Strong solutions to stochastic Volterra equations of convolution type

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Abstract

The lecture concerns the stochastic Volterra equations in a separable Hilbert space H of the form

$$X(t) = X(0) + \int_0^t a(t-\tau)\, AX(\tau) d\tau + \int_0^t \Psi(\tau)\, dW(\tau)\ , \qquad t \geq 0\ . \tag{1}$$

In (1), $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is a scalar kernel, A is a closed linear unbounded operator in H with a dense domain $D(A)$ equipped with the graph norm $|\cdot|_{D(A)}$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis and U be a separable Hilbert space. In (1), $X(0)$ is an H-valued $\mathcal{F}_0$-measurable random variable, $W(t)$, $t \geq 0$, is a so-called cylindrical Wiener process on U with the covariance operator $Q \in L(U)$. $\Psi(t)$, $t \geq 0$, is a process with values in a space L_2^0 of Hilbert-Schmidt operators, providing sense for the stochastic integral $\int_0^t \Psi(\tau) dW(\tau)$.

In our considerstions we use the resolvent approach to the Volterra equation (1), see [4]. So, we assume that the resolvent family $S(t)$, $t \geq 0$, to (1) exists. We want to emphasize that the family $S(t)$, $t \geq 0$, is generated by the operator A and the kernel function $a(t)$, $t \geq 0$, and then it does not create in general any semigroup.

We understand solutions to the equation (1) in the following sense.

Definition 1 *An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* `strong solution` *to (1), if X has a version such that $P(X(t) \in D(A)) = 1$ for almost all $t \in [0,T]$; for any $t \in [0,T]$*

$$\int_0^t |a(t-\tau)AX(\tau)|_H \, d\tau < +\infty, \quad P-a.s.$$

and for any $t \in [0,T]$ the equation (1) holds P-a.s.

Definition 2 *An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* `mild solution` *to the stochastic Volterra equation (1), if*

$$\mathbb{E}\left(\int_0^t |S(t-\tau)\Psi(\tau)|^2_{L_2^0}\, d\tau\right) < +\infty \quad \textit{for} \quad t \leq T$$

and, for arbitrary $t \in [0,T]$*,*

$$X(t) = S(t)X(0) + \int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau), \quad P-a.s.$$

where $S(t)$*,* $t \geq 0$*, is the resolvent family.*

We introduce the stochastic convolution

$$W^{\Psi}(t) := \int_0^t S(t-\tau)\Psi(\tau)\, dW(\tau), \quad t \geq 0. \tag{2}$$

Analogously like in a semigroup case, the convolution (2) is mild solution to (1) (for details, see [1]). If the kernel function $a(t)$, $t \geq 0$, is completely positive and the operator A generates a C_0-semigroup, we can provide sufficient conditions for the convolution (2) to be also strong solution to (1). In order to do this we use convergence of resolvents for the equation (1) (see [2]).

If the kernel function $a(t) = g_\alpha(t)$, where

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \qquad \alpha > 0, \tag{3}$$

the equation (1) is an integral form of the equation

$$D^\alpha u(t) = A\, u(t) + f(t), \qquad t > 0,$$

where D^α, $\alpha > 0$, is a fractional derivative, A is as above and $f(t)$, $t > 0$, is an appropriate H-valued function.

Note that the kernel function $g_\alpha(t)$ is completely positive only for $\alpha \in (0,1]$, but for $\alpha > 1$ is not! Hence, the above mentioned results can not be used directly in the last case. We omit that difficulty using new tools concerning α-times resolvent families.

The talk will present the recent results from [2] and [3] on existence of strong solutions to the equation (1) with completely positive kernel and the kernel (3) for $\alpha \in (0,1]$ and $\alpha \in (0,2)$, as well.

References

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