Relative Entropy and Mixing Properties of Infinite Dimensional Diffusions

A.F. Ramírez*

Centre de Mathématiques Appliquées
École Polytechnique
F 91128 Palaiseau Cedex, France

Abstract

Let η be a diffusion process taking values on the infinite dimensional space $T^{\mathbf{Z}}$, where T is the circle, and with components satisfying the equations $d\eta_i = \sigma_i(\eta)dW_i + b_i(\eta)dt$ for some coefficients σ_i and b_i, $i \in \mathbf{Z}$. Suppose we have an initial distribution μ and a sequence of times $t_n \to \infty$ such that $\lim_{n\to\infty} \mu S_{t_n} = \nu$ exists, where S_t is the semi-group of the process. We prove that if σ_i and b_i are bounded, of finite range, have uniformly bounded second order partial derivatives, and $\inf_{i,\eta} \sigma_i(\eta) > 0$, then ν is invariant.

Mathematics Subject Classification: 60K35, 60J60

1. INTRODUCTION.

This paper is an extension of Mountford using the relative entropy method presented in Ramirez-Varadhan. In [5] and in [8] it was proved that under some very mild restrictions, limit measures of interacting particle systems are invariant. Here we show that a similar result is true for a large class of infinite dimensional diffusions.

The strategy behind the proof presented in [8] was based in the following idea. If S_t is a Feller semi-group with bounded generator

$$\Omega f(x) = \int_X [f(x) - f(y)]\pi(x, dy)$$

*Present address: Département de Mathématiques, EPFL, CH-1015 Lausanne, Switzerland, e-mail: **Alejandro.Ramirez@epfl.ch**.

then for any initial distribution μ and any fixed $0 < \tau < \infty$, $||\mu S_{t+\tau} - \mu S_t|| \to 0$ as $t \to \infty$, where $||\alpha||$ denotes the total variation norm of the signed measure α. In fact, considering for some $\lambda > 1$, the process defined by a speeded up generator $\lambda\Omega$, using a Girsanov type formula one can estimate the relative entropy between this new process and the old one, obtaining a bound of the form $H \leq Ct(\lambda - 1)^2$, where C is a bound for the generator Ω. By making the choice $\lambda = \frac{\tau}{t} + 1$ we get $H = o(1)$ as $t \to \infty$ and fixed τ. Since the marginal at time t of the speeded up process is $\mu S_{t+\tau}$ and the relative entropy controls the variational distance, it follows that $||\mu S_{t+\tau} - \mu S_t|| \to 0$. To adapt this idea to interacting particle systems the first step was to approximate the process to one corresponding to the above description.

When trying to emulate this method in the context of diffusions defined on an infinite dimensional lattice some characteristics appear that make the procedure more difficult. The main one is that a truncation to a finite box does not produce a probability measure corresponding to a generator Ω that is absolutely continuous with respect to the probability measure defined by $\lambda\Omega$ for finite time intervals $[0, t]$. One can get around this problem constructing an auxiliary process with some special properties.

A second difficulty appears when the relative entropy H is bounded via Girsanov: one obtains terms proportional to the relative entropy of the initial marginal with respect to Lebesgue measure, which need to be controlled. In particular it is necessary to use estimates of the heat kernel of elliptic diffusions in terms of the heat kernel of the Laplacian where the dependence of the constants on the dimension appears explicitly (see [1] and [4]).

The construction of the auxiliary process and the control based on the heat kernel estimates are carried out in Section 4. In Section 5 an application is presented.

2. NOTATION AND RESULTS.

Let T be the unit circle. Consider $\Omega := C([0,\infty); T^{\mathbf{Z}^d})$, the space of continuous functions on the interval $[0,\infty)$ taking values in $T^{\mathbf{Z}^d}$, and endow it with the topology of uniform convergence in compact subsets of $[0,\infty)$. Let $\mathcal{B}$ be the corresponding Borel σ-field and define for each finite set $F \subset \mathbf{Z}^d$ the continuous projections π_F from Ω to $C([0,\infty); T^F)$. Then by Kolmogorov extension theorem we know that there exists a unique probability measure $\mathcal{W}$ on $(\Omega, \mathcal{B})$ such that for each non empty finite set $F \subset \mathbf{Z}^d$, the probability measure $\mathcal{W}_F(\cdot) := \mathcal{W}(\pi_F^{-1}\cdot)$ on $C([0,\infty); T^F)$ is an F-dimensional Brownian motion starting from 0. We will call $\mathcal{W}$ a Wiener measure on $(\Omega, \mathcal{B})$.

Identifying T with the unit interval with periodic boundary conditions, we let $\phi : \{w \in C([0,\infty); R^{\mathbf{Z}^d});\ 0 \leq w_k(0) < 1, k \in \mathbf{Z}^d\} \to \Omega$ be the mapping defined by

$$\phi_k(w)(t) := w_k(t) \ mod\ 1$$

where ϕ_k is the k^{th} component of ϕ. Note that this function is $1-1$. Let $b, \sigma : T^{\mathbf{Z}^d} \to R^{\mathbf{Z}^d}$ be Borel-measurable functions. Now for every $\zeta \in T^{\mathbf{Z}^d}$ define $\eta(t,\zeta) := \phi(\psi(t,\zeta))$, where $\psi(t,\zeta)$ solves

$$\psi_i(t) = \zeta_i + \int_0^t \sigma_i(\phi(\psi(s)))dW_s + \int_0^t b_i(\phi(\psi(s)))ds, \tag{1}$$

σ_i and b_i are the i^{th} components of σ and b respectively and W is the coordinate representation process distributed according to $\mathcal{W}$.

We say that a Borel-measurable function $a : T^{\mathbf{Z}^d} \to R^{\mathbf{Z}^d}$ is bounded if

$$\sup_{\substack{i \in \mathbf{Z}^d \\ \zeta \in T^{\mathbf{Z}^d}}} |a_i(\zeta)| < \infty$$

We will say that a has finite range $R \in \mathbf{Z}^+$ if for each $i \in \mathbf{Z}^d$ the function $a_i(t,\zeta)$ depends only on ζ_j for $j \in [i-R, i+R]^d$. We will say that the function a has bounded first order partial derivatives if

$$\sup_{\substack{i,j \in \mathbf{Z}^d \\ \zeta \in T^{\mathbf{Z}^d}}} \left| \frac{\partial a_i(\zeta)}{\partial \zeta_j} \right| < \infty$$

And we will say that it has bounded second order partial derivatives if

$$\sup_{\substack{i,j,k \in \mathbf{Z}^d \\ \zeta \in T^{\mathbf{Z}^d}}} \left| \frac{\partial^2 a_i(\zeta)}{\partial \zeta_k \partial \zeta_j} \right| < \infty$$

It can be shown that for each finite range, bounded functions b and σ with bounded first order partial derivatives, and $\zeta \in T^{\mathbf{Z}^d}$ there exists a unique process $\eta(t,\zeta)$ such that $\phi^{-1}(\eta)(t)$ satisfies (1) (see [2]). We shall call such a process a finite range infinite dimensional diffusion with bounded coefficients with bounded first order partial derivatives. To this process there corresponds a family of probability measures $\{P_\zeta; \zeta \in T^{\mathbf{Z}^d}\}$, defined by $P_\zeta(A) := \mathcal{W}(\eta(\cdot,\zeta) \in A)$ for $A \in \mathcal{B}$, which is Feller continuous. We will call this family an infinite dimensional diffusion family on $T^{\mathbf{Z}^d}$ with coefficients σ and b. If

$$\inf_{\substack{i\in\mathbf{Z}^d\\ \eta\in T^{\mathbf{Z}^d}}} |\sigma_i(\eta)| > 0$$

we will say that this family is uniformly elliptic. By the Feller continuity there is a Markov semi-group $\{S_t, t \geq 0\}$ (see [2] or Liggett [3]) on the space of continuous functions on $T^{\mathbf{Z}^d}$, $C(T^{\mathbf{Z}^d})$, given by $S_t f(\zeta) = E_\zeta(f(\eta(t)))$, where E_ζ is the expectation corresponding to P_ζ. Then for each probability measure μ on $(T^{\mathbf{Z}^d}, \mathcal{B}_{T^{\mathbf{Z}^d}})$, where $\mathcal{B}_{T^{\mathbf{Z}^d}}$ is the Borel σ-field of $T^{\mathbf{Z}^d}$, we can define μS_t as the unique measure satisfying

$$\int_{T^{\mathbf{Z}^d}} S_t f d\mu = \int_{T^{\mathbf{Z}^d}} f d\mu S_t \qquad f \in C(T^{\mathbf{Z}^d})$$

In this paper, in analogy with a recent result for interacting particles (see Mountford [5]), we will prove that when $d = 1$ and under certain conditions on the coefficients, limit measures of infinite dimensional diffusion families are invariant. Our approach will be based on the relative entropy method presented in [8].

Theorem 1. *Let S_t be the semi-group of a finite range uniformly elliptic infinite dimensional diffusion family on $T^{\mathbf{Z}}$ with bounded coefficients b and σ. Assume that they have bounded second order partial derivatives. Let μ be some probability measure on $T^{\mathbf{Z}}$ such that for some $t_n \to \infty$*

$$\lim_{n\to\infty} \mu S_{t_n} = \nu$$

exists. Then ν is invariant.

Remark 1. *Note that if $a : T^{\mathbf{Z}^d} \to R^{\mathbf{Z}^d}$ has bounded second order partial derivatives then $\sup_{i,\eta}\left|\frac{\partial a_i}{\partial \eta_i}\right| \leq \sup_{i,\eta}\left|\frac{\partial^2 a_i}{\partial \eta_i^2}\right|$. Therefore it follows that bounded second order partial derivatives imply bounded first order partial derivatives. For $\sigma_i = 1$, $i \in \mathbf{Z}$, it is enough to assume that b has bounded first order partial derivatives [7].*

Remark 2. *It is possible to replace the finite range assumption by the following weaker exponential decay conditions [7]:*

$$\max\left\{\left|\frac{\partial^2 b_i}{\partial\eta_j\partial\eta_k}\right|, \left|\frac{\partial^2 a_i}{\partial\eta_j\partial\eta_k}\right|\right\} \leq Ce^{-\gamma(|i-j|+|i-k|)}$$

for arbitrary constants $C > 0$ and $\gamma > 0$.

