_{\mathbf{x}\in\mathbb{R}^m} |\nabla V(\mathbf{x})|$, and $\|P_q(\mu)r(\mu)\| = \mathcal{O}(1)$, $\|P_q(\mu)r(\mu)a(\mu)\| = \mathcal{O}(\mu^{1/2})$ as $\mu \to \infty$, we get

$$\|P_q(\mu)r(\mu)[V,a]\| = \mathcal{O}(1),\ \|P_q(\mu)r(\mu)a(\mu)[V,a^*]\| = \mathcal{O}(\mu^{1/2}),\ \mu\to\infty.$$

Similarly, $\|P_q(\mu)r(\mu)V^2\| = \mathcal{O}(1)$, $\|P_q(\mu)g(\mu)V\| = \mathcal{O}(1)$ as $\mu\to\infty$. On the other hand,

$$\|a(\mu)^*r(\mu)(I-P_q(\mu))\| = \mathcal{O}(\mu^{-1/2}),\ \|r(\mu)(I-P_q(\mu))\| = \mathcal{O}(\mu^{-1}),\ \mu\to\infty.$$

Summarizing, we get

$$\|2\text{Re } P_q(\mu)g(\mu)Vr(\mu)(I-P_q(\mu))\| = \mathcal{O}(\mu^{-1/2}), \quad \mu\to\infty. \tag{3.4.16}$$

Combining (3.4.15) and (3.4.16), we find that

$$n_-(s-\varepsilon;(I-P_q(\mu))t(\mu)(I-P_q(\mu))) + n_-(\varepsilon;2\text{Re } P_q(\mu)t(\mu)(I-P_q(\mu))) = 0 \tag{3.4.17}$$

for each $s>0$ and $\varepsilon\in(0,s)$, provided that μ is large enough. Choose a sequence $\{\varepsilon_l\}_{l\geq1}$ such that $\lim_{l\to\infty}\varepsilon_l = 0$, and for each $l\geq1$ we have $\varepsilon_l\in(0,s)$, and $s-\varepsilon_l$ is a continuity point of $\nu_{\lambda_1,\lambda_2}(.;V)$. Then (3.4.14) and (3.4.17) imply that for each fixed $s>0$ and $l\geq1$ we have

$$n_-(s;t(\mu)) \leq n_-(s-\varepsilon_l;P_q(\mu)t(\mu)P_q(\mu)), \tag{3.4.18}$$

provided that μ is large enough. By Proposition 3.7,

$$\lim_{\mu\to\infty}\mu^{-d}n_-(s-\varepsilon_l;P_q(\mu)t(\mu)P_q(\mu)) = \nu_{\lambda_1,\lambda_2}(s-\varepsilon_l;V). \tag{3.4.19}$$

Combining (3.4.18) and (3.4.19), and then letting $l\to\infty$, we get

$$\limsup_{\mu\to\infty}\mu^{-d}n_-(s;t(\mu)) \leq \nu_{\lambda_1,\lambda_2}(s;V). \tag{3.4.20}$$

Putting together (3.4.20) and (3.4.13), we obtain (3.4.12). $\diamondsuit$

Proposition 3.8 *Assume that the hypotheses of Theorem 3.1 hold. Suppose in addition that $V\in L^r(\mathbb{R}^m)$. Then (3.1.3) is valid.*

Proof: By Lemma 3.7,

$$\mathcal{N}(\lambda_1+\mu\Lambda_q;\lambda_2+\mu\Lambda_q;\mathbb{H}(\mu)) = n_-(1;t(\mu;V)). \tag{3.4.21}$$

Fix a sequence $\{\eta_l\}_{l\geq1}$ such that $\lim_{l\to\infty}\eta_l = 0$, and for each $l\geq1$ we have $\eta_l>0$. Write $V = V_0+V_1 = V_0^{(l)}+V_1^{(l)}$ where $V_0^{(l)}\in C_0^\infty(\mathbb{R}^m)$, and $\int_{\mathbb{R}^m}|V_1^{(l)}|^r d\mathbf{x}\leq\eta_l$. Choose a sequence $\{\varepsilon_n\}_{n\geq1}$ such that $\lim_{n\to\infty}\varepsilon_n = 0$, and for each $n\geq1$ we have $\varepsilon_n\in(0,1)$, and $1\pm\varepsilon_n$ are continuity points of the functions $\nu_{\lambda_1,\lambda_2}(.;V_0^{(l)})$ for each $l\geq1$. Notice the operator inequalities

$$(1-\varepsilon_n^2)t(\mu;V_0) - (\varepsilon_n^{-2}-1)r(\mu)V_1^2r(\mu) + 2\text{Re } g(\mu)V_1r(\mu) + 2\varepsilon_n^2\text{Re } g(\mu)V_0r(\mu) \leq$$

$$t(\mu;V)\leq$$

$$(1+\varepsilon_n^2)t(\mu;V_0) + (1+\varepsilon_n^{-2})r(\mu)V_1^2r(\mu) + 2\text{Re } g(\mu)V_1r(\mu) - 2\varepsilon_n^2\text{Re } g(\mu)V_0r(\mu).$$

Therefore, by Lemma 3.4 and Lemma 3.5,

$$n_-(1+\varepsilon_n;(1+\varepsilon_n^2)t(\mu;V_0)) - n_+(\varepsilon_n/3;(1+\varepsilon_n^{-2})r(\mu)V_1^2r(\mu)) -$$

$$n_+(\varepsilon_n/3; 2\mathrm{Re}\, g(\mu)V_1 r(\mu)) - n_-(\varepsilon_n/3; 2\varepsilon_n^2 \mathrm{Re}\, g(\mu)V_0 r(\mu)) \leq$$
$$n_-(1; t(\mu; V)) \leq$$
$$n_-(1-\varepsilon_n; (1-\varepsilon_n^2)t(\mu; V_0)) + n_+(\varepsilon_n/3; (\varepsilon_n^{-2}-1)r(\mu)V_1^2 r(\mu)) +$$
$$n_-(\varepsilon_n/3; 2\mathrm{Re}\, g(\mu)V_1 r(\mu)) + n_-(\varepsilon_n/3; 2\varepsilon_n^2 \mathrm{Re}\, g(\mu)V_0 r(\mu)). \quad (3.4.22)$$

By Proposition 3.7 we have

$$\limsup_{\mu\to\infty} \mu^{-d} n_-(1-\varepsilon_n; (1-\varepsilon_n^2)t(\mu; V_0)) \leq \lim_{\mu\to\infty} \mu^{-d} n_-(1-\varepsilon_n; t(\mu; V_0)) = \nu_{\lambda_1,\lambda_2}(1-\varepsilon_n; t(\mu; V_0)), \quad (3.4.23)$$

$$\liminf_{\mu\to\infty} \mu^{-d} n_-(1+\varepsilon_n; (1+\varepsilon_n^2)t(\mu; V_0)) \geq \lim_{\mu\to\infty} \mu^{-d} n_-(1+\varepsilon_n; t(\mu; V_0)) = \nu_{\lambda_1,\lambda_2}(1+\varepsilon_n; t(\mu; V_0)). \quad (3.4.24)$$

Assume that ε_n is small enough, and denote by $\lambda_{1,2}^{\pm} = \lambda_{1,2}^{\pm}(\varepsilon_n)$ the real solutions of the quadratic equation $\lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2(1\mp\varepsilon_n)$, such that $\lambda_1^{\pm} < \lambda_2^{\pm}$. Evidently, $\lambda_1^{\pm}\lambda_2^{\pm} > 0$, and $\lim_{n\to\infty} \lambda_{1,2}^{\pm}(\varepsilon_n) = \lambda_{1,2}$. Then

$$\nu_{\lambda_1,\lambda_2}(1\mp\varepsilon_n; t_0(\mu; V_0)) = \kappa_q \left(\mathrm{k}_{\mathrm{Schr}}(\lambda_2^{\pm}; V_0) - \mathrm{k}_{\mathrm{Schr}}(\lambda_1^{\pm}; V_0)\right). \quad (3.4.25)$$

Further, for every $s_1, s_2 \in \mathbb{R}$, such that $s_1 < s_2$, $s_1 s_2 > 0$, and $\delta > 0$ small enough, we have

$$\kappa_q \left(\mathrm{k}_{\mathrm{Schr}}(s_2-\delta; V) - \mathrm{k}_{\mathrm{Schr}}(s_1+\delta; V)\right) - \mathcal{C}_q(B)\ \mathrm{vol}\,\{\mathbf{x}\in\mathbb{R}^m | |V_1(\mathbf{x})| > \delta\} \leq$$
$$\kappa_q \left(\mathrm{k}_{\mathrm{Schr}}(s_2; V_0) - \mathrm{k}_{\mathrm{Schr}}(s_1; V_0)\right) \leq$$
$$\kappa_q \left(\mathrm{k}_{\mathrm{Schr}}(s_2+\delta; V) - \mathrm{k}_{\mathrm{Schr}}(s_1-\delta; V)\right) + \mathcal{C}_q(B)\ \mathrm{vol}\,\{\mathbf{x}\in\mathbb{R}^m | |V_1(\mathbf{x})| > \delta\}. \quad (3.4.26)$$

Moreover, for each $\delta > 0$ we have

$$\mathrm{vol}\,\{\mathbf{x}\in\mathbb{R}^m | |V_1(\mathbf{x})| > \delta\} \leq \delta^{-r} \int_{\mathbb{R}^m} |V_1(\mathbf{x})|^r d\mathbf{x} \leq \delta^{-r}\eta_l. \quad (3.4.27)$$

Next, by Corollary 3.9 and $\|g(\mu)\| = \mathcal{O}(1)$ as $\mu\to\infty$, we have

$$\limsup_{\mu\to\infty} \mu^{-d} n_+\left(\varepsilon_n/3; (\varepsilon_n^{-2}\pm 1)r(\mu)V_1^2 r(\mu)\right) = \limsup_{\mu\to\infty} \mu^{-d} n_*\left(1; \sqrt{3\varepsilon^{-1}(\varepsilon_n^{-2}\pm 1)}V_1 r(\mu)\right) = \mathcal{O}(\eta_l), \quad (3.4.28)$$

$$\limsup_{\mu\to\infty} \mu^{-d} n_\pm(\varepsilon_n/3; 2\mathrm{Re}\, g(\mu)V_1 r(\mu)) = \mathcal{O}(\eta_l), \quad (3.4.29)$$

as $l\to\infty$. Similarly,

$$n_+(\varepsilon_n/3; 2\varepsilon^2 \mathrm{Re}\, g(\mu)V_0 r(\mu)) = \mathcal{O}(\varepsilon_n^r) \quad (3.4.30)$$

as $n\to\infty$ uniformly with respect to η_l. The combination of (3.4.21) – (3.4.30) yields

$$\limsup_{\mu\to\infty} \mu^{-d} \mathcal{N}(\lambda_1+\mu\Lambda_q; \lambda_2+\mu\Lambda_q; \mathbb{H}(\mu)) \leq \kappa_q \left(\mathrm{k}_{\mathrm{Schr}}(\lambda_2^+ + \delta; V) - \mathrm{k}_{\mathrm{Schr}}(\lambda_1^+ - \delta; V)\right) + c_6\varepsilon_n^r + c_7\eta_l, \quad (3.4.31)$$

$$\liminf_{\mu\to\infty} \mu^{-d} \mathcal{N}(\lambda_1+\mu\Lambda_q; \lambda_2+\mu\Lambda_q; \mathbb{H}(\mu)) \geq \kappa_q \left(\mathrm{k}_{\mathrm{Schr}}(\lambda_2^- - \delta; V) - \mathrm{k}_{\mathrm{Schr}}(\lambda_1^- + \delta; V)\right) - c_6\varepsilon_n^r - c_7\eta_l, \quad (3.4.32)$$

for $\delta > 0$ small enough, $c_6 > 0$ independent of η_l, and $c_7 = c_7(\varepsilon_n, \delta) > 0$. Letting at first $l\to\infty$ (hence, $\eta_l \downarrow 0$), and then $n\to\infty$ (hence, $\varepsilon_n \downarrow 0$), and $\delta\downarrow 0$ in (3.4.31) – (3.4.32), we arrive at (3.1.3) in the case $V\in L^2(\mathbb{R}^m)$. $\diamondsuit$

In order to prove Theorem 3.1 in the general case $V\in\mathcal{L}_r$, fix $\varepsilon > 0$ small enough, and write $V = V_1 + V_2$ with $V_1 \in L^r(\mathbb{R}^m)$ and $\sup_{\mathbf{x}\in\mathbb{R}^m} |V_2(\mathbf{x})| \leq \varepsilon$. Fix a sequence $\{\varepsilon_n\}_{n\geq 1}$ such that