Idea of Proof. Let $\{P_\zeta; \zeta \in T^{\mathbf{Z}}\}$ be the given infinite dimensional diffusion family and let η be distributed according to P_η. Since η satisfies the Feller property it is enough to show that if f is a local function on Ω with bounded second order partial derivatives, then for any fixed $\tau > 0$:

$$\lim_{t\to\infty} \left| \int S_t f \, d\mu - \int S_{t+\tau} f \, d\mu \right| = 0 \tag{2}$$

To do this we define a diffusion family $\eta^\lambda(s)$ that at time t has the same marginal distribution as $\eta(t+\tau)$. This can be done by "speeding up" the η process so that η^λ is given by

$$\eta^\lambda(s) = \eta(\int_0^s \lambda_{t,\tau}(u)du)$$

where $\lambda_{t,\tau}(s) : [0,\infty) \to [1,\infty)$. Therefore $\lambda_{t,\tau}(s)$ has to satisfy the equation

$$\int_0^t \lambda_{t,\tau}(s)ds = t + \tau.$$

Then (2) can be written as

$$\lim_{t\to\infty} \left| \int S_t f \, d\mu - \int S^\lambda_{0,t} f \, d\mu \right| = 0, \tag{3}$$

where $S^\lambda_{0,s}$ is the time change operator of $\eta^\lambda(s)$, defined for $f \in C(T^{\mathbf{Z}})$ by

$$S^\lambda_{0,s} f(\zeta) := E_\zeta(f(\eta(\int_0^s \lambda_{t,\tau}(u)du))) = E_\zeta(f(\eta^\lambda(s))).$$

We will prove (3) in two steps. We will define two diffusion families η^h and $\eta^{h,\lambda}$ corresponding to η and η^λ respectively with the following two properties:
a) For each local function f with bounded second order partial derivatives we have

$$\lim_{t\to\infty} \sup_{\zeta\in T^{\mathbf{Z}}} |S_t f(\zeta) - S^h_{0,t} f(\zeta)| \;=\; 0 \tag{4}$$

$$\lim_{t\to\infty} \sup_{\zeta\in T^{\mathbf{Z}}} |S^\lambda_{0,t} f(\zeta) - S^{h,\lambda}_{0,t} f(\zeta)| \;=\; 0 \tag{5}$$

where $S^h_{0,s}$ and $S^{h,\lambda}_{0,s}$ are the time change operators corresponding to $\eta^h(s)$ and $\eta^{h,\lambda}(s)$ respectively and defined by $S^h_{0,s}f(\zeta) := E_{\mathcal{W}}(f(\eta^h(s,\zeta));\ S^{h,\lambda}_{0,s}f(\zeta) := E_{\mathcal{W}}(f(\eta^{h,\lambda}(s,\zeta)))$ for all $f \in C(T^{\mathbf{Z}})$ and $\zeta \in T^{\mathbf{Z}}$ (here $E_{\mathcal{W}}$ denotes expectation with respect to the Wiener measure $\mathcal{W}$).

b) The variational norm distance $||\cdot||_M$ between the measures μ evolved up to time t according to η^h and μ evolved up to time t according to $\eta^{h,\lambda}$, both restricted to $T^{[-M,M]}$, converges to 0, i.e.

$$\lim_{t\to\infty} ||\mu S^h_{0,t} - \mu S^{h,\lambda}_{0,t}||_M = 0 \tag{6}$$

where $\mu S^h_{0,s}$ and $\mu S^{h,\lambda}_{0,s}$ are measures on $T^{\mathbf{Z}}$ defined by

$$\begin{aligned} \int_{T^{\mathbf{Z}}} fd(\mu S^h_{0,s}) &:= \int_{T^{\mathbf{Z}}} S^h_{0,s} f d\mu \qquad \forall f \in C(T^{\mathbf{Z}}) \\ \int_{T^{\mathbf{Z}}} fd(\mu S^{h,\lambda}_{0.s}) &:= \int_{T^{\mathbf{Z}}} S^{h,\lambda}_{0,s} f d\mu \qquad \forall f \in C(T^{\mathbf{Z}}), \end{aligned}$$

Clearly equations (4), (5) and the variational limit (6) imply (3). □

In the next section we will carry out the first part of the proof of theorem 1 that has been explained above. We will show how to reduce our problem to a finite dimensional one (i.e. we will prove equations (4) and (5)). In section 4 we will prove the second part of theorem 1 using a relative entropy argument.

3. CONSTRUCTION OF TRUNCATED PROCESS.

Let $C^2(T^{\mathbf{Z}})$ be the set of continuous functions with continuous and uniformly bounded (respect to $\eta \in T^{\mathbf{Z}}$) second order partial derivatives. We define, for $f \in C^2(T^{\mathbf{Z}})$, the following measure of its dependence on the coordinate i,

$$\Delta_f(i) := \sup_{\eta\in T^{\mathbf{Z}}} \left| \frac{\partial^2 f}{\partial \eta_i^2} \right|$$

Then we have the following basic estimate

Theorem 2. *Let S_t be the semi-group of a finite range uniformly elliptic infinite dimensional diffusion family on $T^{\mathbf{Z}}$ with bounded coefficients and bounded second order partial derivatives. Then for each local function f with support on $[-M, M]$ such that $f \in C^2(T^{\mathbf{Z}})$, and for each $\gamma > 0$, there exist constants A and β (depending on M, γ and f) such that,*

$$\Delta_{S_t f}(i) \leq A e^{\beta t} e^{-\gamma |i|}$$

Proof. Let $D(T^{\mathbf{Z}}) = \{g \in C^2(T^{\mathbf{Z}}) : |||g||| = \sum_{i=1}^{\infty} \Delta_g(i) < \infty\}$. Note that the generator Ω of the process acts on functions $g \in D(T^{\mathbf{Z}})$ as

$$\Omega g(\eta) = \sum_{i \in \mathbf{Z}} \left(\frac{1}{2} a_i(\eta) \frac{\partial^2 g(\eta)}{\partial \eta_i^2} + b_i(\eta) \frac{\partial g(\eta)}{\partial \eta_i} \right)$$

where $a_i(\eta) = \sigma_i^2(\eta)$. Now let $u(t, \eta)$ be the solution of the equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Omega u \\ u(0, \eta) &= f(\eta) \end{aligned}$$

where f is a local function with support on $[-M, M]$ such that $f \in C^2(T^{\mathbf{Z}})$. Define $u_i = \frac{\partial u}{\partial \eta_i}$ and $u_{i,j} = \frac{\partial^2 u}{\partial \eta_i \partial \eta_j}$. Then to prove the theorem it is enough to find for each $\gamma > 0$, constants A and β such that

$$|u_{i,j}(t, \eta)| \leq A e^{\beta t} e^{-\frac{\gamma}{2}(|i| + |j|)} \tag{7}$$

We first begin considering truncated approximations of our process. These are the infinite dimensional diffusion families with coefficients σ^L and b^L, where $\sigma_i^L(\eta) = (\sigma_i(\eta) - 1)\theta_{[-L,L]}(i) + 1$, $b_i^L(\eta) = b_i(\eta)\theta_{[-L,L]}(i)$, $L \in \mathbf{N}$ and θ_A is the indicator function of $A \subset \mathbf{Z}$. The corresponding generators indexed by L, act on functions $g \in D(T^{\mathbf{Z}})$ as

$$\Omega^L g(\eta) = \sum_{i \in \mathbf{Z}} \left(\frac{1}{2} a_i^L(\eta) \frac{\partial^2 g(\eta)}{\partial \eta_i^2} + b_i^L(\eta) \frac{\partial g(\eta)}{\partial \eta_i} \right)$$

where $a_i^L(\eta) = (\sigma_i^L(\eta))^2$. Let $u^L(t, \eta)$ be the solution of the equations

$$\begin{aligned} \frac{\partial u^L}{\partial t} &= \Omega^L u^L \\ u^L(0, \eta) &= f(\eta) \end{aligned}$$

and define $u_i^L = \frac{\partial u^L}{\partial \eta_i}$ and $u_{i,j}^L = \frac{\partial^2 u^L}{\partial \eta_i \partial \eta_j}$. Our first step towards the proof of (7) will be to find corresponding bounds uniform in L $u_{i,j}^L$: i.e. for each $\gamma > 0$ there are constants A and β independent of L such that

$$|u^L_{i,j}(t,\eta)| \le Ae^{\beta t}e^{-\frac{\gamma}{2}(|i|+|j|)} \tag{8}$$

The second step will be to deduce the bounds (7) from these inequalities.

Note that $u^L_j(t,\eta)$ is solution of

$$\begin{aligned}\frac{\partial u^L_j}{\partial t} &= \Omega^L u^L_j + \sum_{i\in\mathbf{Z}}\left(\frac{1}{2}\frac{\partial a^L_i}{\partial \eta_j}\frac{\partial u^L_i}{\partial \eta_i} + \frac{\partial b^L_i}{\partial \eta_j}u^L_i\right) \\ u^L_j(0,\eta) &= \frac{\partial f}{\partial \eta_j}(\eta)\end{aligned} \tag{9}$$

From these equations we will be able to find an upper bound uniform in L for $W^L(t,\eta) := \sum_{j\in\mathbf{Z}} e^{2\gamma|j|}(u^L_j)^2(t,\eta)$, which we will need to prove the upper bound (8) for $u^L_{i,j}$. Note that W^L, $\frac{\partial W^L}{\partial t}$ and $\Omega^L W^L$ are well defined, since the corresponding sums are finite. Now, from (9) we obtain that

$$\begin{aligned}\frac{\partial W^L}{\partial t} - \Omega^L W^L &= \sum_{i,j\in\mathbf{Z}} e^{2\gamma|j|}\left(-a^L_i\left(\frac{\partial u^L_j}{\partial \eta_i}\right)^2 + \frac{\partial a^L_i}{\partial \eta_j}u^L_j\frac{\partial u^L_i}{\partial \eta_i} + 2\frac{\partial b^L_i}{\partial \eta_j}u^L_i u^L_j\right) \\ &= \sum_{i\in\mathbf{Z}}\left(-a^L_i\sum_{j\in\mathbf{Z}}e^{2\gamma|j|}\left(\frac{\partial u^L_j}{\partial \eta_i}\right)^2 + \frac{\partial u^L_i}{\partial \eta_i}\sum_{j\in\mathbf{Z}}e^{2\gamma|j|}\frac{\partial a^L_i}{\partial \eta_j}u^L_j\right) + 2\sum_{i,j\in\mathbf{Z}}\frac{\partial b^L_i}{\partial \eta_j}u^L_i u^L_j e^{2\gamma|j|}\end{aligned} \tag{10}$$

At this point let us take a look at the first term on the last expression. We would like to obtain a bound for it not involving second order partial derivatives of u^L. We introduce the constants $A' = \sup_{\substack{i,j\in\mathbf{Z}\\ \eta\in T^{\mathbf{Z}}}}\left|\frac{\partial a_i}{\partial \eta_j}\right|$ and $a = \min\left\{\inf_{\substack{i\in\mathbf{Z}\\ \eta\in T^{\mathbf{Z}}}}|a_i|, 1\right\}$ and note that,

$$\begin{aligned}&\textstyle\sum_{i\in\mathbf{Z}}\left(-a^L_i\sum_{j\in\mathbf{Z}}e^{2\gamma|j|}\left(\frac{\partial u^L_j}{\partial \eta_i}\right)^2 + \frac{\partial u^L_i}{\partial \eta_i}\sum_{j\in\mathbf{Z}}e^{2\gamma|j|}\frac{\partial a^L_i}{\partial \eta_j}u^L_j\right) \\ &\le \textstyle\sum_{i\in\mathbf{Z}}\left(-a^L_i e^{2\gamma|i|}\left(\frac{\partial u^L_i}{\partial \eta_i}\right)^2 + \frac{\partial u^L_i}{\partial \eta_i}\sum_{j\in\mathbf{Z}}e^{2\gamma|j|}\frac{\partial a^L_i}{\partial \eta_j}u^L_j\right) \\ &= \textstyle\sum_{i\in\mathbf{Z}}\left(-a^L_i e^{2\gamma|i|}\left(\frac{\partial u^L_i}{\partial \eta_i} - \frac{1}{2a^L_i e^{2\gamma|i|}}\sum_{j\in\mathbf{Z}}e^{2\gamma|j|}\frac{\partial a^L_i}{\partial \eta_j}u^L_j\right)^2\right. \\ &\qquad \left. + \frac{1}{4a^L_i e^{2\gamma|i|}}\left(\sum_{j\in\mathbf{Z}}e^{2\gamma|j|}\frac{\partial a^L_i}{\partial \eta_j}u^L_j\right)^2\right)\end{aligned}$$

$$\leq \sum_{i\in\mathbf{Z}} \frac{1}{4a_i^L e^{2\gamma|i|}} \left(\sum_{j\in\mathbf{Z}} e^{2\gamma|j|} \frac{\partial a_i^L}{\partial \eta_j} u_j^L\right)^2$$
$$\leq \frac{(A')^2}{4a} \sum_{i\in\mathbf{Z}} e^{-2\gamma|i|} \left(\sum_{j=i-R}^{i+R} e^{2\gamma|j|} |u_j^L|\right)^2$$
$$\leq \frac{(A')^2}{4a} e^{2\gamma R} \sum_{i\in\mathbf{Z}} \left(\sum_{j=i-R}^{i+R} e^{\gamma|j|} |u_j^L|\right)^2$$
$$\leq \frac{(A')^2}{4a} e^{2\gamma R} (2R+1)^2 W^L \tag{11}$$

Here, in the first inequality we have kept only the negative term such that $j = i$. In the equality that follows we have completed the square. Later, in the second to last inequality we used the relation $e^{2\gamma|j|} \leq e^{\gamma|j|} e^{\gamma|i|} e^{\gamma|j-i|}$. And in the last inequality we have used Cauchy-Schwartz. Using (11) as a bound for the first term of (10) and using the inequality $2xy \leq x^2 + y^2$ valid for $x, y \in \mathbf{R}$, we see that,

$$\frac{\partial W^L}{\partial t} - \Omega^L W^L \leq W^L \frac{(A')^2}{4a} e^{2\gamma R}(2R+1)^2 + \sum_{i,j\in\mathbf{Z}} \left|\frac{\partial b_i^L}{\partial \eta_j}\right| ((u_i^L)^2 + (u_j^L)^2) e^{2\gamma|j|} \tag{12}$$

We would also like to obtain a factor of W^L for the last term of the above expression. With that purpose in mind and using the assumption that the coefficients b_i^L are of finite range R and have uniformly bounded first order partial derivatives, we get that,

$$\sum_{i,j\in\mathbf{Z}} \left|\frac{\partial b_i^L}{\partial \eta_j}\right| ((u_i^L)^2 + (u_j^L)^2) e^{2\gamma|j|}$$
$$\leq \sum_{i\in\mathbf{Z}} (u_i^L)^2 \sum_{j\in\mathbf{Z}} \left|\frac{\partial b_i^L}{\partial \eta_j}\right| e^{2\gamma|j|} + B'(2R+1) W^L$$
$$\leq W^L \left(B' e^{2\gamma R}(2R+1) + B'(2R+1)\right)$$

where $B' = \sup_{\substack{i,j\in\mathbf{Z} \\ \eta\in T^{\mathbf{Z}}}} \left|\frac{\partial b_i}{\partial \eta_j}\right|$. Now we can use this estimate on (12) to obtain,

$$\begin{aligned} \frac{\partial W^L}{\partial t} - \Omega^L W^L &\leq K_1 W^L \\ W^L(0,\eta) &= \sum_{i\in\mathbf{Z}} e^{2\gamma|i|} \left(\frac{\partial f(\eta)}{\partial \eta_i}\right)^2 \end{aligned} \tag{13}$$

where $K_1 = \left(\frac{(A')^2}{4a}(2R+1) + B'\right)(2R+1)e^{2\gamma R} + (2R+1)B'$.

From inequality (13) it follows that

$$W^L(t,\eta) \le \left(\sup_{i\in[-M,M]} \left\| \frac{\partial f}{\partial \eta_i} \right\|_\infty^2 (2M+1)e^{2\gamma M} \right) e^{K_1 t} \tag{14}$$

Now, to derive (8) we define $V^L(t,\eta) := \sum_{j,k\in\mathbf{Z}} (u^L_{k,j})^2 e^{\gamma(|k|+|j|)}$. This time note that

$$\begin{aligned}
\frac{\partial u^L_{k,j}}{\partial t} &= \Omega^L u^L_{k,j} + \sum_{i\in\mathbf{Z}} \left(\frac{1}{2}\frac{\partial a^L_i}{\partial \eta_j}\frac{\partial u^L_{i,k}}{\partial \eta_i} + \frac{1}{2}\frac{\partial a^L_i}{\partial \eta_k}\frac{\partial u^L_{i,j}}{\partial \eta_i} \right. \\
&+ \left. \frac{1}{2}\frac{\partial^2 a^L_i}{\partial \eta_j \partial \eta_k} u^L_{i,i} + \frac{\partial b^L_i}{\partial \eta_k} u^L_{i,j} + \frac{\partial^2 b^L_i}{\partial \eta_j \partial \eta_k} u^L_i + \frac{\partial b^L_i}{\partial \eta_j} u^L_{i,k} \right) \\
u^L_{k,j}(0,\eta) &= \frac{\partial^2 f}{\partial \eta_k \partial \eta_j}(\eta)
\end{aligned}$$

from where we have,

$$\begin{aligned}
\frac{\partial V^L}{\partial t} - \Omega^L V^L = \sum_{i,j,k\in\mathbf{Z}} 2e^{\gamma(|k|+|j|)} & \left(\frac{\partial a^L_i}{\partial \eta_j} u^L_{k,j} \frac{\partial u^L_{i,k}}{\partial \eta_i} - \frac{a^L_i}{2} \left(\frac{\partial u^L_{k,j}}{\partial \eta_i} \right)^2 + 2\frac{\partial b^L_i}{\partial \eta_j} u^L_{k,j} u^L_{i,k} \right. \\
& \left. + \frac{1}{2} \frac{\partial^2 a^L_i}{\partial \eta_j \partial \eta_k} u^L_{k,j} u^L_{i,i} + \frac{\partial b^L_i}{\partial \eta_k \partial \eta_j} u^L_{k,j} u^L_i \right) \\
= \sum_{i,k\in\mathbf{Z}} e^{\gamma|k|} & \left(-a^L_i \sum_{j\in\mathbf{Z}} e^{\gamma|j|} \left(\frac{\partial u^L_{k,j}}{\partial \eta_i} \right)^2 + 2\frac{\partial u^L_{i,k}}{\partial \eta_i} \sum_{j\in\mathbf{Z}} e^{\gamma|j|} \frac{\partial a^L_i}{\partial \eta_j} u^L_{k,j} \right) && (15) \\
+ \sum_{i,j,k\in\mathbf{Z}} e^{\gamma(|k|+|j|)} & \left(4\frac{\partial b^L_i}{\partial \eta_j} u^L_{k,j} u^L_{i,k} + \frac{\partial^2 a^L_i}{\partial \eta_j \partial \eta_k} u^L_{k,j} u^L_{i,i} + 2\frac{\partial b^L_i}{\partial \eta_k \partial \eta_j} u^L_{k,j} u^L_i \right) && (16)
\end{aligned}$$

Here, in the same spirit as the calculations involving W^L, we want to obtain a bound for this expression not involving third order partial derivatives of u^L. Thus, we first deal with the term (15) obtaining the following bound,

$$\begin{aligned}
\textstyle\sum_{i,k\in\mathbf{Z}} e^{\gamma|k|} & \left(-a^L_i \textstyle\sum_{j\in\mathbf{Z}} e^{\gamma|j|} \left(\frac{\partial u^L_{j,k}}{\partial \eta_i} \right)^2 + 2\frac{\partial u^L_{i,k}}{\partial \eta_i} \textstyle\sum_{j\in\mathbf{Z}} e^{\gamma|j|} \frac{\partial a^L_i}{\partial \eta_j} u^L_{j,k} \right) \\
&\le \textstyle\sum_{i,k\in\mathbf{Z}} e^{\gamma(|k|-|i|)} \frac{1}{a^L_i} \left(\textstyle\sum_{j\in\mathbf{Z}} e^{\gamma|j|} u^L_{k,j} \frac{\partial a^L_i}{\partial \eta_j} \right)^2 \\
&\le \frac{(A')^2}{a} e^{\gamma R} (2R+1)^2 V^L && (17)
\end{aligned}$$

Here in analogy with the steps leading to expression (11), in the first inequality we have kept only the negative term with $j = i$, then completed the square and finally used Cauchy-Schwartz.