$\lim_{n\to\infty}\varepsilon_n = 0$ and for each $n \geq 1$ we have $\varepsilon_n \in (0,\varepsilon)$, and the points $\lambda_1 \pm \varepsilon \pm \varepsilon_n$ and $\lambda_2 \pm \varepsilon \pm \varepsilon_n$ are continuity points of $\mathrm{k}_{\mathrm{Schr}}(.;V_1)$. By Proposition 3.8 we have

$$\limsup_{\mu\to\infty}\mu^{-d}\mathcal{N}(\lambda_1+\mu\Lambda_q;\lambda_2+\mu\Lambda_q;\mathbb{H}(\mu)) \leq \limsup_{\mu\to\infty}\mu^{-d}\mathcal{N}(\lambda_1-\varepsilon+\mu\Lambda_q;\lambda_2+\varepsilon+\mu\Lambda_q;\mathbb{H}_0(\mu)+V_1) \leq$$

$$\lim_{\mu\to\infty}\mu^{-d}\mathcal{N}(\lambda_1-\varepsilon-\varepsilon_n+\mu\Lambda_q;\lambda_2+\varepsilon+\varepsilon_n+\mu\Lambda_q;\mathbb{H}_0(\mu)+V_1) =$$

$$\kappa_q\left(\mathrm{k}_{\mathrm{Schr}}(\lambda_2+\varepsilon+\varepsilon_n;V_1)-\mathrm{k}_{\mathrm{Schr}}(\lambda_1-\varepsilon-\varepsilon_n;V_1)\right) \leq$$

$$\kappa_q\left(\mathrm{k}_{\mathrm{Schr}}(\lambda_2+2\varepsilon;V_1)-\mathrm{k}_{\mathrm{Schr}}(\lambda_1-2\varepsilon;V_1)\right) \leq \kappa_q\left(\mathrm{k}_{\mathrm{Schr}}(\lambda_2+3\varepsilon;V)-\mathrm{k}_{\mathrm{Schr}}(\lambda_1-3\varepsilon;V)\right),$$

and analogously

$$\liminf_{\mu\to\infty}\mu^{-d}\mathcal{N}(\lambda_1+\mu\Lambda_q;\lambda_2\mu\Lambda_q;\mathbb{H}(\mu)) \geq \kappa_q\left(\mathrm{k}_{\mathrm{Schr}}(\lambda_2-3\varepsilon;V)-\mathrm{k}_{\mathrm{Schr}}(\lambda_1+3\varepsilon;V)\right).$$

Letting $\varepsilon \downarrow 0$, we obtain (3.1.3) in the general case.

3.5 Sketch of the proof of Theorem 3.4

In this subsection we follow the proof of [R 5, Theorem 2.2], shedding additional light on some obscure details. On the other hand, we only mention the steps in the proof of Theorem 3.4 which are the same as in the proof of Theorem 3.1 contained in the previous subsection.
Throughout the subsection we assume $m = 2$.

Lemma 3.11 [R 5, Lemma 4.1] *Assume that (2.4.4) holds. Let $W \in L^2(\mathbb{R}^2)$. Then*

$$\|WP(\mu)\|_2^2 \leq \frac{ec_2}{4\pi}\mu\int_{\mathbb{R}^2}|W|^2dX. \tag{3.5.1}$$

Let μ be large enough. Let $\lambda_1 \leq \lambda_2$, $\lambda_1\lambda_2 > 0$. If $\lambda_1 < \lambda_2$, set $r^{\pm}(\mu) = r^{\pm}_{\lambda_1,\lambda_2}(\mu) = \tilde{\mathcal{R}}(\lambda_1,\lambda_2;\mathbb{P}_0^{\pm})$ with $\mathbb{P}_0^{\pm} = \mathbb{H}_0(\mu) \pm \mu b$ (see (3.3.5) and (2.4.1)); if $\lambda_1 = \lambda_2$, extend the definition by continuity.

Corollary 3.11 [R 5, Corollary 4.1] *Assume that the hypotheses of Lemma 3.11 hold. Let $\lambda_1 \leq \lambda_2$, $\lambda_1\lambda_2 > 0$. Then we have*

$$\|Wr^{-}_{\lambda_1,\lambda_2}(\mu)P(\mu)\|_2^2 \leq c_8\mu\int_{\mathbb{R}^2}|W|^2dX, \tag{3.5.2}$$

and, therefore, $n_*(\varepsilon; Wr^{-}_{\lambda_1,\lambda_2}(\mu)P(\mu)) \leq c_8\epsilon^{-2}\mu\int_{\mathbb{R}^2}|W|^2dX$ for each $\varepsilon > 0$ with $c_8 := \frac{ec_2}{4\pi\lambda_1\lambda_2}$.

Proof: As in the proof of Corollary 3.8, it suffices to notice that $Wr^{-}_{\lambda_1,\lambda_2}(\mu)P(\mu) = (\lambda_1\lambda_2)^{-1/2}WP(\mu)$, and to apply (3.5.1). $\diamondsuit$

Lemma 3.12 (cf. [R 5, Lemma 4.2]) *Under the hypotheses of Corollary 3.11 there exists a constant c_9 independent of μ and W such that*

$$\|Wr^{-}_{\lambda_1,\lambda_2}(\mu)(I-P(\mu))\|_2^2 \leq c_9\mu^{-1}\int_{\mathbb{R}^2}|W|^2dX, \tag{3.5.3}$$

and, therefore, $n_*(\varepsilon; Wr^{-}_{\lambda_1,\lambda_2}(\mu)(I-P(\mu)))c_9\varepsilon^{-2}\mu^{-1}\int_{\mathbb{R}^2}|W|^2dX$ for each $\varepsilon > 0$.

Proof: Write $r^-(\mu)(I-P(\mu)) = (\mathbb{H}_0(\mu)+\mu)^{-1}(\mathbb{H}_0(\mu)+\mu)r^-(\mu)(I-P(\mu))$. On the other hand,

$$\|(\mathbb{H}_0(\mu)+\mu)r^-(\mu)(I-P(\mu))\| = \|(\mathbb{P}_0^-(\mu)+\mu b+\mu)r^-(\mu)(I-P(\mu))\| \leq$$

$$\|\mathbb{P}_0^-(\mu)r^-(\mu)(I-P(\mu))\| + \mu(c_2+1)\|r^-(\mu)(I-P(\mu))\|.$$

Evidently, there exists a constant $c_{10} > 0$ independent of μ large enough such that $\|\mathbb{P}_0^-(\mu)r^-(\mu)(I - P(\mu))\| + \mu(c_2+1)\|r^-(\mu)(I-P(\mu))\| \le c_{10}$. Hence, $\|Wr^-(\mu)(I-P(\mu))\|_2^2 \le c_{10}^2\|W(\mathbb{H}_0(\mu)+\mu)^{-1}\|_2^2$. The diamagnetic inequality (see Lemma 2.1) and the Parseval identity entail

$$\|W(\mathbb{H}_0(\mu)+\mu)^{-1}\|_2^2 \le \|W(-\Delta+\mu)^{-1}\|_2^2 = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}|W|^2dX\int_{\mathbb{R}^2}\frac{d\xi}{(|\xi|^2+\mu)^2} = \frac{1}{4\pi\mu}\int_{\mathbb{R}^2}|W|^2dX.$$

Consequently, (3.5.3) holds with $c_9 = \frac{c_{10}^2}{4\pi}$. $\diamondsuit$

Corollary 3.12 *Under the hypotheses of Corollary 3.11 we have*

$$\|Wr^-_{\lambda_1,\lambda_2}(\mu)\|_2^2 \le 2(c_8\mu + c_9\mu^{-1})\int_{\mathbb{R}^2})|W|^2dX, \tag{3.5.4}$$

and, therefore,

$$n_*(\varepsilon; Wr^-_{\lambda_1,\lambda_2}(\mu)) \le 2(c_8\mu + c_9\mu^{-1})\varepsilon^{-2}\int_{\mathbb{R}^2}|W|^2dX,\ \forall\varepsilon>0. \tag{3.5.5}$$

Proof: It suffices to write $\|Wr^-_{\lambda_1,\lambda_2}(\mu)\|_2^2 \le 2\|Wr^-_{\lambda_1,\lambda_2}(\mu)P(\mu)\|_2^2 + 2\|Wr^-_{\lambda_1,\lambda_2}(\mu)(I-P(\mu))\|_2^2$, and to apply (3.5.3) and (3.5.5). $\diamondsuit$

Lemma 3.13 [R 5, Lemma 4.3] *Under the hypotheses of Corollary 3.11 we have*

$$n_*(\varepsilon; Wr^+_{\lambda_1,\lambda_2}(\mu)) \le c_9\varepsilon^{-2}\mu^{-1}\int_{\mathbb{R}^2}|W|^2dX,\ \forall\varepsilon>0.$$

The proof is completely analogous with that of (3.5.3).

Lemma 3.14 [R 5, Lemma 6.3] *Assume that (2.4.4) holds. Let W be in the Sobolev space* $\mathrm{H}^1(\mathbb{R}^2)$. *Then there exists a constant c_{11} independent of μ and W such that*

$$\|[W,P(\mu)]\|_2^2 \le c_{11}\int_{\mathbb{R}^2}|\nabla W|^2dX. \tag{3.5.6}$$

Proof: For $\mu>0$ large enough we have $P(\mu) = -\frac{1}{2\pi i}\int_{\mathbb{S}^1}(\mathbb{P}_0^-(\mu)-\omega)^{-1}d\omega$ with $\mathbb{S}^1 := \{\omega\in\mathbb{C}||\omega|=1\}$ oriented in the anti-clockwise direction. Therefore,

$$[W,P(\mu)] = \frac{1}{2\pi i}\int_{\mathbb{S}^1}(\mathbb{P}_0^-(\mu)-\omega)^{-1}[W,\mathbb{P}_0^-(\mu)](\mathbb{P}_0^-(\mu)-\omega)^{-1}d\omega =$$

$$\frac{1}{2\pi i}\int_{\mathbb{S}^1}(\mathbb{P}_0^-(\mu)-\omega)^{-1}\left(\partial W a^* + a\partial^* W\right)(\mathbb{P}_0^-(\mu)-\omega)^{-1}d\omega$$

where $\partial W := i\frac{\partial W}{\partial x} + \frac{\partial W}{\partial y}$, $\partial^* W := i\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}$. Hence

$$\|[W,P(\mu)]\|_2 \le 2\sup_{\omega\in\mathbb{S}^1}\|(\mathbb{P}_0^-(\mu)-\omega)^{-1}\partial W a^*(\mathbb{P}_0^-(\mu)-\omega)^{-1}\|_2 \le$$

$$2\sup_{\omega\in\mathbb{S}^1}\|(\mathbb{P}_0^-(\mu)-\omega)^{-1}\partial W\|_2\sup_{\omega\in\mathbb{S}^1}\|a^*(\mathbb{P}_0^-(\mu)-\omega)^{-1}\|. \tag{3.5.7}$$

Estimate (3.5.4) with $\lambda_1=\lambda_2=-1$ easily implies

$$\|(\mathbb{P}_0^-(\mu)-\omega)^{-1}\partial W\|_2^2 \le 2(c_8\mu + c_9\mu^{-1})\int_{\mathbb{R}^2}|\nabla W|^2dX, \quad \forall\omega\in\mathbb{S}^1, \tag{3.5.8}$$

provided that $\mu>0$ is large enough. On the other hand,

$$\|a^*(\mathbb{P}_0^-(\mu)-\omega)^{-1}\|^2 = \|(\mathbb{P}_0^-(\mu)-\overline{\omega})^{-1}\mathbb{P}_0^-(\mu)(\mathbb{P}_0^-(\mu)-\omega)^{-1}\| = \mathcal{O}(\mu^{-1}), \quad \mu\to\infty, \tag{3.5.9}$$

uniformly with respect to $\omega\in\mathbb{S}^1$. The combination of (3.5.8) and (3.5.9) yields (3.5.6). $\diamondsuit$

Proposition 3.9 [R 5, Proposition 1] *Assume that (2.4.4) is valid with* $c_1 > 0$. *Let* $W \in C_0^\infty(\mathbb{R}^2)$. *Then* $P(\mu)WP(\mu) \in S_1$, *and we have*

$$\lim_{\mu\to\infty} \mu^{-1}\mathrm{Tr}\,(P(\mu)WP(\mu))^l = \frac{1}{2\pi}\int_{\mathbb{R}^2} W(X)^l b(X)dX, \quad l \geq 1. \tag{3.5.10}$$