Now let us define $A'' = \sup_{\substack{i,j,k\in\mathbf{Z}\\ \eta\in T^{\mathbf{Z}}}} \left|\frac{\partial^2 a_i}{\partial\eta_j\partial\eta_k}\right|$ and $B'' = \sup_{\substack{i,j,k\in\mathbf{Z}\\ \eta\in T^{\mathbf{Z}}}} \left|\frac{\partial^2 b_i}{\partial\eta_j\partial\eta_k}\right|$. Then we have the following bound for the second term (16),

$$\sum_{i,j,k\in\mathbf{Z}} \left(4u^L_{k,j}\frac{\partial b^L_i}{\partial\eta_j}u^L_{i,k} + \frac{\partial^2 a^L_i}{\partial\eta_j\eta_k}u^L_{k,j}u^L_{i,i} + 2\frac{\partial^2 b^L_i}{\partial\eta_k\partial\eta_j}u^L_{k,j}u^L_i\right) e^{\gamma(|k|+|j|)}$$
$$\leq V^L\Big(2B'(2R+1)+2B'(2R+1)e^{\gamma R}+\tfrac{A''}{2}(2R+1)(1+(2R+1)e^{2\gamma R}) + B''(2R+1)\Big)$$
$$+W^L B''(2R+1)^2 e^{2\gamma R}$$

Using this as a bound for expression (16) and bounding (15) by (17) we obtain the following inequality,

$$\begin{aligned} \frac{\partial V^L}{\partial t} - \Omega^L V^L &\leq K_2 V^L + K_3 W^L \\ V^L(0,\eta) &= \sum_{j,k\in\mathbf{Z}} e^{\gamma(|k|+|j|)} \left(\frac{\partial^2 f(\eta)}{\partial\eta_k\partial\eta_j}\right)^2 \end{aligned} \tag{18}$$

where

$$\begin{aligned} K_2 &= e^{\gamma R}(2R+1)\left(\frac{(A')^2}{a}(2R+1) + 2B' + \frac{A''}{2}e^{\gamma R}(2R+1)\right) \\ &\quad +(2R+1)(2B' + \frac{A''}{2} + B'') \\ K_3 &= B'' e^{2\gamma R}(2R+1)^2 \end{aligned}$$

Finally, from (14) and (18) we obtain the following bound for V^L

$$\begin{aligned} V^L(t,\eta) &\leq \left(\sup_{i,j\in[-M,M]}\left\|\frac{\partial^2 f}{\partial\eta_i\partial\eta_j}\right\|^2_\infty (2M+1)^2 e^{2\gamma M}\right) e^{K_2 t} \\ &+ \tfrac{K_3}{K_1+K_2}\left(\sup_{i\in[-M,M]}\left\|\frac{\partial f}{\partial\eta_i}\right\|^2_\infty (2M+1)e^{2\gamma M}\right) e^{2(K_1+K_2)t} \end{aligned} \tag{19}$$

It follows from the inequality (19) involving V^L that the estimates (8) are satisfied with

$$
\begin{aligned}
A^2 &= \left(\frac{K_1+K_2+K_3}{K_1+K_2}\right)(2M+1)^2 e^{2\gamma M} \sup_{i,j\in[-M,M]} \left\| \frac{\partial^2 f}{\partial\eta_i \partial\eta_j} \right\|_\infty^2 \\
\beta &= K_1+K_2
\end{aligned}
$$

At this point, note that in particular we have obtained uniform bounds in L for $u^L_{i,i}$, which in turn imply the equicontinuity in the variable η_i of the family of functions u^L_i, indexed by L. Now, by remark (1), u^L_i are also uniformly bounded by the right hand side of inequality (8) with $i=j$. Therefore, we can apply the theorem of Ascoli-Arzelá to extract subsequences that converge uniformly in η_i. Let $u^{L_n}_i$ be such a subsequence and call v its limit. Then, since $\lim_{L\to\infty} u^L(t,\eta) = u(t,\eta)$ (see [2]), we can conclude that

$$
\begin{aligned}
u(t,\eta) - u(t,\eta^0) &= \lim_{L\to\infty} \int_0^{\eta_i} u^L_i(t,\eta')d\eta'_i \\
&= \int_0^{\eta_i} \lim_{n\to\infty} u^{L_n}_i(t,\eta')d\eta'_i \\
&= \int_0^{\eta_i} v(t,\eta')d\eta'_i
\end{aligned}
$$

where we have defined $\eta^0_i = 0$ and $\eta^0_j = \eta'_j = \eta_j$ for $j \neq i$. Thus, every subsequence of u^L_i has a uniformly (in η_i) convergent subsequence whose limit is u_i. It follows that,

$$
\lim_{L\to\infty} u^L_i(t,\eta) = u_i(t,\eta) \tag{20}
$$

uniformly in η_i. Note also that the uniform bounds on L for $u^L_{i,j}$ imply that the convergence in (20) is uniform in η_j for $j \neq i$.

In order to prove the inequalities (7) for the second order partial derivatives of u we will use (20) and the following argument. First let $\varphi(x) : T \to [0,\infty]$ be a smooth function such that $\int_T \varphi dx = 1$, and define $\varphi_{\epsilon,y}(x) = \frac{1}{\epsilon}\varphi\left(\frac{x-y}{\epsilon}\right)$. Now, note that since the convergence in (20) is uniform in η_j, it follows that,

$$
\lim_{L\to\infty} \int_T \varphi_{\epsilon,\eta_j}(\eta'_j) u^L_{i,j}(t,\eta')d\eta'_j = \int_T \varphi_{\epsilon,\eta_j}(\eta'_j) u_{i,j}(t,\eta')d\eta'_j
$$

This combined with the bound (8) gives us

$$
\left| \int_T \varphi_{\epsilon,\eta_j}(\eta'_j) u_{i,j}(t,\eta')d\eta'_j \right| \leq A e^{\beta t} e^{-\frac{1}{2}\gamma(|i|+|j|)}
$$

Letting $\epsilon \to 0$ we obtain the desired bound (7). This completes the proof. $\square$

In what follows, given an infinite dimensional diffusion family with coefficients σ and b, and a function $h : [0,\infty) \to [0,\infty)$ we will define a truncated version of this process as the infinite dimensional diffusion family with coefficients σ^h and b^h, where $\sigma_i^h(\eta, s) = (\sigma_i(\eta) - 1)\vartheta_{h(s)}(i) + 1$, $b_i^h(\eta, s) = b_i(\eta)\vartheta_{h(s)}(i)$, and $\vartheta_c(x) : R \to [0,1]$ is a smooth function such that $\vartheta_c(x) = 1$ if $|x| < c$ and $\vartheta_c(x) = 0$ if $|x| \geq c+1$.

As a corollary of theorem 2 we are able to prove that for long times, our infinite dimensional diffusion family differs little from its truncated version when looked at in a finite box.

Theorem 3. *Let S_t be the semi-group of a finite range, uniformly elliptic infinite dimensional diffusion family on $T^{\mathbf{Z}}$ with bounded coefficients having bounded second order partial derivatives. Let $S_{0,t}^h$ be the time change operator of its truncated version, corresponding to a function $h(s) \geq c(t-s) + p(t)$. Assume that $\lim_{t\to\infty} p(t) = \infty$ and that $c > \beta/\gamma$. Then, for any local function $f \in C^2(T^{\mathbf{Z}})$ we have*

$$\lim_{t\to\infty} \sup_{\zeta \in T^{\mathbf{Z}}} |S_{0,t}^h f(\zeta) - S_t f(\zeta)| = 0$$

Proof. First note that

$$S_{0,t}^h f(\zeta) - S_t f(\zeta) = \int_0^t S_{0,t}^h(\Omega_s^h - \Omega) S_{t-s} f(\zeta) ds$$

where Ω and Ω_s^h are the infinitesimal generators corresponding to the semi-group S_t and to $S_{0,t}^h$ respectively. Now note that for any function $g \in D(T^{\mathbf{Z}})$ we have

$$|(\Omega_s^h - \Omega) g(s)| \leq D \sum_{|i| \geq h(s)} \Delta_g(i)$$

where $D = \frac{1}{2}\max\{1, \sup_{i,\zeta} |\sigma_i(\zeta)|^2\} + \sup_{i,\zeta} |b_i(\zeta)|)$. Therefore, from this relationship and theorem 2, we see that,

$$
\begin{aligned}
|S^h_{0,t}f(\zeta) - S_t f(\zeta)| &\leq D\int_0^t \sum_{|i|\geq h(s)} \Delta_{S_{t-s}f}(i)ds \\
&\leq 2DA\int_0^t e^{\beta(t-s)} \sum_{i\geq h(s)} e^{-\gamma i}ds \\
&\leq 2DA\frac{e^{-\gamma p(t)}}{c\gamma-\beta}\frac{1}{1-e^{-\gamma}}
\end{aligned}
$$

□

For the proof of theorem 1 we choose the truncated process η^h as a truncation of η with coefficients $\sigma^h(\eta, s)$ and $b^h(\eta, s)$, and with a function h satisfying the hypothesis of theorem 3. Then we let $\eta^{h,\lambda}(s) = \eta^{h'}(\int_0^s \lambda_{t,\tau}ds')$, where $h'(s) = h(\beta_{t,\tau}^{-1}(s))$ and $\beta_{t,\tau}^{-1}$ is the inverse of $\beta_{t,\tau} = \int_0^s \lambda_{t,\tau}(s')ds'$. Notice that the generator of $\eta^{h,\lambda}$ is equal to the generator of η^h times $\lambda_{t,\tau}(s)$.

4. RELATIVE ENTROPY OF TRUNCATED PROCESS.

The following two lemmas will be needed to prove the convergence to 0 of the variational limit (6). We will need to control the relative entropy of the marginal distribution of an n-dimensional diffusion with respect to Lebesgue measure, in terms of time and dimension. In turn, for this we need the following fundamental estimate, which has been proved by Fabes and Stroock[1]. Following Lu [4], we have kept track of the dependence of the constants on the dimension. We define $C^{\infty,2}([0,\infty), T^n)$ as the space of functions $f(t,\eta)$ which are smooth in the first variable and have continuous second order partial derivatives.

Lemma 1. *Consider the second order parabolic operator* $\Omega = \sum_{i=1}^n \frac{1}{2}\frac{\partial}{\partial\eta_i}a_i(t,\eta)\frac{\partial}{\partial\eta_i} - \frac{\partial}{\partial t}$, *where* $\eta \in T^n$. *Assume that for* $1 \leq i \leq n$, $a_i(t,\eta) \in C^{\infty,2}([0,\infty), T^n)$, *and that there is a positive constant* v *such that* $v < a_i(t,\eta) < \frac{1}{v}$. *Then the fundamental solution* $\Gamma(t,\zeta,\eta)$ *of the parabolic operator* $\Omega - \frac{\partial}{\partial t}$ *satisfies*

$$\Gamma(t,\zeta,\eta) \leq (u_2)^{n/2}\Gamma_0(u_1 t,\zeta,\eta)$$

where $\Gamma_0(t,\zeta,\eta)$ *is the fundamental solution of* $\frac{1}{2}\nabla^2 - \frac{\partial}{\partial t}$ *on* T^n, $u_1 = \frac{16}{v}$ *and* $u_2 = \frac{4096\pi}{v^2}$.

As a corollary of this, we are able to obtain a bound on the relative entropy of the one dimensional marginal distributions of a diffusion process with respect to Lebesgue measure.

Lemma 2. *Let $\{P_\zeta;\zeta \in T^n\}$ be an n-dimensional diffusion family with time dependent coefficients coefficients $b(t,\eta)$ and $\sigma(t,\eta)$ and call $S_{0,t}$ the corresponding time change operators. Assume that $\sigma(t,\eta)$, $b(t,\eta) \in C^{\infty,2}([0,\infty),T^{\mathbf{Z}})$, that $B := \sup_{\substack{t\geq 0,\eta\in T^{\mathbf{Z}}\\ 1\leq i\leq n}} |b_i(t,\eta)| < \infty$, $A' := \sup_{\substack{t\geq 0,\eta\in T^{\mathbf{Z}}\\ 1\leq i\leq n}} \left|\frac{\partial\sigma_i}{\partial\eta_i}(t,\eta)\right| < \infty$ and that $v < \sigma_i^2(t,\eta) < \frac{1}{v}$ for some constant v. Then for each probability measure μ on $(T^n,\mathcal{B}_{T^n})$, and $t>0$, $\mu S_{0,t}$ has a Radon-Nikodym derivative $g(t,\eta)$ with respect to Lebesgue measure m that satisfies the following inequality,*

$$\int g(t,\eta)\ln g(t,\eta)dm \leq E_1 n + E_2 n\frac{1}{\sqrt{t}} \qquad t>0$$

where $E_1 = \left(\frac{B^2}{v} + (A')^2 + \sqrt{\frac{v}{8}} + \frac{1}{2}\ln\left(\frac{4096\pi}{v^2}\right)\right)$ *and* $E_2 = \sqrt{\frac{v}{8}}$.

Proof. Let $h(t,\eta)$ be the marginal distribution at time t of an n-dimensional Brownian motion on $(C([0,\infty);T^n),\mathcal{B}_{T^n})$, with initial distribution μ. We know that $h(t,\eta)$ has the representation

$$h(t,\eta) = \sum_{\vec{n}\in\mathbf{Z}^n} e^{-2\pi^2\vec{n}^2 t} e^{2\pi\vec{n}\cdot\eta i}\int e^{-2\pi\vec{n}\cdot\zeta i}d\mu(\zeta)$$

Therefore we obtain the bound

$$h(t,\eta) \leq \sum_{\vec{n}\in\mathbf{Z}^n} e^{-2\pi^2\vec{n}^2 t} \leq \left(1+\sqrt{\frac{2}{t}}\right)^n \tag{21}$$

On the other hand we can always write the second order operator Ω associated to the given n-dimensional diffusion family $\{P_\zeta;\zeta\in T^n\}$ in the form

$$\Omega = \sum_{i=1}^n \left(\frac{1}{2}\frac{\partial}{\partial\eta_i}\sigma_i^2(t,\eta)\frac{\partial}{\partial\eta_i} + \left(b_i(t,\eta) - \sigma_i(t,\eta)\frac{\partial\sigma_i}{\partial\eta_i}(t,\eta)\right)\frac{\partial}{\partial\eta_i}\right)$$

Now, consider the n-dimensional diffusion family $\{P^L_\zeta;\zeta\in T^n\}$ corresponding to the operator $L = \sum_{i=1}^n \frac{1}{2}\frac{\partial}{\partial\eta_i}\sigma_i^2(t,\eta)\frac{\partial}{\partial\eta_i}$. Call $e(t,\eta)$ the Radon-Nikodym derivative of the marginal distribution at time t of this process, with initial distribution μ, with respect to Lebesgue measure. And let E^L_μ denote the expectation with respect to this process with initial distribution μ. Note that by Girsanov's theorem we have,

$$\frac{dP_\zeta|_t}{dP^L_\zeta|_t} = \exp\left(\sum_{i=1}^n \left(\int_0^t \frac{1}{\sigma_i^2}(b_i - \sigma_i \frac{\partial \sigma_i}{\partial \eta_i}) d\eta_i(s) - \frac{1}{2}\int_0^t \frac{1}{\sigma_i^2}(b_i^2 - \left(\sigma_i \frac{\partial \sigma_i}{\partial \eta_i}\right)^2) ds\right)\right)$$

where the notation $\cdot|_t$ denotes restriction of the corresponding measure to the σ-algebra generated by $\eta(s)$ up to time t. Therefore it follows that,

$$E^L_\mu\left(\frac{dP_\zeta|_t}{dP^L_\zeta|_t}(\eta)\ln\left(\frac{dP_\zeta|_t}{dP^L_\zeta|_t}(\eta)\right)\right) = \frac{1}{2}E_\mu\left(\int_0^t \sum_{i=1}^n \frac{1}{\sigma_i^2}\left(b_i - \sigma_i \frac{\partial \sigma_i}{\partial \eta_i}\right)^2 ds\right) \tag{22}$$

where E_μ denotes expectation with respect to the original process with generator Ω and initial distribution μ. On the other hand, by the convexity of the relative entropy we know that,

$$\int g(t,\eta) \ln\left(\frac{g(t,\eta)}{e(t,\eta)}\right) dm \leq E^L_\mu\left(\frac{dP_\zeta|_t}{dP^L_\zeta|_t}(\eta) \ln\left(\frac{dP_\zeta|_t}{dP^L_\zeta|_t}(\eta)\right)\right) \tag{23}$$

Thus, combining the estimates (22) with (23) we derive the following bound

$$\int g(t,\eta) \ln g(t,\eta) dm \leq \frac{1}{2}E_\mu\left(\int_0^t \sum_{i=1}^n \frac{1}{\sigma_i^2(s,\eta(s))}\left(b_i(s,\eta(s)) - \sigma_i(s,\eta(s))\frac{\partial \sigma_i}{\partial \eta_i}(s,\eta(s))\right)^2 ds\right) + \int g(t,\eta) \ln e(t,\eta) dm \tag{24}$$

Now, by lemma 1 and estimate (21) we have

$$e(t,\eta) \leq u_2^{n/2}\left(1 + \sqrt{\frac{2}{u_1 t}}\right)^n$$

It follows using (24) that

$$\int g(t,\eta) \ln g(t,\eta) dm(\eta) \leq n\left(\frac{B^2}{v} + (A')^2\right) t + n\sqrt{\frac{2}{u_1}} \frac{1}{\sqrt{t}} + \frac{n}{2}\ln u_2$$

where in the second term we have used the inequality $\ln(1+x) \leq x$. In particular if $\Gamma(t,\zeta,\eta)$ is the fundamental solution of the given diffusion process,

$$\int \Gamma(t,\zeta,\eta)\ln\Gamma(t,\zeta,\eta)dm(\eta) \leq n\left(\frac{B^2}{v}+(A')^2\right)t + n\sqrt{\frac{2}{u_1}}\frac{1}{\sqrt{t}} + \frac{n}{2}\ln u_2 \tag{25}$$

Now, by Chapman-Kolmogorov we have $\Gamma(t+1,\zeta,\eta) = \int \Gamma(1,\zeta,\chi)\Gamma(t,\chi,\eta)dm(\chi)$. Therefore, by the convexity of relative entropy and inequality (25) we obtain

$$\int \Gamma(t,\zeta,\eta)\ln\Gamma(t,\zeta,\eta)dm(\eta) \leq \left(\frac{B^2}{v}+(A')^2+\sqrt{\frac{2}{u_1}}+\frac{1}{2}\ln u_2\right)n \qquad t \geq 1 \tag{26}$$

Now, note that $E_1 n + E_2 n\frac{1}{\sqrt{t}}$ is a bound for the right hand side of (25) when $0 < t \leq 1$ and for the right hand side of (26) which is valid for $t \geq 1$. Therefore, for $t > 0$ we have,

$$\int \Gamma(t,\zeta,\eta)\ln\Gamma(t,\zeta,\eta)dm(\eta) \leq E_1 n + E_2 n\frac{1}{\sqrt{t}}$$

The proof can now be finished noting that $g(t,\eta) = \int \Gamma(t,\zeta,\eta)d\mu(\zeta)$ and using again the convexity of relative entropy.

□

Remark 3. *Let $\tilde{h}(s) = [h(s)]+R+1$. By shifting time if necessary, by lemma 2 it follows that without loss of generality we can assume for the restrictions to the interval $[-\tilde{h}(0),\tilde{h}(0)]$ of the measures $\mu S^h_{0,s}$ and $\mu S^{h,\lambda}_{0,s}$, where μ is the probability measure of theorem 1, the following linear growth condition,*

$$\begin{aligned}\int g^h(s,\eta)\ln g^h(s,\eta)dm &\leq 2C_1\tilde{h}(0) \qquad s \geq 0\\ \int g^{h,\lambda}(s,\eta)\ln g^{h,\lambda}(s,\eta)dm &\leq 2C_1\tilde{h}(0) \qquad s \geq 0\end{aligned} \tag{27}$$

where $g^h(\eta,s)$ and $g^{h,\lambda}(\eta,s)$ are the Radon-Nikodym derivatives of μS^h_s and $\mu S^{h,\lambda}_s$ respectively, restricted to $[-\tilde{h}(0),\tilde{h}(0)]$, with respect to Lebesgue measure m and C_1 is a constant.