Proof: In contrast to the proof of Lemma 3.3, here we follow the strategy of [W] where we do not rely on an explicit expression of the diagonal value of the kernel of $P(\mu)$, but rather on the commutator estimate contained in the previous lemma.
By Lemma 3.11, $W \in C_0^\infty(\mathbb{R}^2)$ implies $|W|^{1/2}P(\mu) \in S_2$; hence $P(\mu)WP(\mu) \in S_1$. By Proposition 2.2 we have

$$\lim_{\mu\to\infty} \mu^{-1}\mathrm{Tr}\,P(\mu)W^lP(\mu) = \lim_{\mu\to\infty} \mu^{-1}\int_{\mathbb{R}^2} W(X)^l K_{\mu,\mathrm{Pauli}}(X;X)dX = \frac{1}{2\pi}\int_{\mathbb{R}^2} W(X)^l b(X)dX, \ l \geq 1. \tag{3.5.11}$$

Further, for $l \geq 2$ we have

$$P(\mu)W^lP(\mu) - (P(\mu)WP(\mu))^l = \sum_{s=1}^{l-1} P(\mu)W^s[W,P(\mu)](P(\mu)W)^{l-s-1}P(\mu).$$

Hence, $|\mathrm{Tr}\ P(\mu)W^lP(\mu) - \mathrm{Tr}\ (P(\mu)WP(\mu))^l| \leq \sum_{s=1}^{l-1} \|P(\mu)W^s\|_2 \|[W,P(\mu)]\|_2 \|W\|_{L^\infty(\mathbb{R}^2)}^{l-s-1}$. By Lemma 3.11 we have $\|P(\mu)W^s\|_2 = \mathcal{O}(\mu^{1/2})$, $\mu \to \infty$, for any $s \geq 1$, and by Lemma 3.14 – $\|[W,P(\mu)]\|_2 = \mathcal{O}(1)$, $\mu \to \infty$. Therefore,

$$\mathrm{Tr}\,P(\mu)W^lP(\mu) - \mathrm{Tr}\,(P(\mu)WP(\mu))^l = \mathcal{O}(\mu^{1/2}), \quad \mu \to \infty, \quad l \geq 2. \tag{3.5.12}$$

Putting together (3.5.11) and (3.5.12), we get (3.5.10). ♢
Remark: Since $P(\mu) = P_{\mathrm{Pauli}}(\mu)$ is the orthogonal projection on the weighted holomorphic space $\mathcal{H}(\mathbb{R}^2; e^{-2\mu\varphi})$ with $\Delta\varphi = b$, $P(\mu)WP(\mu)$ is a classical Toeplitz operator (see [Ha, Section 8]).

Let $W = \overline{W}$ be a measurable function. For $s > 0$ set

$$\nu_{\mathrm{Pauli}}(s;W) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta(-W(X)-s)b(X)dX.$$

Corollary 3.13 *Under the hypotheses of Proposition 3.9 we have*

$$\lim_{\mu\to\infty} \mu^{-1} n_-(s; P_{\mathrm{Pauli}}(\mu)WP_{\mathrm{Pauli}}(\mu)) = \nu_{\mathrm{Pauli}}(s;W)$$

provided that $s > 0$ *is a continuity point of* $\nu_{\mathrm{Pauli}}(.;W)$.

Let $0 < \lambda_1 < \lambda_2$. Put $t_{\mathrm{Pauli}}(\mu) := \tilde{\mathcal{T}}(\lambda_1,\lambda_2;\mathbb{P}_0(\mu),VI_2)$, $t^\pm_{\mathrm{Pauli}}(\mu) := \tilde{\mathcal{T}}(\lambda_1,\lambda_2;\mathbb{P}_0^\pm,VI_2)$ (see (3.3.7)).

Proposition 3.10 *Assume that the hypotheses of Theorem 3.4 hold. Suppose in addition* $V \in C_0^\infty(\mathbb{R}^2)$. *Then we have*

$$\lim_{\mu\to\infty} \mu^{-1} n_-(s; t^-_{\mathrm{Pauli}}(\mu)) = \nu_{\mathrm{Pauli}}\left(.;(V^2 - (\lambda_1+\lambda_2)V)/\lambda_1\lambda_2\right),$$

provided that $s > 0$ *is a continuity point of* $\nu_{\mathrm{Pauli}}\left(s;(V^2 - (\lambda_1+\lambda_2)V)/\lambda_1\lambda_2\right)$.

The proof is completely analogous to that of Proposition 3.7 but now we have to use Corollary 3.13 instead of Corollary 3.10.

Proposition 3.11 *Assume that the hypotheses of Theorem 3.4 hold. Suppose in addition* $V \in L^2(\mathbb{R}^2)$. *Then (3.1.12) is valid.*

Sketch of the proof: The proof is very similar to that of Proposition 3.8. By Lemma 3.7,

$$\mathcal{N}(\lambda_1, \lambda_2; \mathbb{P}(\mu)) = n_-(1; t_{\rm Pauli}(\mu)) = n_-(1; t^-_{\rm Pauli}(\mu)) + n_-(1; t^+_{\rm Pauli}(\mu)). \tag{3.5.13}$$

By Lemma 3.13, we find that

$$n_-(1; t^+_{\rm Pauli}(\mu)) = 0, \tag{3.5.14}$$

provided that $\mu > 0$ is large enough. By analogy with the proof of Proposition 3.8 we obtain

$$\lim_{\mu\to\infty} \mu^{-1} n_-(1; t^-_{\rm Pauli}(\mu)) = {\rm k}_{\rm Pauli}(\lambda_2) - {\rm k}_{\rm Pauli}(\lambda_1). \tag{3.5.15}$$

The combination of (3.5.13), (3.5.14), and (3.5.15) yields (3.1.12) in the case $V \in L^2(\mathbb{R}^2)$. $\diamondsuit$
The passage from $V \in L^2(\mathbb{R}^2)$ to general $V \in \mathcal{L}_2$ in the proof of Theorem 3.4 is elementary and is quite similar to the final step of the proof of Theorem 3.1 (see the end of the previous subsection).

3.6 Proof of Propositions 3.1 – 3.3 and 3.5

Lemma 3.15 [Re.S, vol.3, Theorem XI.20] *Let* $k \geq 1$, $m = k + 2d$, $d \geq 1$. *Assume that* $w \in L^{m/2}(\mathbb{R}^k)$. *Then for each* $\lambda < 0$ *we have*

$$\| |w|^{1/2}(-\Delta - \lambda)^{-1/2} \|_m^m \leq c_{12} |\lambda|^{-d} \int_{\mathbb{R}^k} |w(z)|^{m/2} dz \tag{3.6.1}$$

with $c_{12} := (2\pi)^{-k} \int_{\mathbb{R}^k} \frac{dz}{(|z|^2+1)^{m/2}}$.

Proof of Proposition 3.1: Let $V \in L^{\frac{m}{2}}(\mathbb{R}^m)$. Set $\Omega(V) := \left\{X_\perp \in \mathbb{R}^{2d} | \int_{\mathbb{R}^k} |V(X_\perp, z)|^{\frac{m}{2}} dz < \infty\right\}$. Evidently, vol $\left\{\mathbb{R}^{2d} \setminus \Omega(V)\right\} = 0$. By Lemma 3.15, $|V(X_\perp, .)|^{1/2}(-\Delta_z + 1)^{-1/2} \in \mathcal{S}_m \subset \mathcal{S}_\infty$ if $X_\perp \in \Omega(V)$. Let now $V \in \mathcal{L}_{m/2}$. Fix a sequence $\{\varepsilon_l\}_{l\geq 1}$ such that $\varepsilon_l \downarrow 0$. Write $V = V_{1,l} + V_{2,l}$ with $V_{1,l} \in L^{m/2}(\mathbb{R}^m)$ and $\sup_{\mathbf{x}\in\mathbb{R}^m} |V_2(\mathbf{x})| \leq \varepsilon_l$. Set $\tilde{\Omega} := \bigcap_{l=1}^\infty \Omega(V_{1,l})$. Obviously, vol $\left\{\mathbb{R}^{2d} \setminus \tilde{\Omega}\right\} = 0$, and $|V_{1,l}(X_\perp, .)|^{1/2}(-\Delta_z + 1)^{-1/2} \in S_\infty$ for each $l \geq 1$ and $X_\perp \in \tilde{\Omega}$. On the other hand,

$$\| |V(X_\perp, .)|^{1/2}(-\Delta_z + 1)^{-1/2} - |V_{1,l}(X_\perp, .)|^{1/2}(-\Delta_z + 1)^{-1/2} \| \leq \varepsilon_l^{1/2}, \quad X_\perp \in \tilde{\Omega}.$$

Since the operator $|V(X_\perp, .)|^{1/2}(-\Delta_z + 1)^{-1/2}$, $X_\perp \in \tilde{\Omega}$, can be approximated uniformly by compact operators, it is a compact operator itself. Since $\tilde{\Omega} \subseteq \Omega_{\rm Schr}$, and vol $\left\{\mathbb{R}^{2d} \setminus \tilde{\Omega}\right\} = 0$, we find that vol $\left\{\mathbb{R}^{2d} \setminus \Omega_{\rm Schr}\right\} = 0$ if $V \in \mathcal{L}_{m/2}$ which completes the proof of Proposition 3.1. $\diamondsuit$
Proof of Proposition 3.2: Let $w = \bar{w} \in \mathcal{L}_{m/2}(\mathbb{R}^k)$. For $\lambda < 0$ set $\mathbf{t}_\lambda(w) := (-\Delta - \lambda)^{-1/2} w (-\Delta - \lambda)^{-1/2}$. By Lemma 2.2, the operator $\mathbf{t}_\lambda(w)$ is compact and self-adjoint in $L^2(\mathbb{R}^k)$. Let $\{\mathbf{a}_j(w, \lambda)\}_{j\geq 1}$ be the negative eigenvalues of the operator $\mathbf{t}_\lambda(w)$ enumerated in a non-decreasing order, i.e. $\mathbf{a}_1(w, \lambda) < \mathbf{a}_2(w, \lambda) \leq \ldots < 0$. If $\mathbf{t}_\lambda(w)$ has exactly J negative eigenvalues, $0 \leq J < \infty$, we set $\mathbf{a}_j(w, \lambda) = 0$ for $j \geq J + 1$.
Let $V \in \mathcal{L}_{m/2}(\mathbb{R}^m)$. For $\lambda < 0$, and $X_\perp \in \Omega_{\rm Schr}$, set $\tau_\lambda(X_\perp) := \mathbf{t}_\lambda(V(X_\perp, .))$. If eventually, $\mathbb{R}^{2d} \neq \Omega_{\rm Schr}$, put $\tau_\lambda(X_\perp) = 0$ for $X_\perp \in \mathbb{R}^{2d} \setminus \Omega_{\rm Schr}$. Set $\alpha_j(X_\perp, \lambda) := \mathbf{a}_j(V(X_\perp, .), \lambda)$, $j \geq 1$. By Lemma 3.6,

$$N(\lambda; \chi(X_\perp)) = n_-(1; \tau_\lambda(X_\perp)), \ \lambda < 0, \ X_\perp \in \mathbb{R}^{2d}.$$

Therefore, the measurability of the function $N(\lambda; \chi(X_\perp))$, $X_\perp \in \mathbb{R}^{2d}$, is equivalent to the measurability of $n_-(1; \tau_\lambda(X_\perp))$, $X_\perp \in \mathbb{R}^{2d}$, $\lambda < 0$ being a fixed parameter. On the other hand, $n_-(1; \tau_\lambda(.)) : \mathbb{R}^{2d} \to \mathbb{N}$, and we have

$$\left\{X_\perp \in \mathbb{R}^{2d} | n_-(1; \tau_\lambda(X_\perp)) = 0\right\} = \left\{X_\perp \in \mathbb{R}^{2d} | \alpha_1(X_\perp, \lambda)) \geq -1\right\},$$

$$\{X_\perp \in \mathbb{R}^{2d} | n_-(1; \tau_\lambda(X_\perp)) = l\} =$$

$$\{X_\perp \in \mathbb{R}^{2d} | \alpha_l(X_\perp, \lambda)) < -1\} \bigcap \{X_\perp \in \mathbb{R}^{2d} | \alpha_{l+1}(X_\perp, \lambda)) \geq -1\}, \quad l \in \mathbb{N}_*.$$

Hence, the measurability of $n_-(1; \tau_\lambda(.))$ will follow from the measurability of $\alpha_j(., \lambda))$, $j \geq 1$.
Let at first $V \in C_0^\infty(\mathbb{R}^m)$. In this case $\alpha_j(X_\perp, \lambda))$, $j \geq 1$, are continuous and hence measurable with respect to $X_\perp \in \mathbb{R}^{2d}$. Assume now $V \in L^{m/2}(\mathbb{R}^m)$. Choose a functional sequence $V_{0,l} = \bar{V}_{0,l} \in C_0^\infty(\mathbb{R}^m)$ such that $V \to V_{0,l}$ as $l \to \infty$ in $L^{m/2}(\mathbb{R}^m)$. Set

$$U_l(X_\perp) := \int_{\mathbb{R}^k} |V(X_\perp, z) - V_{0,l}(X_\perp, z)|^{m/2} dz, X_\perp \in \mathbb{R}^{2d}.$$

Then $U_l \to 0$ as $l \to \infty$ in $L^1(\mathbb{R}^{2d})$. Passing, if necessary, to a subsequence, we might assume that $U_l(X_\perp) \to 0$ as $l \to \infty$ for almost every $X_\perp \in \mathbb{R}^{2d}$. For $X_\perp \in \mathbb{R}^{2d}$, $\lambda < 0$, and $l \geq 1$ set

$$\tau_{\lambda,l}^{(0)}(X_\perp) := \mathbf{t}_\lambda(V_{0,l}(X_\perp, .)), \quad \alpha_{j,l}^{(0)}(X_\perp, \lambda) := \mathbf{a}_j(V_{0,l}(X_\perp, .), \lambda), \ j \geq 1.$$

By the minimax principle and Lemma 3.15, for each $j \geq 1$

$$|\alpha_j(X_\perp, \lambda) - \alpha_{j,l}^{(0)}(X_\perp, \lambda)| \leq \|\tau_\lambda(X_\perp) - \tau_{\lambda,l}^{(0)}(X_\perp)\| \leq \||V(X_\perp, .) - V_{0,l}(X_\perp, .)|^{1/2}(-\Delta_z - \lambda)^{-1/2}\|^2 \leq$$

$$\||V(X_\perp, .) - V_{0,l}(X_\perp, .)|^{1/2}(-\Delta_z - \lambda)^{-1/2}\|_m^2 \leq c_{12}^{2/m} |\lambda|^{-2d/m} U_l(X_\perp)^{2/m}.$$

Therefore, $\lim_{l\to\infty} \alpha_{j,l}^{(0)}(X_\perp, \lambda) = \alpha_j(X_\perp, \lambda)$ for almost every $X_\perp \in \mathbb{R}^{2d}$ and each $\lambda < 0$, and $j \geq 1$. Since $\alpha_{j,l}^{(0)}(., \lambda)$, $j \geq 1$, are measurable functions, $\alpha_j(., \lambda)$ are measurable as well.
Let now $V \in \mathcal{L}_{m/2}$. Choose a sequence ε_l such that $\varepsilon_l \downarrow 0$. Write $V = V_{1,l} + V_{2,l}$ with $V_{1,l} \in L^{m/2}(\mathbb{R}^m)$ and $\sup_{\mathbf{x}\in\mathbb{R}^m} |V_2(\mathbf{x})| \leq \varepsilon_l$. Set $\tilde{\Omega} := \bigcap_{l=1}^\infty \Omega(V_{1,l})$. For $\lambda < 0$, $X_\perp \in \tilde{\Omega}$, and $l \geq 1$ set $\tau_{\lambda,l}^{(1)}(X_\perp) := \mathbf{t}_\lambda(V_{1,l}(X_\perp, .))$. If eventually $\tilde{\Omega} \neq \mathbb{R}^{2d}$ set $\tau_{\lambda,l}^{(1)}(X_\perp) = 0$ for $X_\perp \in \mathbb{R}^{2d} \setminus \tilde{\Omega}$. For $X_\perp \in \mathbb{R}^{2d}$, $\lambda < 0$, $l \geq 1$, and $j \geq 1$ put $\alpha_{j,l}^{(1)}(X_\perp, \lambda) := \mathbf{a}_j(V_{1,l}(X_\perp, .), \lambda)$. The functions $\alpha_{j,l}^{(1)}(X_\perp, \lambda)$ are measurable on $\mathbb{R}^{2d}$ and, moreover, $|\alpha_j(X_\perp, \lambda) - \alpha_{j,l}^{(1)}(X_\perp, \lambda)| \leq \|\tau_\lambda(X_\perp) - \tau_{\lambda,l}^{(1)}(X_\perp)\| \leq |\lambda|^{-1}\varepsilon_l$, $X_\perp \in \tilde{\Omega}$. In other words, we have $\lim_{l\to\infty} \alpha_{j,l}^{(1)}(X_\perp, \lambda) = \alpha_j(X_\perp, \lambda)$, $j \geq 1$, for almost every $X_\perp \in \mathbb{R}^{2d}$. Therefore, the functions $\alpha_j(., \lambda)$ are measurable under the general assumption $V \in \mathcal{L}_{m/2}$. ♢

Proof of Proposition 3.3: Fix $\varepsilon \in (0, |\lambda|)$. Write $V = V_1 + V_2$ with $V_1 \in L^{m/2}(\mathbb{R}^m)$ and $\sup_{\mathbf{x}\in\mathbb{R}^m} |V_2(\mathbf{x})| \leq \varepsilon$. By the minimax principle and Lemma 3.6, we have

$$N(\lambda; \chi(X_\perp)) \leq N(\lambda + \varepsilon; -\Delta_z + V_1(X_\perp, .)) =$$

$$n_-(1; (-\Delta_z - \lambda - \varepsilon)^{-1/2} V_1(X_\perp, .)(-\Delta_z - \lambda - \varepsilon)^{-1/2}) \leq n_*(1; |V_1(X_\perp, .)|^{1/2}(-\Delta_z - \lambda - \varepsilon)^{-1/2}) \quad (3.6.2)$$

By Lemma 3.15, we have

$$n_*(1; |V_1(X_\perp, .)|^{1/2}(-\Delta_z - \lambda - \varepsilon)^{-1/2}) \leq c_{12}|\lambda + \varepsilon|^{-d} \int_{\mathbb{R}^k} |V_1(X_\perp, z)|^{m/2} dz, \ X_\perp \in \tilde{\Omega}. \quad (3.6.3)$$

Combining (3.6.2) and (3.6.3), and integrating with respect to $X_\perp$, we get $\int_{\mathbb{R}^{2d}} N(\lambda; \chi(X_\perp)) dX_\perp \leq c_{12}|\lambda + \varepsilon|^{-d} \int_{\mathbb{R}^m} |V_1(\mathbf{x})|^{m/2} d\mathbf{x}$. ♢

Sketch of the proof of Proposition 3.5: The argument is close to that of the proof of Proposition 3.2. Let $m = 3$, $V \in \mathcal{L}_3$, $-1 < \lambda_1 < \lambda_2 < 1$. Set $\tilde{\tau}_{\lambda_1,\lambda_2}(V(X_\perp, .)) := \tilde{T}(\lambda_1, \lambda_2; \partial_0, V(X_\perp, .)I_4)$, $X_\perp \in \Omega_{\rm Dirac}$. By Lemma 3.7,

$$\mathcal{N}(\lambda_1, \lambda_2; \partial(X_\perp)) = n_-(1; \tilde{\tau}_{\lambda_1,\lambda_2}(V(X_\perp, .))), \quad X_\perp \in \mathbb{R}^2. \quad (3.6.4)$$

Hence, the measurability of $\mathcal{N}(\lambda_1, \lambda_2; \partial(X_\perp))$ will follow from the measurability with respect to $X_\perp \in \mathbb{R}^2$ of the negative eigenvalues of the operator $\tilde{\tau}_{\lambda_1,\lambda_2}(V(X_\perp, .))$. At first, we assume $V \in C_0^\infty(\mathbb{R}^3)$. Then the eigenvalues of $\tilde{\tau}_{\lambda_1,\lambda_2}(V(X_\perp, .))$ are continuous and hence measurable with respect to $X_\perp \in \mathbb{R}^2$. Next, we assume $V \in L^3(\mathbb{R}^3)$, approximate it in $L^3(\mathbb{R}^3)$ by a sequence $V_{0,l} \in C_0^\infty(\mathbb{R}^3)$, $l \geq 1$, and show that $\lim_{l\to\infty} \|\tilde{\tau}_{\lambda_1,\lambda_2}(V(X_\perp, .)) - \tilde{\tau}_{\lambda_1,\lambda_2}(V_{0,l}(X_\perp, .))\| = 0$ for almost every $X_\perp \in \mathbb{R}^2$. Here we make use of the estimate $\|\tilde{\tau}_{\lambda_1,\lambda_2}(V(X_\perp, .)) - \tilde{\tau}_{\lambda_1,\lambda_2}(V_{0,l}(X_\perp, .))\| \leq \{\|\varrho(V(X_\perp, .) + V_{0,l}(X_\perp, .))\| + 2\|\gamma\|\}\ \|(V(X_\perp, .) - V_{0,l}(X_\perp, .))\varrho\|$ with $\varrho := \tilde{\mathcal{R}}(\lambda_1, \lambda_2; \partial_0)$, $\gamma := \mathcal{G}(\lambda_1, \lambda_2; \partial_0)$ (see (3.3.5) – (3.3.6)), and the following

Lemma 3.16 [R 4, Lemma 7.2] *Let $w \in L^3(\mathbb{R})$, $-1 < \lambda_1 < \lambda_2 < 1$. Then we have*

$$\|w\varrho\|_3^3 \leq c_{13} \int_{\mathbb{R}} |w(z)|^3 dz$$

where c_{13} does not depend on w.

Therefore, the eigenvalues of $\tilde{\tau}_{\lambda_1,\lambda_2}(V(X_\perp, .))$, $X_\perp \in \mathbb{R}^2$, are measurable if $V \in L^3(\mathbb{R}^3)$.
Finally, we suppose that $V \in \mathcal{L}_3$, approximate it in $L^\infty(\mathbb{R}^3)$ by a sequence $V_{1,l} \in L^3(\mathbb{R}^3)$, $l \geq 1$, and show that $\lim_{l\to\infty} \|\tilde{\tau}_{\lambda_1,\lambda_2}(V(X_\perp, .)) - \tilde{\tau}_{\lambda_1,\lambda_2}(V_{1,l}(X_\perp, .))\| = 0$. Hence, the eigenvalues of $\tilde{\tau}_{\lambda_1,\lambda_2}(V(X_\perp, .))$ are measurable with respect to $X_\perp \in \mathbb{R}^2$ under the general assumption $V \in \mathcal{L}_3$. By (3.6.4), $\mathcal{N}(\lambda_1, \lambda_2; \partial(.))$ is measurable as well. ♢

3.7 Sketch of the proof of Theorem 3.6

In this subsection we follow quite closely the exposition of the original paper [R 5]. Throughout the subsection we assume $m = 3$.
Define the orthogonal projection $\mathbf{P}(\mu) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ by

$$(\mathbf{P}(\mu)u)(X_\perp, z) := \int_{\mathbb{R}^2} K_\mu(X_\perp; X'_\perp) u(X'_\perp, z) dX'_\perp,$$

the function K_μ being defined in (2.4.9). For $\lambda < 0$ set $\mathbf{R}^\pm_\lambda := \mathcal{R}(\lambda; \mathbb{P}_0^\pm(\mu))$ (see (3.3.3)), where, as usually, $\mathbb{P}_0^\pm(\mu) := \mathbb{H}_0(\mu) \pm \mu b$.

Proposition 3.12 [R 5, Corollary 4.4] *Let $W \in L^3(\mathbb{R}^3)$, $\lambda < 0$, $\varepsilon > 0$. Then there exists a constant c_{14} independent of W, μ, and ε such that we have*

$$n_*(\varepsilon; W\mathbf{R}^-_\lambda \mathbf{P}(\mu)) \leq c_{14}\mu\varepsilon^{-3} \int_{\mathbb{R}^3} |W(\mathbf{x})|^3 d\mathbf{x}. \tag{3.7.1}$$

Proposition 3.13 [R 5, Lemmas 4.5–4.6] *Under the assumptions of Proposition 3.12 there exists $\mu_0 = \mu_0(\lambda, \varepsilon)$ such that $\mu > \mu_0$ entails*

$$n_*(\varepsilon; W\mathbf{R}^-_\lambda (I - \mathbf{P}(\mu))) = n_*(\varepsilon; W\mathbf{R}^+_\lambda) = 0. \tag{3.7.2}$$

Let $\lambda < 0$. Define the operator $\mathbf{R}^{(0)}_\lambda := \left(\Pi_3^2 - \lambda\right)^{-1/2} = \left(-\frac{\partial^2}{\partial z^2} - \lambda\right)^{-1/2}$, self-adjoint in $L^2(\mathbb{R}^3)$. Assume $V \in C_0^\infty(\mathbb{R}^3)$, $\lambda < 0$, and introduce the operator

$$\mathbf{T}_\lambda(\mu) := \mathbf{P}(\mu)\mathbf{R}^{(0)}_\lambda V \mathbf{R}^{(0)}_\lambda \mathbf{P}(\mu),$$

compact in $L^2(\mathbb{R}^3)$, and the operator

$$\tau_\lambda(X_\perp) := \left(-\frac{d^2}{dz^2} - \lambda\right)^{-1/2} V(X_\perp, .) \left(-\frac{d^2}{dz^2} - \lambda\right)^{-1/2} \equiv \mathcal{T}(\lambda; -\frac{d^2}{dz^2}, V(X_\perp, .))$$

(see (3.3.4)), compact in $L^2(\mathbb{R}_z)$ and depending on the parameter $X_\perp \in \mathbb{R}^2$.