Consider the processes $\zeta_1 = \{\eta^h_i : |i| \leq \tilde{h}(0)\}$ and $\zeta_2 = \{\eta^{h,\lambda}_i : |i| \leq \tilde{h}(0)\}$ taking values on $Y = T^{[-\tilde{h}(0),\tilde{h}(0)]}$. By their independence with respect to the components of η^h and $\eta^{h,\lambda}$ outside the interval $[-\tilde{h}(0), \tilde{h}(0)]$, they are finite dimensional diffusion processes and define for any initial distribution μ, probability measures P and Q respectively on $(C([0,\infty);Y), \mathcal{B}_Y)$, where $\mathcal{B}_Y$ is the corresponding Borel σ-algebra. These measures are associated to the second order differential operators

$$\Omega_s^{(1)} = \frac{1}{2} \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} (\sigma_i^h)^2(\eta, s) \frac{\partial^2}{\partial \eta_i^2} + \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} b_i^h(\eta, s) \frac{\partial}{\partial \eta_i}$$

and

$$\Omega_s^{(2)} = \lambda_{t,\tau}(s) \Omega_s^{(1)}$$

respectively, defined for $\mathcal{D} = \{f \in C(Y) : f \in C^2(Y)\}$. Now define the natural projections $\pi_{s_1,\ldots,s_n}$ as the continuous mappings from $C([0,\infty);Y)$ to Y^n given by $\pi_{s_1,\ldots s_n}\zeta = (\zeta_1, \ldots, \zeta_n)$, and let $\mathcal{F}_s$ be the collection of subsets of $C([0,\infty);Y)$ of the form $\pi^{-1}_{s_1,\ldots,s_n} H$ for some $s_1, \ldots, s_n \in [0,s]$ and $H \in \mathcal{B}_{Y^n}$, where $\mathcal{B}_{Y^n}$ is the Borel σ-algebra of Y^n. Then $\mathcal{B} = \bigcup_{s \geq 0} \sigma(\mathcal{F}_s)$. We will call P_s and Q_s the restrictions of P and Q to $\sigma(\mathcal{F}_s)$. Note that these restrictions are singular with respect to each other. To prove the variational limit (6), we will define a probability measure R on $(C([0,\infty);Y), \mathcal{B}_Y)$ with the following properties:

(i) For every $s \geq 0$, P_s and R_s are absolutely continuous with respect to each other. Here R_s is the restriction of R to $\sigma(\mathcal{F}_s)$.
(ii) The one dimensional marginals of R coincide with the one dimensional marginals of Q.
(iii)For an appropriate choice of the function $\lambda_{t,\tau}(s)$, the relative entropy between R_t and P_t converges to 0,

$$\lim_{t \to \infty} H(R_t | P_t) = 0 \tag{28}$$

Then the inequalities

$$||\mu S_{0,t}^{h,\lambda} - \mu S_{0,t}^h||_M \leq 6 H(\mu S_{0,t}^{h,\lambda}|_M | \mu S_{0,t}^h|_M) \tag{29}$$

and

$$H(\mu S_{0,t}^{h,\lambda}|_M | \mu S_{0,t}^h|_M) \leq H(R_t|P_t) \tag{30}$$

together with the property (iii) above, imply that $\lim_{t\to\infty} ||\mu S_{0,t}^{h,\lambda} - \mu S_{0,t}^h||_M = 0$. Here $\nu|_N$ denotes the restriction of ν to $T^{[-N,N]}$.

Let m be Lebesgue measure on Y. By remark 3, we can assume that $\mu S_{0,s}^{h.\lambda}|_{\tilde{h}(0)}$ is absolutely continuous with respect to m for $s \geq 0$. Let $g(s,\eta)$ be the corresponding Radon-Nikodym derivative. We define R as the probability measure on $(C([0,\infty);Y), \mathcal{B}_Y)$ with initial distribution μ, defined by the generator

$$\Omega_s^{(3)} = \frac{1}{2} \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} (\sigma_i^h)^2(\eta,s) \frac{\partial^2}{\partial \eta_i^2} + \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} \beta_i(\eta,s) \frac{\partial}{\partial \eta_i}$$

with domain $\mathcal{D}$. Here

$$\beta_i(\eta,s) = \frac{(1-\lambda_{t,\tau}(s))}{2} \frac{1}{g(\eta,s)} \frac{\partial((\sigma_i^h)^2(\eta,s)g(\eta,s))}{\partial \eta_i} + \lambda_{t,\tau}(s) b_i^h(\eta,s)$$

We will denote by $\vec{\beta}$ the corresponding vector with components β_i, and $-\tilde{h}(0) \leq i \leq \tilde{h}(0)$.

That R satisfies property (i) is a consequence of Girsanov's theorem. On the other hand,

$$\frac{\partial g(\eta,s)}{\partial s} = \Omega_s^{(2,*)} g(\eta,s) = \Omega_s^{(3,*)} g(\eta,s)$$

where $\Omega_s^{(2,*)}$ and $\Omega_s^{(3,*)}$ are the adjoints of $\Omega_s^{(2)}$ and $\Omega_s^{(3)}$ respectively. Therefore property (ii) is satisfied. To prove condition (iii) for a good choice of $\lambda_{t,\tau}$, or the convergence to 0 of the relative entropy in equation (28) we will need the following lemma,

Lemma 3. *Let $h(s) = c(t-s) + t^\alpha$ where $c > 0$. Assume that $\lambda_{t,\tau}(s) : [0,\infty) \to [1,2]$ is a monotone increasing, differentiable function of s. Then for $t \geq \max\{\frac{1}{c^{1/(1-\alpha)}}, \frac{R+1}{c}\}$ we have*

$$H(R_t|P_t) \leq C_2 \int_0^t (1-\lambda_{t,\tau})^2 h(s) ds + C_3 t (1-\lambda_{t,\tau}(t))^2$$

Here $C_2 := \frac{1}{a}\left(\frac{4B'A^4}{a} + 2B^2 + 4\frac{A^4}{a^2}(AA')^2 + 2(AA')^2\right)$, $C_3 := \frac{12A^4}{a^2}C_1 c + 2C_2$, $B := \sup_{i,\eta}|b_i(\eta)|$, $A := \sup_{i,\eta}|\sigma_i(\eta)|$, $B' := \sup_{i,j,\eta}\left|\frac{\partial b_i(\eta)}{\partial \eta_j}\right|$, $A' := \sup_{i,j,\eta}\left|\frac{\partial \sigma_i(\eta)}{\partial \eta_j}\right|$, $a := \inf_{i,\eta}\sigma_i(\eta)^2$ *and* C_1 *is the constant on inequality (27).*

Proof. By Girsanov's theorem, the Radon-Nikodym derivative of R_t with respect to P_t is given by

$$\frac{dR_t}{dP_t} = e^{\int_0^t \mathbf{a}^{-1}(\vec{\beta}-\vec{b}^h)\cdot d\eta^h - \frac{1}{2}\int_0^t(\vec{\beta}\cdot\mathbf{a}^{-1}\vec{\beta}-\vec{b}^h\cdot\mathbf{a}^{-1}\vec{b}^h)ds}$$

where $\mathbf{a}$ is the diagonal matrix with entries $(\sigma_i^h)^2$, for $-\tilde{h}(0) \leq i \leq \tilde{h}(0)$. It follows that

$$H(R_t|P_t) = \frac{1}{2}E^{R_t}(\int_0^t (1-\lambda_{t,\tau}(s))^2||\vec{c}-\vec{d}||^2 ds)$$

where

$$c_i(\eta,s) = \frac{1}{2\sigma_i^h(\eta,s)}\frac{1}{g(\eta,s)}\frac{\partial}{\partial \eta_i}((\sigma_i^h(\eta,s))^2 g(\eta,s))$$

and

$$d_i(\eta,s) = \frac{1}{\sigma_i^h(\eta,s)}b_i^h(\eta,s)$$

Therefore,

$$H(R_t|P_t) \leq \frac{1}{a}E^{R_t}\left(\int_0^t (1-\lambda_{t,\tau}(s))^2 \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} b_i^h(\eta,s)^2 ds\right)$$
$$+\frac{1}{4a}E^{R_t}\left(\int_0^t (1-\lambda_{t,\tau}(s))^2 \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} \frac{1}{g^2}\left(\frac{\partial}{\partial \eta_i}(\sigma_i^h(\eta,s)^2 g)\right)^2 ds\right) \qquad (31)$$

For the first term note that,

$$E^{R_t}\left(\int_0^t (1-\lambda_{t,\tau}(s))^2 \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} b_i^h(\eta,s)^2 ds\right) \leq 2B^2\int_0^t (1-\lambda_{t,\tau}(s))^2 h^+(s) ds \qquad (32)$$

where $h^+(s) := [h(s)] + 2$. And to analyse the second one note that

$$\left(\frac{\partial}{\partial \eta_i}((\sigma_i^h(\eta,s)^2 g(\eta,s))\right)^2 \leq 4(AA')^2 g^2(\eta,s)\theta_{[-h^+(s),h^+(s)]}(i) + A^4\left(\frac{\partial g}{\partial \eta_i}\right)^2$$

where $\theta_A(i)$ is the indicator function of $A \in \mathbf{Z}$. Therefore,

$$\frac{1}{a}E^{R_t}\left(\int_0^t (1-\lambda_{t,\tau})^2 \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} \frac{1}{g^2}\left(\frac{\partial}{\partial \eta_i}((\sigma_i^h)^2 g)\right)^2 ds\right) \leq 8\frac{(AA')^2}{a}\int_0^t (1-\lambda_{t,\tau})^2 h^+(s)ds$$
$$+ \quad \frac{A^4}{a}\int\int_0^t (1-\lambda_{t,\tau})^2 \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} \frac{1}{g}\left(\frac{\partial g}{\partial \eta_i}\right)^2 dsdm \tag{33}$$

For the second term in (33) we use the following relation,

$$\frac{\partial}{\partial s}(g(\eta,s)\ln g(\eta,s)) = \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} \lambda_{t,\tau}\left(\frac{1}{2}\frac{\partial^2}{\partial \eta_i^2}((\sigma_i^h)^2 g) - \frac{\partial}{\partial \eta_i}(b_i^h g)\right)\ln g + \frac{\partial g}{\partial s}$$

to obtain that

$$\begin{aligned}
\sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} \int_0^t (1-\lambda_{t,\tau})^2\lambda_{t,\tau}\int \frac{1}{g}\frac{\partial((\sigma_i^h)^2 g)}{\partial \eta_i}\frac{\partial g}{\partial \eta_i}dmds \leq \\
2\int\int_0^t (1-\lambda_{t,\tau})^2\lambda_{t,\tau} g \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)}\left|\frac{\partial b_i}{\partial \eta_i}\right| dmds \\
+2(1-\lambda_{t,\tau}(0))^2\int g(\eta,0)\ln g(\eta,0)dm + \\
4\int_0^t (\lambda_{t,\tau}-1)\lambda'_{t,\tau}\int g(\eta,s)\ln g(\eta,s)dmds
\end{aligned} \tag{34}$$

where $\lambda'_{t,\tau} = \frac{d\lambda_{t,\tau}(s)}{ds}$. Let $\epsilon > 0$ be arbitrary. Then note that,

$$\begin{aligned}
\frac{1}{g}\frac{\partial((\sigma_i^h)^2 g)}{\partial \eta_i}\frac{\partial g}{\partial \eta_i} &= \frac{(\sigma_i^h)^2}{g}\left(\frac{\partial g}{\partial \eta_i}\right)^2 + \frac{\partial(\sigma_i^h)^2}{\partial \eta_i}\frac{\partial g}{\partial \eta_i} \\
\left|\int \frac{\partial(\sigma_i^h)^2}{\partial \eta_i}\frac{\partial g}{\partial \eta_i}dm\right| &\leq \frac{1}{2\epsilon}\int\left(\frac{\partial(\sigma_i^h)^2}{\partial \eta_i}\right)^2 gdm + \frac{\epsilon}{2}\int\left(\frac{\partial g}{\partial \eta_i}\right)^2\frac{1}{g}dm
\end{aligned}$$

Making use of these relations in (34) we obtain

$$a\int_0^t (1-\lambda_{t,\tau})^2\lambda_{t,\tau}\int \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} \frac{1}{g}\left(\frac{\partial g}{\partial \eta_i}\right)^2 dmds \leq 4B'\int_0^t (1-\lambda_{t,\tau})^2\lambda_{t,\tau}h^+(s)ds +$$

$$2(1-\lambda_{t,\tau}(0))^2\int g(\eta,0)\ln g(\eta,0)dm + 4\int_0^t (\lambda_{t,\tau}-1)\lambda'_{t,\tau}\int g(\eta,s)\ln g(\eta,s)dmds$$

$$+\frac{4(AA')^2}{\epsilon}\int_0^t (1-\lambda_{t,\tau})\lambda_{t,\tau}h^+(s)ds + \frac{\epsilon}{2}\int_0^t (1-\lambda_{t,\tau})^2\lambda_{t,\tau}\int \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} \frac{1}{g}\left(\frac{\partial g}{\partial \eta_i}\right)^2 dmds$$

So that choosing $\epsilon = a$ and using the fact that $1 \leq \lambda_{t,\tau}(s) \leq 2$, we obtain

$$a\int_0^t (1-\lambda_{t,\tau})^2\int \sum_{i=-\tilde{h}(0)}^{\tilde{h}(0)} \frac{1}{g}\left(\frac{\partial g}{\partial \eta_i}\right)^2 dmds \leq 16\left(B' + \frac{(AA')^2}{a}\right)\int_0^t (1-\lambda_{t,\tau})^2 h^+(s)ds$$

$$+4(1-\lambda_{t,\tau}(0))^2\int g(\eta,0)\ln g(\eta,0)dm + 8\int_0^t (\lambda_{t,\tau}-1)\lambda'_{t,\tau}\int g(\eta,s)\ln g(\eta,s)dmds$$

By remark 3 which follows lemma 2, we can assume without loss of generality that there is a constant C_1 depending only on B, A' and a, such that

$$\int g(\eta,s)\ln g(\eta,s)ds \leq 2C_1\tilde{h}(0) \qquad s \geq 0$$

Thus, for $t \geq t^\alpha/c$ and $t \geq (R+1)/c$, since $2\tilde{h}(0) \leq 6ct$, we get the following bound after substituting in (33)

$$\frac{1}{a}\quad E^{P_t}\left(\int_0^t (1-\lambda_{t,\tau})^2 \sum_{i=-[h(0)]}^{[h(0)]} \frac{1}{g^2}\left(\frac{\partial}{\partial \eta_i}((\sigma_i^h)^2 g)\right)^2\right) \leq$$

$$\left(8\frac{(AA')^2}{a} + 16\frac{A^4 B'}{a^2} + 16\frac{A^4(AA')^2}{a^3}\right)\int_0^t (1-\lambda_{t,\tau})^2 h^+ ds + 48\frac{A^4}{a^2}cC_1 t(1-\lambda_{t,\tau}(t))^2$$

where we have used the fact that $\lambda_{t,\tau}(s)$ is monotone increasing. The proof now follows by a substitution of this estimate and estimate (32) on the bound (31) on the relative entropy $H(R_t|P_t)$. □

The following lemma tells us that for an appropriate choice of λ condition (iii) for the relative entropy of the process R with respect to P is satisfied.

Lemma 4. *Let* $h(s) = c(t-s) + t^{\alpha}$, $\lambda_{t,\tau}(s) = \frac{c\tau}{c(t-s)+t^{\alpha}} \frac{1}{\ln(ct^{1-\alpha}+1)} + 1$ *and* $\frac{1}{2} < \alpha < 1$. *Then*

$$\lim_{t \to \infty} H(R_t|P_t) = 0$$

Proof. The lemma follows from the equations

$$\int_0^t (1 - \lambda_{t,\tau}(s))^2 h(s) ds = \frac{c\tau^2}{\ln(ct^{1-\alpha}+1)}$$

and $(1 - \lambda_{t,\tau}(t))^2 t = \frac{1}{\ln^2(ct^{1-\alpha}+1)} (c\tau)^2 t^{1-2\alpha}$. □

5. AN APPLICATION.

In this section we present an infinite dimensional diffusion process for which it can be proved that there is a unique invariant measure. As a consequence of Theorem 1 and the compactness of the state space, it follows that when the underlying lattice has dimension $d = 1$, any initial probability measure evolved according to the semi-group of the process converges weakly to the unique invariant measure.

Theorem 4. *Let* S_t *be the semi-group of an infinite dimensional diffusion family on* $T^{\mathbf{Z}^d}$ *with coefficients* σ *and* b *such that* $\sigma_i(\eta) = 1$ *for* $i \in \mathbf{Z}^d$. *Assume that* b *is finite range, uniformly bounded and has bounded first order partial derivatives. Also, suppose that* b_i *are smooth and that* $\frac{\partial b_i}{\partial \eta_i} = 0$ *for* $i \in \mathbf{Z}^d$. *Then the Lebesgue measure* m *on* $T^{\mathbf{Z}^d}$ *is the unique invariant measure for this process. Moreover, when* $d = 1$, *given any probability measure* μ *on* $T^{\mathbf{Z}}$, *one has that*

$$\lim_{t \to \infty} \mu S_t = m$$

To prove this theorem we will use the following auxiliary lemma,

Lemma 5. *Let* $f(x) : T \to R$ *be a continuous function and* $g(x) : T \to [0, \infty)$ *be an absolutely continuous function. Assume that* $\int_T f dx = 0$. *Then,*

$$\left| \int_T f g^2 dx \right| \le \frac{5}{4} \int_T f^2 g^2 dx + \frac{5}{4} \int_T \left(\frac{dg}{dx} \right)^2 dx \tag{35}$$

proof of Lemma 5. Since the inequality is true for $g = 0$, we can without loss of generality assume that $\int_T g^2 dx = 1$. Let a be such that $a > 1$. First assume that

$$\inf_{x\in[0,1]} g(x) \geq \frac{1}{a}. \tag{36}$$

Then,

$$\begin{aligned} \left|\int_T fg^2 dx\right| &= \left|\int_T f(g^2-1)dx\right| \\ &\leq a\int_T f^2 g^2 dx + \frac{1}{4a}\int_T \frac{(g^2-1)^2}{g^2}dx \end{aligned}$$

where the inequality is a consequence of the fact that $|f(g^2-1)| = \left|\sqrt{2a}fg \cdot \frac{(g^2-1)}{\sqrt{2a}g}\right| \leq a(fg)^2 + \frac{1}{4a}\left(\frac{(g^2-1)}{g}\right)^2$. Now, since the average of the square of g equals 1, there must be some $x_0 \in T$ such that $g(x_0) = 1$. Therefore,

$$\begin{aligned} \int_T \frac{(g^2(x)-1)^2}{g^2(x)}dx &= \int_T \frac{1}{g^2(x)}\left(\int_{x_0}^{x} \frac{dg^2(x')}{dx'}dx'\right)^2 dx \\ &\leq \int_T \frac{1}{g^2}\left(\frac{dg^2}{dx}\right)^2 dx \int_T \frac{1}{g^2}dx \end{aligned}$$

where the inequality is a consequence of Cauchy-Schwartz and the fact that the square of g has mean 1: $\left(\int_{x_0}^{x} \frac{dg^2(x')}{dx'}dx'\right)^2 \leq \left(\int_T \frac{1}{g^2}\left(\frac{dg^2}{dx}\right)^2 dx\right) \cdot \left(\int_T g^2 dx\right) = \int_T \frac{1}{g^2}\left(\frac{dg^2}{dx}\right) dx$. Thus, from the assumption (36), we conclude that

$$\begin{aligned} \left|\int_T fg^2 dx\right| &\leq a\int_T f^2 g^2 dx + \frac{1}{4a}\int_T \frac{1}{g^2}\left(\frac{dg^2}{dx}\right)^2 dx \int_T \frac{1}{g^2}dx \\ &\leq a\int_T f^2 g^2 dx + a\int_T \left(\frac{dg}{dx}\right)^2 dx \end{aligned} \tag{37}$$

Now suppose that $\inf_{x\in[0,\infty)} g(x) \leq \frac{1}{a}$. By our previous observation we know that $g(x_0) = 1$ for some $x_0 \in T$. Then, the total variation T_g of g satisfies $T_g \geq 2(1-1/a)$. Thus, $\int_T \left(\frac{dg}{dx}\right)^2 dx \geq T_g^2 \geq 4(1-1/a)^2$. From here it follows that

$$\left|\int_T fg^2dx\right| \le a\int_T f^2g^2dx + \frac{1}{4a} \le a\int_T f^2g^2dx + \frac{1}{16}\frac{a}{(1-a)^2}\int_T\left(\frac{dg}{dx}\right)^2 dx \tag{38}$$

Letting $a = 5/4$ in (37) and (38) it follows that independently of the lower bound on g, inequality (35), which is what we wanted to prove, is satisfied. □

proof of Theorem 4. First note that the second statement of the Theorem, for $d = 1$, follows from the uniqueness statement, Theorem 1 and the compactness of the state space $T^{\mathbf{Z}^d}$. We therefore proceed to prove the uniqueness statement. We have followed some ideas presented in Holley-Stroock [2].