Proposition 3.14 [R 5, Corollary 6.6] *Let $V \in C_0^\infty(\mathbb{R}^3)$, $\lambda < 0$. Then $\mathbf{T}_\lambda(\mu) \in \mathcal{S}_1$ and we have*

$$\lim_{\mu\to\infty} \mu^{-1}\mathrm{Tr}\;\mathbf{T}_\lambda(\mu)^l = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}\;\tau_\lambda(X_\perp)^l b(X_\perp)dX_\perp,\; l\in\mathbb{N}_*.$$

For $s > 0$ set

$$\tilde{\mathcal{K}}_\lambda(s) := \frac{1}{2\pi}\int_{\mathbb{R}^2} n_-(s;\tau_\lambda(X_\perp))b(X_\perp)dX_\perp.$$

Combining Lemma 3.8 and Proposition 3.14, we get

Corollary 3.14 *Under the hypotheses of Proposition 3.14*

$$\lim_{\mu\to\infty} \mu^{-1}n_-(s;\mathbf{T}_\lambda(\mu)) = \tilde{\mathcal{K}}_\lambda(s), \tag{3.7.3}$$

provided that $s > 0$ is a continuity point of $\tilde{\mathcal{K}}_\lambda$.

Notice that by Lemma 3.6, we have

$$\tilde{\mathcal{K}}_\lambda(1) = \frac{1}{2\pi}\int_{\mathbb{R}^2} N(\lambda;\chi(X_\perp))b(X_\perp)dX_\perp = \mathcal{K}_{\mathrm{Pauli}}(\lambda).$$

Moreover, $s = 1$ is a continuity point of $\tilde{\mathcal{K}}_\lambda$ if and only if $\lambda < 0$ is a continuity point of $\mathcal{K}_{\mathrm{Pauli}}$.

Proposition 3.15 *Assume that the hypotheses of Theorem 3.6 hold. Suppose in addition $V \in C_0^\infty(\mathbb{R}^3)$. Then (3.1.15) is valid.*

Proof: For $\lambda < 0$ set $\mathbf{T}_\lambda^\pm(\mu) := \mathcal{T}(\lambda;\mathbb{P}_0^\pm(\mu),V)$ (see (3.3.4)). By Lemma 3.6

$$N(\lambda;\mathbb{P}(\mu)) = n_-(1;\mathbf{T}_\lambda^+(\mu)) + n_-(1;\mathbf{T}_\lambda^-(\mu)). \tag{3.7.4}$$

By Lemmas 3.4 and 3.5, for each $\varepsilon \in (0,1)$ we have

$$n_-(1;\mathbf{P}(\mu)\mathbf{T}_\lambda^-(\mu)\mathbf{P}(\mu)) \le n_-(1;\mathbf{T}_\lambda^-(\mu)) \le n_-(1-\varepsilon;\mathbf{P}(\mu)\mathbf{T}_\lambda^-(\mu)\mathbf{P}(\mu)) +$$

$$n_-(1-\varepsilon;(I-\mathbf{P}(\mu))\mathbf{T}_\lambda^-(\mu)(I-\mathbf{P}(\mu))) + n_-(\varepsilon;2\mathrm{Re}\,(I-\mathbf{P}(\mu))\mathbf{T}_\lambda^-(\mu)\mathbf{P}(\mu)). \tag{3.7.5}$$

Assume that $\mu > 0$ is large enough. Then Proposition 3.13 entails

$$n_-(1;\mathbf{T}_\lambda^+(\mu)) = n_-(1-\varepsilon;(I-\mathbf{P}(\mu))\mathbf{T}_\lambda^-(\mu)(I-\mathbf{P}(\mu))) = n_-(\varepsilon;2\mathrm{Re}\,(I-\mathbf{P}(\mu))\mathbf{T}_\lambda^-(\mu)\mathbf{P}(\mu)) = 0. \tag{3.7.6}$$

Putting together (3.7.4)–(3.7.6), we get

$$N(\lambda;\mathbb{P}(\mu)) \ge n_-(1;\mathbf{P}(\mu))\mathbf{T}_\lambda^-(\mu)\mathbf{P}(\mu)), \tag{3.7.7}$$

$$N(\lambda;\mathbb{P}(\mu)) \le n_-(1-\varepsilon;\mathbf{P}(\mu)\mathbf{T}_\lambda^-(\mu)\mathbf{P}(\mu)), \forall\varepsilon\in(0,1). \tag{3.7.8}$$

Now notice that we have $\mathbf{P}(\mu)\mathbf{T}_\lambda^-(\mu)\mathbf{P}(\mu) = \mathbf{T}_\lambda(\mu)$. By assumption, $\lambda < 0$ is a continuity point of $\mathcal{K}_{\mathrm{Pauli}}$. Hence, $s = 1$ is continuity point of $\tilde{\mathcal{K}}_\lambda$, and estimate (3.7.7) combined with Corollary 3.14 yields

$$\liminf_{\mu\to\infty} \mu^{-1}N(\lambda;\mathbb{P}(\mu)) \ge \mathcal{K}_{\mathrm{Pauli}}(\lambda). \tag{3.7.9}$$

Choose a sequence $\{\varepsilon_l\}_{l\ge1}$ such that $\lim_{l\to\infty}\varepsilon_l = 0$, and for each $l \ge 1$ we have $\varepsilon_l \in (0,1)$, and $s = 1-\varepsilon_l$ is a continuity point of $\tilde{\mathcal{K}}_\lambda$. Then (3.7.8) entails

$$\limsup_{\mu\to\infty} \mu^{-1}N(\lambda;\mathbb{P}(\mu)) \le \tilde{\mathcal{K}}_\lambda(1-\varepsilon_l), \quad l\ge1.$$

Letting $l \to \infty$, and taking into account that $s = 1$ is a continuity pont of $\tilde{\mathcal{K}}_\lambda$, we get

$$\limsup_{\mu \to \infty} \mu^{-1} N(\lambda; \mathbb{P}(\mu)) \leq \tilde{\mathcal{K}}_\lambda(1) = \mathcal{K}_{\text{Pauli}}(\lambda). \tag{3.7.10}$$

Combining (3.7.9) and (3.7.10), we obtain (3.1.15) in the case $V \in C_0^\infty(\mathbb{R}^3)$. $\diamondsuit$

In order to prove (3.1.15) in full generality, we assume at first that $V \in L^{3/2}(\mathbb{R}^3)$, approximate it by an appropriate sequence $V_{0,l} \in C_0^\infty(\mathbb{R}^3)$ and apply Proposition 3.15. Here we use also estimates (3.7.1)–(3.7.2), as well as (3.6.1). Finally, we assume $V \in \mathcal{L}_{3/2}$, approximate it in $L^\infty(\mathbb{R}^3)$ by an appropriate sequence $V_{1,l} \in L^{3/2}(\mathbb{R}^3)$, and apply the previous result. The details concerning these two approximations can be found in [R 5, Section 8].

4 Spectral asymptotics for quantum Hamiltonians in strong magnetic fields: precise theorems

The method used to compute the main term in the asymptotic of the counting spectral function in Section 3 is simple, works in general situations, and seems to be the best if one does not care about precise remainder estimates (see [Iv, Section 0.2, p.2]).
On the contrary, to get an asymptotic formula involving sharp remainder estimates, one needs to consider some function $\Phi(T, t)$ of an operator T and an auxiliary parameter t. One then tries to construct $\Phi(T, t)$ by means of pseudo-differential operator theory, which involves some regularity conditions. The case where T is a ΨDO depending on the small semiclassical parameter $\hbar$ (or, in brief, a $\hbar$-ΨDO) has been studied extensively by this method during the last thirty years. Asymptotic expansions containing several terms were obtained in many papers [Iv], [He.Ro], [Ch], [D.Sj].
In the two-dimensional and three-dimensional cases, the asymptotics as $\hbar \downarrow 0$ and $\hbar\mu < 1$ with precise remainder estimate for the counting spectral function of the operator $\mathbb{H}(\hbar, \mu)$, have been obtained by Ivrii [Iv]. The method of Ivrii uses strongly the fact that $\kappa_q = 1$ (see (2.3.23)) if $d = 1$, i.e. if $m = 2, 3$. Ivrii constructs a microlocal canonical form for $\mathbb{H}(\hbar, \mu)$, which leads to the sharp remainder estimates. For $m \geq 4$, a complete reduction similar to the one obtained in [Iv] seems to be difficult [Iv, Problem 6.2, p.365]. Then, one can not hope to apply directly the methods of [Iv] for $\mathbb{H}(\mu)$.
Here, we shall make use of a strong field reduction onto the qth eigenfunction of the harmonic oscillator and a well-posed Grushin problem for $\mathbb{H}(\mu)$. We show that the spectral study of $\mathbb{H}(\mu)$ near some energy level z can be reduced to the study of an $\frac{1}{\mu}$-ΨDO $E_{-+}(z)$ called *the effective Hamiltonian.* Then, sharp remainder estimates can be obtained by the methods of [D.Sj, Chapter 12] and [D].

4.1 Formulations of main results

Throughout the section we assume that, $k \equiv \dim \operatorname{Ker} B = 0$ and V is a smooth potential tending to zero at the infinity and bounded with all its derivatives.
Fix the real numbers λ_1 and λ_2 such that $\lambda_1 < \lambda_2$ and $\lambda_1\lambda_2 > 0$. For $q \geq 1$, set $\mathbb{H}_0^{(q)}(\mu) = \mathbb{H}_0(\mu) - \mu\Lambda_q$, $\mathbb{H}^{(q)}(\mu) = \mathbb{H}(\mu) - \mu\Lambda_q$. In this section, we will give an asymptotic expansion in powers of μ^{-1} of $\operatorname{Tr}(\phi(\mathbb{H}^{(q)}(\mu), \mu))$ in the two following cases:
a) $\Phi(x, \mu) = \phi(x)$, where $\phi \in C_0^\infty((\lambda_1, \lambda_2)\,; \mathbb{R})$.
b) $\Phi(x, \mu) = \phi(x)\widehat{\theta}(\mu(x - \tau))$, where $\theta \in C_0^\infty(\mathbb{R}; \mathbb{R})$, $\tau \in \mathbb{R}$, and $\widehat{\theta}$ is the Fourier transform of θ.
As a consequence, we get a sharp remainder estimates for the counting spectral function of $\mathbb{H}^{(q)}$ when $\mu \to \infty$. Let us state the results precisely.

Theorem 4.1 *Let $\phi \in C_0^\infty((\lambda_1, \lambda_2)\,;\mathbb{R})$. There exists a sequence of real numbers $(a_j)_{j\in\mathbb{N}}$, such that*

$$\mathrm{Tr}\big(\phi(\mathbb{H}^{(q)}(\mu))\big) \sim \mu^d \sum_{j=0}^{\infty} a_j \mu^{-j}, \;\; \mu \to \infty, \tag{4.1.1}$$

with

$$a_0 = \kappa_q \frac{b_1 ... b_d}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \phi(V(X))dX, \tag{4.1.2}$$

$$a_1 = \frac{b_1 ... b_d}{(2\pi)^d} \sum_{j=1}^{\kappa_q} \sum_{i=1}^{d} \frac{2n_i^j + 1}{4b_i} \int_{\mathbb{R}^{2d}} \Delta_{x_i, y_i} V(X) \phi'(V(X))dX, \tag{4.1.3}$$

the d-uplet $(n_1^j, ..., n_d^j)$, $j \in \{1, ..., \kappa_q\}$ satisfying $\sum_{i=1}^d b_i(2n_i^j + 1) = \Lambda_q$.

Let $\theta \in C_0^\infty(\mathbb{R})$, and let ϵ be a positive constant. Set

$$\breve{\theta}(\tau) = \frac{1}{2\pi} \int e^{it\tau} \theta(t)dt, \quad \breve{\theta}_\epsilon(t) = \frac{1}{\epsilon} \breve{\theta}(\frac{t}{\epsilon}).$$

In the sequel we shall say that λ is not a critical value of V if and only if $V(X) = \lambda$ for some $X \in \mathbb{R}_{2d}$ implies $\nabla V(X) \neq 0$.

Theorem 4.2 *Fix $\lambda \neq 0$ which is not a critical value of V. Let $\phi \in C_0^\infty((\lambda - \epsilon, \lambda + \epsilon)\,;\mathbb{R})$ and $\theta \in C_0^\infty(\big(-\frac{1}{C}, \frac{1}{C}\big)\,;\mathbb{R})$. Then there exist $\epsilon > 0$, $C > 0$ and a functional sequence $c_j \in C^\infty(\mathbb{R};\mathbb{R})$, $j \in \mathbb{N}$, such that for all $M, N \in \mathbb{N}$, we have*

$$\mathrm{Tr}\big(\phi(\mathbb{H}^{(q)}(\mu))\breve{\theta}_{\frac{1}{\mu}}(\tau - \mathbb{H}^{(q)}(\mu))\big) = \mu^d \Big(\sum_{j=0}^{M} c_j(\tau)\mu^{-j} + \mathcal{O}\left(\frac{\mu^{-M-1}}{\langle \tau \rangle^N}\right) \Big), \;\; \mu \to \infty, \tag{4.1.4}$$

uniformly in $\tau \in \mathbb{R}$. In particular,

$$c_0(\tau) = \kappa_q \frac{b_1 ... b_d}{(2\pi)^d} f(\tau)\, \theta(0) \int_{\{X \in \mathbb{R}^{2d} | V(X) = \tau\}} \frac{dS_\tau}{|\nabla_X V(X)|}. \tag{4.1.5}$$

We recall that $\kappa_q \frac{b_1 ... b_d}{(2\pi)^d} = \mathcal{C}_q(B)$ (see (2.3.22)).

Corollary 4.1 *Choose $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 < \lambda_2$, $\lambda_1\lambda_2 > 0$, and assume that λ_1, λ_2 are not critical values of V. Then we have*

$$\mathcal{N}(\mu\Lambda_q + \lambda_1, \mu\Lambda_q + \lambda_2; \mathbb{H}(\mu)) = \mu^d \kappa_q \big(k_{\mathrm{Schr}}(\lambda_2) - k_{\mathrm{Schr}}(\lambda_2)\big) + \mathcal{O}(\mu^{d-1}), \;\; \mu \to \infty. \tag{4.1.6}$$

Asymptotic formula (4.1.6) is a sharper version of (3.1.3) since it contains the remainder estimate $\mathcal{O}(\mu^{d-1})$. However, the hypotheses of Corollary 4.1 are much more restrictive than those of Theorem 3.1. In particular, the assumption that λ_j, $j = 1, 2$ are not critical points, is much stronger than the assumption that λ_j are continuity points of $\mathrm{k_{Schr}}$, i.e. that $\mathrm{vol}\, \{X \in \mathbb{R}^{2d} | V(X) = \lambda_j\} = 0$.

4.2 Classes of symbols

Let $\hbar \in (0, \hbar_0)$, where $\hbar_0$ is a small positive constant. For $k \in \mathbb{R}$ and $\delta \in (0,1)$ we denote by $S_\delta^k(\mathbb{R}^{2d})$ the space of functions $a(y, \eta; \hbar)$ defined on $\mathbb{R}^{2d} \times (0, \hbar_0)$ which are smooth in (y, η), and for all $\alpha, \beta \in \mathbb{N}^d$ we have

$$|\partial_y^\alpha \partial_\eta^\beta a(y, \eta; \hbar)| \le C_{\alpha,\beta}\, \hbar^{-\delta(|\alpha|+|\beta|)-k} \tag{4.2.1}$$

with $C_{\alpha,\beta} > 0$ independent of $\hbar \in (0, \hbar_0)$. By $S^0(\mathbb{R}^{2d})$ we denote the space $S^0_0(\mathbb{R}^{2d})$. If $a = a(y, \eta; z, \hbar)$ depends also on $z \in \Omega \subset \mathbb{C}$, we say that $a \in S^k_\delta(\mathbb{R}^{2d})$, if (4.2.1) holds uniformly for $z \in \Omega$.
Let $a(y, \eta; z, \hbar) \in S^0(\mathbb{R}^{2d})$. We shall say that $a(y, \eta; z, \hbar)$ has an asymptotic expansion in powers of $\hbar$ in $S^0(\mathbb{R}^{2d})$, and we write $a(y, \eta; z, \hbar) \sim \sum_{j=0}^{\infty} a_j(y, \eta; z)\hbar^j$ in $S^0(\mathbb{R}^{2d})$, if for every $N \in \mathbb{N}$ we have $\hbar^{-(N+1)}(a - \sum_{j=0}^{N} a_j\hbar^j) \in S^0(\mathbb{R}^{2d})$.
In what follows, we denote by $a(y, \hbar D_y; \hbar)$ the $\hbar$-ΨDO with Weyl symbol a (see (1.2.4)). In order to prove the above results, we shall recall some well-known results [D.Sj, Chapters 7-8].

Proposition 4.1 *Let $a_i \in S^{m_i}_\delta(\mathbb{R}^{2d})$, $i = 1, 2$, $\delta \in [0, \frac{1}{2})$. Then $b(y, \hbar D_y; \hbar) = a_1(y, \hbar D_y) \circ a_2(y, \hbar D_y)$ is an $\hbar$-pseudodifferential operator, and*

$$b(y, \eta; \hbar) \sim \sum_{j=0}^{\infty} b_j(y, \eta)\hbar^j, \text{ in } S^{m_1+m_2}_\delta(\mathbb{R}^{2d}) \tag{4.2.2}$$

with $b_j(y, \eta) = \sum_{|\alpha|+|\beta|=j} \frac{(-1)^\beta}{2^j \alpha!\beta!} \partial^\alpha_\eta D^\beta_y a_1 \partial^\alpha_\eta D^\beta_y a_2$.

Proposition 4.2 *Let $A = A_\hbar : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, $0 < \hbar \leq 1$. The following two statements are equivalent:*
(1) $A = a(y, \hbar D_y; \hbar)$, for some $a = a(y, \eta; \hbar) \in S^0(1)$.
(2) For every $N \in \mathbb{N}$ and for every sequence $l_1(y, \eta), \ldots, l_N(y, \eta)$ of linear forms on $\mathbb{R}^{2d}$, the operator $\mathrm{ad}_{l_1(y,\hbar D_y)} \circ \ldots \circ \mathrm{ad}_{l_N(y,\hbar D_y)} A_\hbar$ belongs to $\mathcal{L}(L^2, L^2)$ and is of norm $\mathcal{O}(\hbar^N)$ in that space. Here, $\mathrm{ad}_A B := [A, B] = AB - BA$.

4.3 Reduction to a semi-classical problem

For the simplicity of the notations, we only treat the two-dimensional case $m = 2$. Without any loss of generality we may assume that $b_1 = 1$.

Lemma 4.1 *There exists a unitary operator $\widetilde{W}_\mu : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ such that*

$$\mathbb{H}^{(q)}(\mu) = \widetilde{\mathcal{W}}_\mu \hat{\mathbb{H}}^{(q)}(\mu) \widetilde{\mathcal{W}}^*_\mu \tag{4.3.1}$$

where $\hat{\mathbb{H}}^{(q)}(\mu) := \mu\hat{\mathbb{H}}^{(q)}_0 + \mathcal{V}(\hbar^{1/2})$, $\hat{\mathbb{H}}_{0,q} := \hat{\mathbb{H}}_0 - \Lambda_q \equiv \hat{\mathbb{H}}_0 - (2q-1)$, $q \in \mathbb{N}_*$, the operator $\hat{\mathbb{H}}_0 := -\frac{\partial^2}{\partial x^2} + x^2$ is described in (2.3.10) and (2.3.12), $\mathcal{V}(\hbar^{1/2}) := V(\hbar D_y + \hbar^{1/2} D_x, y - \hbar^{1/2} x)$, and $\hbar := \mu^{-1}$ is our effective *semi-classical parameter.*

Proof: The linear symplectic mapping

$$\widetilde{S}_\mu(\mathbf{x}, \mathbf{p}) := \left(\frac{1}{\sqrt{\mu}}\xi + \frac{1}{\mu}\eta, y - \frac{1}{\sqrt{\mu}x}, -\frac{\sqrt{\mu}}{2}x - \frac{\mu}{2}y, -\frac{\sqrt{\mu}}{2}\xi + \frac{1}{2}\eta\right), \ \mathbf{x} = (x, y), \ \mathbf{p} = (\xi, \eta),$$

maps the Weyl symbol of the operator $\mathbb{H}^{(q)}(\mu)$ into the Weyl symbol of the operator $\hat{\mathbb{H}}^{(q)}(\mu)$ By Lemma 2.5, there exists a unitary operator $\widetilde{W}_\mu : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ such that $\widetilde{W}_\mu : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is valid. $\diamondsuit$
Remark: We have $\widetilde{S}_\mu(\mathbf{x}, \mathbf{p}) = (S_\mu \circ S'_\mu)(\mathbf{x}, \mathbf{p})$ where S_μ is the linear symplectic mapping defined in (2.3.9), and $S'_\mu(\mathbf{x}, \mathbf{p}) := \left(\xi, -\sqrt{\mu}y, -x, -\frac{1}{\sqrt{\mu}}\eta\right)$, $\mathbf{x} = (x, y)$, $\mathbf{p} = (\xi, \eta)$. Then the operator $\mathcal{W}'_\mu : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined by

$$(\mathcal{W}'_\mu u)(x, y) = \frac{1}{(2\pi\sqrt{\mu})^{1/2}} \int_{\mathbb{R}} e^{-ixx'} u(x', -\sqrt{\mu}y) dx',$$

is a unitary operator generated by the linear symplectic mapping S'_μ according to Lemma 2.5, and $\widetilde{\mathcal{W}}_\mu = \mathcal{W}'_\mu \mathcal{W}_\mu$. Notice that the operator $\mathcal{W}'_\mu$ commutes with $\hat{\mathbb{H}}_0$. Therefore, we have $\mu\hat{\mathbb{H}}_0(\mu) = \mathcal{W}^*_\mu \hat{\mathbb{H}}_0(\mu)\mathcal{W}_\mu$ *and* $\mu\hat{\mathbb{H}}_0(\mu) = \widetilde{\mathcal{W}}^*_\mu \hat{\mathbb{H}}_0(\mu)\widetilde{\mathcal{W}}_\mu$ (see the remark after Lemma 2.5).