Let R be the range of the process and define $\Lambda_N = [-NR, NR]^d$ and $\delta\Lambda_N = \Lambda_N/\Lambda_{N-1}$. Let μ be a probability measure on $T^{\mathbf{Z}^d}$. From [2] we know the existence of the Radon-Nikodym derivative $u_N(t,\eta)$ of μS_t restricted to T^{Λ_N} with respect to Lebesgue measure on T^{Λ_N} which for fixed $t > 0$ has continuous first and second order partial derivatives with respect to η and is strictly positive.

We begin with some preliminary calculations involving the marginals $u_N(t,\eta)$. Let ϕ be any local function depending on coordinates on Λ_N and belonging to $C^2(T^{\mathbf{Z}^d})$. Then,

$$\begin{aligned}\int \phi\frac{\partial u_N}{\partial t}dm &= \sum_{i\in\Lambda_N}\int\left(\frac{1}{2}\frac{\partial^2\phi}{\partial\eta_i^2}u_N + b_i\frac{\partial\phi}{\partial\eta_i}u_{N+1}\right)dm \\ = \frac{1}{2}\sum_{i\in\Lambda_N}\int\phi\frac{\partial^2 u_N}{\partial\eta_i^2}dm &- \sum_{i\in\Lambda_{N-1}}\int\phi b_i\frac{\partial u_N}{\partial\eta_i}dm - \sum_{i\in\delta\Lambda_N}\int\phi b_i\frac{\partial u_{N+1}}{\partial\eta_i}dm\end{aligned}$$

where in the second equality we have used integration by parts. Choosing $\phi = \ln u_N$, we obtain the following evolution equation for the relative entropy $h_N(\mu) = \int u_N \ln u_N dm$ of μ restricted to Λ_N with respect to Lebesgue measure:

$$\begin{aligned}\frac{dh_N}{dt} &= \frac{1}{2}\sum_{i\in\Lambda_N}\int\frac{\partial^2 u_N}{\partial\eta_i^2}\ln u_N dm - \sum_{i\in\Lambda_{N-1}}\int b_i\frac{\partial u_N}{\partial\eta_i}\ln u_N dm - \sum_{i\in\delta\Lambda_N}\int b_i\frac{\partial u_{N+1}}{\partial\eta_i}\ln u_N dm \\ &= -\frac{1}{2}\sum_{i\in\Lambda_N}\int\left(\frac{\partial u_N}{\partial\eta_i}\right)^2\frac{1}{u_N^2}d\mu + \sum_{i\in\delta\Lambda_N}\int b_i\frac{\partial u_N}{\partial\eta_i}\frac{1}{u_N}d\mu\end{aligned}$$

Thus, in the case in which μ is an invariant measure it follows that

$$\frac{1}{2}\sum_{i\in\Lambda_N}\int\left(\frac{\partial u_N}{\partial\eta_i}\right)^2\frac{1}{u_N^2}d\mu=\sum_{i\in\delta\Lambda_N}\int b_i\frac{\partial u_N}{\partial\eta_i}\frac{1}{u_N}d\mu \tag{39}$$

We divide the rest of the proof into three steps.

STEP 1. Firstly, note that from the equation (39) and Lemma 5 we obtain the following inequality,

$$\sum_{i\in\Lambda_N}\int\left(\frac{\partial u_N}{\partial\eta_i}\right)^2\frac{1}{u_N^2}d\mu\le B^2\left[\sum_{i\in\delta\Lambda_N}\int\left(\frac{\partial u_N}{\partial\eta_i}\right)^2\frac{1}{u_N^2}d\mu+\sum_{i\in\delta\Lambda_N}\int\left(\frac{\partial u_{N+1}}{\partial\eta_i}\right)^2\frac{1}{u_{N+1}^2}d\mu\right] \tag{40}$$

where $B^2=\frac{5}{2}\max\{1,\sup_{i,\eta}|b_i(\eta)|^2\}$. Now note that the following convexity inequality is true,

$$\int\left(\frac{\partial u_m}{\partial\eta_i}\right)^2\frac{1}{u_m^2}d\mu\le\int\left(\frac{\partial u_n}{\partial\eta_i}\right)^2\frac{1}{u_n^2}d\mu \tag{41}$$

whenever $m\le n$ and $i\in\Lambda_m$. Therefore, defining $b_n=\sum_{i\in\delta\Lambda_n}\int\left(\frac{\partial u_{n+1}}{\partial\eta_i}\right)^2\frac{1}{u_{n+1}^2}d\mu$ we obtain from (40) that

$$\sum_{n=1}^{N-1}b_n\le Cb_N \qquad N\ge 2 \tag{42}$$

where $C=2B^2$.

STEP 2. Here we will prove the following polynomial bound,

$$b_n\le Kn^{d-1} \tag{43}$$

for some constant K. From the equality (39), note that,

$$\sum_{n=1}^{N}\sum_{i\in\delta\Lambda_N}\int\left(\frac{\partial u_N}{\partial\eta_i}\right)^2\frac{1}{u_N^2}d\mu\le U_N\sqrt{\sum_{i\in\delta\Lambda_N}\int\left(\frac{\partial u_N}{\partial\eta_i}\right)^2\frac{1}{u_N^2}d\mu} \tag{44}$$

where $U_N=2(\sup_{i,\eta}|b_i(\eta)|)(2R+1)^{d/2}\sqrt{(2NR+1)^{d-1}}$. Clearly this implies that

$$\sum_{i \in \delta\Lambda_N} \int \left(\frac{\partial u_N}{\partial \eta_i}\right)^2 \frac{1}{u_N^2} d\mu \leq U_N^2$$

Substituting this estimate in equation (44) and using the convexity (41) we conclude that $b_n \leq K n^{d-1}$ for some constant K.

STEP 3. We finish the proof by a contradiction argument. Assume that $b_n > 0$ for some $n \geq 1$. Let n_0 be the smallest such n. Then, from inequality (42) we see that

$$\sum_{n=1}^{N-1} b_{n-1+n_0} \leq C b_{N-1+n_0} \qquad N \geq 2$$

Now let c_n, $n \geq 1$, be the unique sequence of real numbers that is a solution to the system of equations,

$$\begin{aligned} \sum_{n=1}^{N-1} c_n &= C c_N \qquad N \geq 2 \\ c_1 &= b_{n_0} \end{aligned}$$

It is easy to check that $c_n = \frac{b_{n_0}}{C}\left(1 + \frac{1}{C}\right)^{n-2}$, for $n \geq 2$. But by induction on n it is clear that $b_{n-1+n_0} \geq c_n$, for $n \geq 1$. Thus,

$$b_{n+n_0} \geq \frac{b_{n_0}}{C}\left(1 + \frac{1}{C}\right)^{n-1} \qquad n \geq 1$$

However this contradicts the estimate (43) of step 2. It follows that $b_n = 0$ for $n \geq 1$. It is easy to see that this implies $\frac{\partial u_N}{\partial \eta_i} = 0$ whenever $i \in \Lambda_N$ and $N \geq 1$. □

Remark 4. *Theorem 4 can be strengthened as follows: the conditions $\sigma_i = 1$ and $\frac{\partial b_i}{\partial \eta_i} = 0$ can be replaced by $\frac{\partial \sigma_i}{\partial \eta_j} = 0$ for $i \neq j$ and the hypothesis that the Lebesgue measure is an invariant measure for the process [6].*

Acknowledgement: I am very grateful to my thesis advisor Professor S.R.S. Varadhan for suggesting this problem and for his constant help and encouragement in this work. I would like to note that the idea of constructing the process R_t to apply Girsanov's theorem, is due to him. I would also like to acknowledge the Courant Institute of Mathematical Sciences, New York University, for their financial support. Most of this work was done there as part of my Ph.D. thesis. Finally I would like to acknowledge Professor H.T. Yau for useful discussions related to the proof of Theorem 4.

References

[1] E.B. Fabes and D.W. Stroock, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash.*, Arch. Rational Mech. Anal., **96** No. 4, 327 (1986).

[2] R. Holley and D. Stroock, *Diffusions on an Infinite Dimensional Torus.*, J. Functional Analysis **42**,29 (1981).

[3] T.M. Liggett, *Interacting Particle Systems.* Springer-Verlag, New York (1985).

[4] S. Lu, *Hydrodynamic Scaling Limits with Deterministic Initial Configurations.*, Thesis, New York University (1992).

[5] T.S. Mountford, *A Coupling of Infinite Particle Systems*, J. Math. Kyoto Univ. **35** (1995), no. 1, 43–52.

[6] A.F. Ramírez, in preparation.

[7] A.F. Ramírez, *Relative Entropy and Limit Measures for some Infinite Dimensional Processes*, Thesis, New York University (1996).

[8] A.F. Ramírez and S.R.S. Varadhan, *Relative Entropy and Mixing Properties of Interacting Particle Systems*, J. Math. Kyoto Univ. **36** (1996), no. 4, 869-875.