Introduce the operator $R_-^{(q)} : L^2(\mathbb{R}_x) \to L^2(\mathbb{R}^2_{x,y})$ by

$$(R_-^{(q)} v)(x,y) = f_q(x)v(y),$$

where f_q is the qth normalized eigenfunction of harmonic oscillator (see (2.3.11)). Further, the operator $R_+^{(q)} : L^2(\mathbb{R}^2_{x,y}) \to L^2(\mathbb{R}_y)$ is defined by

$$(R_+^{(q)} u)(y) = \int f_q(x)u(x,y)dx.$$

Notice that $R_+^{(q)}$ is the adjoint of $R_-^{(q)}$ and, moreover, $R_+^{(q)}R_-^{(q)} = I_{L^2(\mathbb{R})}$ and $R_-^{(q)}R_+^{(q)} = \hat{p}_q$ (see 2.3.13).
Consider the Grushin operator associated with $\hbar\hat{\mathbb{H}}_0^{(q)}(\mu)$

$$\mathcal{H}_{0,q}(z) := \begin{pmatrix} \hbar\hat{\mathbb{H}}_0^{(q)}(\mu) - z & R_-^{(q)} \\ R_+^{(q)} & 0 \end{pmatrix} : \mathrm{Dom}(\hat{\mathbb{H}}_0^{(q)}(\mu)) \oplus L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R})$$

For $|z| < 1$, let $E_0(z)$ denote the reduced resolvent of $(I - \hat{p}_q)\hat{\mathbb{H}}_0^{(q)}(\mu)(I - \hat{p}_q)$ on the range of $I - \hat{p}_q$, i.e $E_0(z) = (z - (I - \hat{p}_q)\hat{\mathbb{H}}_0^{(q)}(\mu)(I - \hat{p}_q))^{-1}(I - \hat{p}_q)$. Evidently, $\mathcal{H}_{0,q}(z)$ is invertible with inverse

$$\mathcal{E}_{0,q}(z) := \begin{pmatrix} E_0(z) & R_-^{(q)} \\ R_+^{(q)} & z \end{pmatrix}.$$

Now, consider the Grushin operator associated with $\hbar\hat{\mathbb{H}}^{(q)}(\mu)$, i.e.

$$\mathcal{H}_q(z) := \begin{pmatrix} \hbar\hat{\mathbb{H}}^{(q)}(\mu) - z & R_-^{(q)} \\ R_+^{(q)} & 0 \end{pmatrix} : \mathrm{Dom}(\hat{\mathbb{H}}_0^{(q)}(\mu)) \oplus L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R})$$

Proposition 4.3 *For $\hbar$ small enough, the operator $\mathcal{H}_q(z)$ is invertible for $|z| < 1$, and the inverse is given by*

$$\mathcal{E}_q(z) := \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}$$

where $E_{-+}(z)$ is an $\hbar$-ΨDO, and $\hbar^{-1}E_{-+}(\hbar z)$ has an asymptotic expansion in powers of $\hbar$ in $S^0(\mathbb{R}^2)$:

$$G(z,\hbar) := \hbar^{-1}E_{-+}(\hbar z) \sim \sum_{j\geq 0} \hbar^j Q_j(y,\eta,z), \tag{4.3.2}$$

with

$$Q_0(y,\eta,z) = z - V(\eta,y), \tag{4.3.3}$$

$$Q_1(y,\eta,z) = \frac{2q-1}{4}\left(\frac{\partial^2 V}{\partial^2 y}(\eta,y) + \frac{\partial^2 V}{\partial^2 \eta}(\eta,y)\right). \tag{4.3.4}$$

Proof: Since $\hbar\hat{\mathbb{H}}^{(q)}(\mu) = \hat{\mathbb{H}}_0^{(q)}(\mu) + \hbar V(\hbar^{\frac{1}{2}})$, for $\hbar$ small enough $\mathcal{H}_q(z)$ is invertible, and

$$\mathcal{H}_q(z)^{-1} = \mathcal{E}_q(z) = \left(I + \hbar\mathcal{E}_{0,q}(z)\begin{pmatrix} V(\hbar^{\frac{1}{2}}) & 0 \\ 0 & 0 \end{pmatrix}\right)^{-1} \mathcal{E}_{0,q}(z). \tag{4.3.5}$$

So we can give explicit formulae for $E(z), E_\pm(z)$ and $E_{-+}(z)$. In what follows , the explicit formulae for $E(z)$ and $E_\pm(z)$ are not needed. We just indicate that they are holomorphic in z. Let us write down the formula for $E_{-+}(z)$. Employing (4.3.5), we deduce that

$$E_{-+}(z) - z = \sum_{j=1}^{\infty} (-\hbar)^j R_+ V(\hbar^{\frac{1}{2}}) \big(E^0(z) V(\hbar^{\frac{1}{2}})\big)^{j-1} R_-.$$

Making use of Proposition 4.2, we prove easily that $G(z,\hbar) \in S^0(\mathbb{R}^{2n})$.
On the other hand, writing

$$V(h^{\frac{1}{2}}) = V(\hbar D_y, y) + \hbar^{\frac{1}{2}} D_x \frac{\partial V}{\partial x}(\hbar D_y, y) - \hbar^{\frac{1}{2}} x \frac{\partial V}{\partial y}(\hbar D_y, y) + \hbar(\,) + ...,$$

and using (4.2.2), we see that $G(z,\hbar)$ has an asymptotic expansion in powers of $\hbar$. Notice that the half-integer powers of $\hbar$ disappear, due to the special properties of eigenfunction of the harmonic oscillator $f_q(x)$ (see [D, Proposition 2.5]). $\diamondsuit$
Using the fact that $\mathcal{E}_q(z)$ is a left and right inverse of $\mathcal{H}_q(z)$ as well as the fact that $R_\pm^{(q)}$ are independent of z, we establish

Lemma 4.2 (cf. [He.Sj]) *We have*

$$\partial_z E_{-+}(z) = E_-(z) E_+(z). \tag{4.3.6}$$

If $z \notin \sigma(\hbar \hat{\mathbb{H}}^{(q)}(\mu))$, *then*

$$(z - \hbar\hat{\mathbb{H}}^{(q)}(\mu))^{-1} = E(z) - E_+(z) E_{-+}(z)^{-1} E_-(z). \tag{4.3.7}$$

4.4 Proof of Theorem 4.1

Let T be a self-adjoint operator, and $\phi \in C_0^\infty((\lambda_1,\lambda_2)\,;\mathbb{R})$. Then the functional calculus due to Helffer-Sjöstrand (see e.g. [D.Sj, Chapter 8]) yields

$$\phi(T) = -\frac{1}{\pi} \int \overline{\partial}\widetilde{\phi}(z)(z-T)^{-1} L(dz). \tag{4.4.1}$$

Here $L(dz) = dxdy$ is the Lebesgue measure on the complex plane $\mathbb{C} \sim \mathbb{R}^2_{x,y}$, and $\widetilde{\phi} \in C_0^\infty((\lambda_1,\lambda_2) + i]-1,1[)$ is an almost analytic extension of ϕ, i.e. $\widetilde{\phi} = \phi$ on $\mathbb{R}$ and $\overline{\partial}\widetilde{\phi}$ vanishes on $\mathbb{R}$ to infinite order, i.e. $\overline{\partial}\widetilde{\phi}(z) = \mathcal{O}_N(|\mathrm{Im}\, z|^N)$ for all $N \in \mathbb{N}$.

Proposition 4.4 *For* $\mu = \hbar^{-1}$ *large enough,*

$$\mathrm{Tr}\,(\phi(\mathbb{H}^{(q)}(\mu))) = \mathrm{Tr}\,\left(-\frac{1}{\pi}\int \overline{\partial}\widetilde{\phi}(z) G(z,\hbar)^{-1} \partial_z G(z,\hbar) L(dz)\, \chi(y,\hbar D_y)\right) + \mathcal{O}(\hbar^\infty),$$

where $\chi \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$ *is equal to* 1 *in a neighbourhood of* $\Sigma_{\lambda_1,\lambda_2} := \{(y,\eta) \in \mathbb{R}^2; V(\eta,y) \in (\lambda_1,\lambda_2)\}$.

Proof: Applying (4.4.1) to $\hat{\mathbb{H}}^{(q)}(\mu)$, and using that $\hat{\mathbb{H}}^{(q)}(\mu) = \widetilde{\mathcal{W}}_\mu^* \mathbb{H}^{(q)}(\mu) \widetilde{\mathcal{W}}_\mu$ (see 4.3.1), we get:

$$\widetilde{\mathcal{W}}_\mu^* \phi(\mathbb{H}^{(q)}(\mu)) \widetilde{\mathcal{W}}_\mu = \phi(\hat{\mathbb{H}}^{(q)}(\mu)) = -\frac{\hbar}{\pi} \int \overline{\partial}\widetilde{\phi}(z)\,(\hbar z - \hbar\hat{\mathbb{H}}^{(q)}(\mu))^{-1} L(dz).$$

Replacing $(\hbar z - \hbar\hat{\mathbb{H}}^{(q)}(\mu))^{-1}$ by the right hand side of (4.3.7), and using the fact that $E(\hbar z)$ is holomorphic in z, we obtain

$$\phi(\hat{\mathbb{H}}^{(q)}(\mu)) = -\frac{1}{\pi} \int \overline{\partial}\widetilde{\phi}(z) E_+(\hbar z,\hbar) G(z,\hbar)^{-1} E_-(\hbar z,\hbar) L(dz). \tag{4.4.2}$$

Let $\widetilde{V} \in S^0(\mathbb{R}^2)$ be a real-valued function coinciding with V for large X, and having the property that $|z - \widetilde{V}| > c > 0$ uniformly on $z \in \text{supp}\, \widetilde{\phi}$ and $X \in \mathbb{R}^2$. Then for sufficiently small $\hbar > 0$, the operator $\widetilde{G}(z,\hbar) := G(z,\hbar) + V(\hbar D_y, y) - \widetilde{V}(\hbar D_y, y)$, is elliptic, and $(z - \widetilde{G})^{-1}$ is well defined and holomorphic for z in some fixed complex neighbourhood of supp $\widetilde{\phi}$. Hence, by an integration by parts, we get

$$-\frac{1}{\pi}\int \overline{\partial}\widetilde{\phi}(z) E_+(\hbar z,\hbar)\widetilde{G}(z,\hbar)^{-1}E_-(\hbar z,\hbar)L(dz) = 0.$$

Combining this with (4.4.2) and using the resolvent identity for Im $z \neq 0$

$$G(z,\hbar)^{-1} = \widetilde{G}(z,\hbar)^{-1} + G(z,\hbar)^{-1}(\widetilde{G}(z,\hbar) - G(z,\hbar))\widetilde{G}(z,\hbar)^{-1},$$

we obtain

$$\text{Tr}(\phi(\mathbb{H}^{(q)}(\mu))) = \text{Tr}\big(-\frac{1}{\pi}\int \overline{\partial}\widetilde{\phi}(z)E_+(\hbar z,\hbar)G^{-1}(\widetilde{G} - G)\widetilde{G}^{-1}E_-(\hbar z,\hbar)L(dz)\big). \tag{4.4.3}$$

Since the symbol of $G - \widetilde{G} = V - \widetilde{V}$ is in $C_0^\infty(\mathbb{R}^2)$, we have $G - \widetilde{G} \in \mathcal{S}_1$. It is then clear that we can permute integration and the operator "Tr" in (4.4.3). Using the property of cyclic invariance of the trace, and applying (4.3.4), we get

$$\text{Tr}(E_+(\hbar z,\hbar)G^{-1}(\widetilde{G} - G)\widetilde{G}^{-1}E_-(\hbar z,\hbar)) = \text{Tr}(G^{-1}(\widetilde{G} - G)\widetilde{G}^{-1}\partial_z G(z,\hbar)).$$

Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be equal to 1 in a neighbourhood of supp $(G - \widetilde{G})$. Using the composition formula for two $\hbar - \Psi$DOs with Weyl symbols (see for instance [D.Sj, Chapter 7]), we see that all the derivatives of the symbol of the operator $(G(z,\hbar) - \widetilde{G}(z,\hbar))(z - \widetilde{G}(z,\hbar))^{-1}(1 - \chi(y,\hbar D_y))$ are $\mathcal{O}(\hbar^N\langle(y,\eta)\rangle^{-N})$ for every $N \in \mathbb{N}$. The $\mathcal{S}_1$-norm of this expression is therefore $\mathcal{O}(\hbar^\infty)$, and consequently

$$\text{Tr}(E_+(\hbar z,\hbar)G^{-1}(\widetilde{G} - G)\widetilde{G}^{-1}E_-(\hbar z,\hbar)) = \text{Tr}(G^{-1}(\widetilde{G} - G)\widetilde{G}^{-1}\partial_z G\chi) + \mathcal{O}(\hbar^\infty). \tag{4.4.4}$$

Inserting (4.4.4) in (4.4.2), and using the fact that $\widetilde{G}(z,\hbar)^{-1}\partial_z G(z,\hbar)$ is holomorphic on z we obtain Proposition 4.4. $\diamondsuit$

Proposition 4.5 *Fix δ in $\left(0,\frac{1}{2}\right)$. Then $A_{\phi,G} := -\frac{1}{\pi}\int_{|\text{Im}z|\geq\hbar^\delta}\overline{\partial}\widetilde{\phi}(z)G(z,\hbar)^{-1}\partial_z G(z,\hbar)L(dz)$ is an $\hbar$-ΨDO, and its symbol admits the following asymptotic expansion*

$$A_{\phi,G} = \phi(V(\eta,y)) + \hbar\phi'(V(\eta,y))Q_1(y,\eta)) + \hbar^2 ..., \tag{4.4.5}$$

Proof: For $|\text{Im } z| > \hbar^\delta, (0 < \delta < \frac{1}{2})$, $G(z,\hbar)^{-1}$ is a $\hbar$-ΨDO whose symbol admits the asymptotic expansion $G(z,\hbar)^{-1} \sim \sum_{j\geq 0}\hbar^j F_j$ in the sense that $G(z,\hbar)^{-1} - \sum_{j=0}^N \hbar^j F_j \in S_\delta^{-(N+1)(1-2\delta)}$ for each $N \in \mathbb{N}$. Here $F_0 = \frac{1}{V-z} \in S_\delta^\delta$, $F_j = \frac{\widetilde{Q}_j}{(V-z)^{2j}} \in S_\delta^{-j(1-2\delta)}$, $j > 0$, $\widetilde{Q}_j$ are functions of Q_i with $i \leq j$, and their derivatives. In particular, $\widetilde{Q}_1 = -Q_1$. Recall that $G(z,\hbar) \sim \sum_{j\geq 0}\hbar^j Q_j$. Since $\partial_z G(z,\hbar)$ is also in class S^0 and has the asymptotic expansion

$$\partial_z G(z,\hbar) = 1 + \hbar\partial_z Q_1(y,\eta,z) + \hbar^2\partial_z Q_2(y,\eta,z) + ...,$$

while

$$G(z,\hbar)^{-1} \circ \partial_z G(z,\hbar)) = (z - V(\eta,y))^{-1} + h((z - V(\eta,y))^{-2}Q_1(y,\eta) + ... + \hbar^m\widetilde{F}_m(y,\eta;z) + ... \tag{4.4.6}$$

where $\widetilde{F}_m$ has the same properties as F_m. Since $\overline{\partial}\widetilde{\phi}(z) = \mathcal{O}(|\text{Im}z|^\infty)$, it follows that

$$\frac{1}{\pi}\int_{|\text{Im } z|\geq\hbar^\delta}\overline{\partial}\widetilde{\phi}(z)\widetilde{F}_m(y,\eta;z)L(dz) \in S^0(\mathbb{R}^2),$$

for all $m \in \mathbb{N}$. Combining this with (4.4.6) and using the fact that

$$-\frac{1}{\pi}\int \overline{\partial}\widetilde{\phi}(z-V)^{-2j-1}L(dz) = \frac{1}{(2j)!}\phi^{(2j)}(V),$$

we get (4.4.5). $\diamondsuit$

End of the proof of Theorem 4.1: If we restrict the integral in Proposition 4.4 to the domain $|\text{Im } z| \leq \hbar^{\delta}$ we get a term of order $\mathcal{O}(\hbar^{\infty})$. This follows easily from the fact that $\overline{\partial}\widetilde{\phi}(z) = \mathcal{O}(|\text{Im } z|^{\infty})$. Hence, it suffices to compute the trace of the operator $A_{\phi,G}$ given by Proposition 4.5.
Recall that, if $a \in S^0(\mathbb{R}^{2d})$ with $\partial_y^{\alpha}\partial_{\eta}^{\beta} a \in L^1(\mathbb{R}^{2d})$ for all $\alpha, \beta \in \mathbb{N}^d$ such that $|\alpha| + |\beta| \leq 2d+1$, then $a(y, hD_y) \in \mathcal{S}_1$, and

$$\text{Tr } a(y,\hbar D_y) = \frac{1}{(2\pi\hbar)^d}\int_{Y\in\mathbb{R}^{2d}} a(Y)dY \tag{4.4.7}$$

(see [D.Sj, Chapter 9]). Now, (4.1.1), (4.1.2) and (4.1.3) follow from (4.3.4), (4.4.5) and (4.4.7). $\diamondsuit$
The proof of Theorem 4.2 is similar to that of Theorem 4.1. We refer the reader to [D] for details.

4.5 Proof of Corollary 4.1

Pick $\sigma > 0$ small enough, so that $(\lambda_1 - \sigma)(\lambda_2 + \sigma) > 0$. Let $\phi_1 \in C_0^{\infty}((\lambda_1 - \sigma, \lambda_1 + \sigma); (0,1))$, $\phi_2 \in C_0^{\infty}(\big(\lambda_1 + \frac{\sigma}{2}, \lambda_2 - \frac{\sigma}{2}\big); (0,1))$, $\phi_3 \in C_0^{\infty}((\lambda_2 - \sigma, \lambda_2 + \sigma); (0,1))$ satisfy $\phi_1 + \phi_2 + \phi_3 = 1$ on $\big(\lambda_1 - \frac{\sigma}{2}, \lambda_2 + \frac{\sigma}{2}\big)$. Let $\gamma_0(\hbar) \leq \gamma_1(\hbar) \leq .. \leq \gamma_N(\hbar)$ be the eigenvalues of $\mathbb{H}^{(q)}(\mu)$ counted with the multiplicities and lying in the interval $(\lambda_1 - \sigma, \lambda_2 + \sigma)$. We have

$$\begin{aligned}\mathcal{N}(\mu\Lambda_q + \lambda_1, \mu\Lambda_q + \lambda_2; \mathbb{H}(\mu)) &= \sum_{\lambda_1 \leq \gamma_j(\hbar) \leq \lambda_2} (\phi_1 + \phi_2 + \phi_3)(\gamma_j(\hbar)) \\ &= \sum_{\lambda_1 \leq \gamma_j(\hbar)} \phi_1(\gamma_j(\hbar)) + \sum \phi_2(\gamma_j(\hbar)) + \sum_{\gamma_j(\hbar) \leq \lambda_2} \phi_3(\gamma_j(\hbar)) \\ &= \sum_{\lambda_1 \leq \gamma_j(\hbar)} \phi_1(\gamma_j(\hbar)) + \text{Tr}(\phi_2(\mathbb{H}^{(q)}(\mu)) + \sum_{\gamma_j(\hbar) \leq \lambda_2} \phi_3(\gamma_j(\hbar)).\end{aligned} \tag{4.5.1}$$

According to Theorem 4.1, we have

$$\text{Tr}\big(\phi_i(\mathbb{H}^{(q)}(\mu))\big) = \kappa_q \frac{b_1 ... b_d}{(2\pi\hbar)^d}\Big(\int_{\mathbb{R}^{2d}} \phi_i(V(X))dX + \mathcal{O}(\hbar)\Big), \quad i = 1,2,3; \tag{4.5.2}$$

we recall that $\hbar = \mu^{-1}$. Set $M(\tau;\hbar) := \sum_{\gamma_j(\hbar)\leq\tau} \phi_3(\gamma_j(\hbar))$. Evidently:

$$\mathcal{M}(\tau) := M'(\tau,\hbar) = \sum_j \delta(\tau - \mu_j(\hbar))\phi_3(\gamma_j(\hbar)). \tag{4.5.3}$$

In what follows, we choose $\theta \in C_0^{\infty}(\big(-\frac{1}{C}, \frac{1}{C}\big); [0,1])$, $(C >> 1\ )$, such that,

$$\begin{cases} \theta(0) = 1, \\ \breve{\theta}(t) \geq 0, t \in \mathbb{R}, \\ \breve{\theta}(t) \geq \epsilon_0, t \in [-\delta_0, \delta_0] \quad \text{for some} \quad \delta_0 > 0, \epsilon_0 > 0. \end{cases} \tag{4.5.4}$$

Corollary 4.2 *There is $C_0 > 0$, such that, for all $(\lambda, \hbar) \in \mathbb{R} \times (0, \hbar_0)$, we have:*

$$|M(\lambda + \delta_0\hbar, \hbar) - M(\lambda - \delta_0\hbar, \hbar)| \leq C_0\hbar^{1-d}.$$

Proof: Since $\phi_3 \geq 0$, it follows from (4.5.4) that

$$\frac{\epsilon_0}{\hbar} \sum_{\lambda-\delta_0\hbar \leq \gamma_j(\hbar) \leq \lambda+\delta_0\hbar} \phi_3(\gamma_j(\hbar)) \leq \sum_{|\lambda-\gamma_j(\hbar)|<\delta_0\hbar} \breve{\theta}_\hbar(\lambda-\gamma_j(\hbar))\phi_3(\gamma_j(\hbar)) \leq$$

$$\sum_j \breve{\theta}_\hbar(\lambda-\gamma_j(\hbar))\phi_3(\gamma_j(\hbar)) = \breve{\theta}_\hbar \star \mathcal{M}(\tau) = \mathrm{Tr}(\phi_3(\mathbb{H}^{(q)})\breve{\theta}_\hbar(\tau-\mathbb{H}^{(q)})). \tag{4.5.5}$$

Now Corollary 4.2 follows from (4.1.4). $\diamondsuit$

According to Corollary 4.2, we have

$$\int \langle \frac{\tau-\lambda}{\hbar} \rangle^{-2} \mathcal{M}(\tau)d\tau = \sum_{k\in\mathbb{Z}} \int_{\{k \leq \frac{\tau-\lambda}{\hbar} \leq k+1\}} \langle \frac{\tau-\lambda}{\hbar} \rangle^{-2} \mathcal{M}(\tau)d\tau \leq C_0(\sum_{k\in\mathbb{Z}} \langle k \rangle^{-2})\hbar^{1-d}. \tag{4.5.6}$$

On the other hand, since $\breve{\theta} \in S(\mathbb{R})$ and $\theta(0) = 1$, there exists $C_1 > 0$ such that:

$$|\int_{-\infty}^{\lambda} \breve{\theta}_\hbar(\tau-y)dy - 1_{(-\infty,\lambda)}(\tau)| = |\int_{\frac{\tau-\lambda}{\hbar}}^{+\infty} \breve{\theta}(y)dy - 1_{(-\infty,\lambda)}(\tau)| \leq C_1 \langle \frac{\tau-\lambda}{\hbar} \rangle^{-2},$$

uniformly on $\tau \in \mathbb{R}$ and $\hbar \in (0, \hbar_0)$. Consequently,

$$|\int_{-\infty}^{\lambda} \breve{\theta}_\hbar \star \mathcal{M}(\tau)d\tau - \int_{-\infty}^{\lambda} \mathcal{M}(\tau)d\tau| \leq C_1 \int \langle \frac{\tau-\lambda}{\hbar} \rangle^{-2} \mathcal{M}(\tau)d\tau. \tag{4.5.7}$$

Putting together (4.5.3), (4.5.5) and (4.5.6), we get

$$\int_{-\infty}^{\lambda} \breve{\theta}_\hbar \star \mathcal{M}(\tau)d\tau = M(\lambda,\hbar) + \mathcal{O}(\hbar^{1-d}). \tag{4.5.8}$$

Note that $\breve{\theta}_h \star \mathcal{M}(\tau) = \mathrm{Tr}(\phi_3(\mathbb{H}^{(q)})\breve{\theta}_h(\tau-\mathbb{H}^{(q)}))$.
As a consequence of (4.1.4), (4.1.5) and (4.5.7) we obtain

$$M(\lambda,\hbar) = \hbar^{-d}(m(\lambda) + \mathcal{O}(\hbar)), \tag{4.5.9}$$

where

$$m(\lambda) = \int_{-\infty}^{\lambda} c_0(\tau)d\tau = \kappa_q \frac{b_1...b_d}{(2\pi)^d} \int_{\{X\in\mathbb{R}^{2d}|V(X)\leq\lambda\}} \phi_3(V(X))dX. \tag{4.5.10}$$

Here we have the fact that if E is not a critical value of $V(X)$, then

$$\frac{\partial}{\partial E}\left(\int_{\{X\in\mathbb{R}^{2d}|V(X)\leq E\}} dX\right) = \int_{S_E} \frac{dS_E}{|\nabla_X V|},$$

where $S_E = V^{-1}(E)$ (see [Ro, Lemma V-9]).
Applying (4.5.2), (4.5.8) and (4.5.9) to ϕ_1 and writing:

$$\sum_{\lambda_1\leq\gamma_j(\hbar)} \phi_1(\gamma_j(\hbar)) = \sum_j \phi_1(\gamma_j(\hbar)) - \sum_{\gamma_j(\hbar)<\lambda_1} \phi_1(\gamma_j(\hbar)),$$

we get

$$\sum_{\lambda_1\leq\gamma_j(\hbar)} \phi_1(\mu_j(\hbar)) = \hbar^{-d}(m_1(\lambda_1) + \mathcal{O}(\hbar)), \tag{4.5.11}$$

with

$$m_1(\lambda_1) = \kappa_q \frac{b_1 ... b_d}{(2\pi)^d} \int_{\{X \in \mathbb{R}^{2d} | V(X) \geq \lambda_1\}} \phi_1(V(X)) dX. \tag{4.5.12}$$

Now (4.1.5) results from (4.5.2), (4.5.8) and (4.5.10).

Acknowledgements

The idea of writing this article was born during second author's visit to the University of Nantes, France, in 2000. He is deeply grateful to his colleagues for their warm hospitality. Both authors attended EURESCO Conference "Geometry, Analysis and Mathematical Physics", September 22-27, 2000, in San Feliu de Guixols, Spain, where the work on this article marked a substantial progress. The authors are grateful to the Organizing Committee for financial support. The second author was supported by the Bulgarian Science Foundation under Grant MM 706/97. The authors thank Professor M.Engliš and Professor A.V.Sobolev for several stimulating discussions.

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