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An elementary proof of the uniqueness of invariant product measures for some infinite dimensional processes

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Abstract.

Consider an infinite dimensional diffusion process with state space $T^{\mathbb{Z}^d}$, where T is the circle, and defined by an infinitesimal generator L which acts on local functions f as $Lf(\eta) = \sum_{i \in \mathbb{Z}^d} (\frac{\sigma_i^2(\eta_i)}{2} \frac{\partial^2 f}{\partial \eta_i^2} + b_i(\eta) \frac{\partial f}{\partial \eta_i})$. Suppose that the coefficients σ_i and b_i are smooth, bounded, of finite range, have uniformly bounded second order partial derivatives, that σ_i are uniformly bounded from below by some strictly positive constant, and that σ_i is a function only of η_i . Suppose that there is a product measure ν which is invariant. Then if ν is the Lebesgue measure or if $d = 1, 2$, it is the unique invariant measure. Furthermore, if ν is translation invariant, it is the unique invariant, translation invariant measure. The proofs are elementary. Similar results can be proved in the context of an interacting particle system with state space $D^{\mathbb{Z}^d}$, D finite, with uniformly positive bounded flip rates which are finite range. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Une preuve élémentaire de l'unicité des mesures invariantes produits pour certains processus en dimension infinie

Résumé.

Considérons une diffusion en dimension infinie avec un espace d'état $T^{\mathbb{Z}^d}$, où T est le cercle, et définie par un générateur infinitésimal L qui agit sur les fonctions locales f de la façon suivante : $Lf(\eta) = \sum_{i \in \mathbb{Z}^d} (\frac{\sigma_i^2(\eta_i)}{2} \frac{\partial^2 f}{\partial \eta_i^2} + b_i(\eta) \frac{\partial f}{\partial \eta_i})$. Supposons que les coefficients σ_i et b_i sont C^∞ , bornés, de portée finie, ont des dérivées partielles du deuxième ordre qui sont uniformément bornées, que les σ_i ont une borne inférieure uniforme qui est une constante strictement positive, et que σ_i est une fonction seulement de η_i . Supposons qu'il y ait une mesure produit ν qui est invariante. Nous démontrons que si ν est égale à la mesure de Lebesgue ou si $d = 1, 2$, alors ν est la seule mesure invariante. D'autre part, si ν est invariante par translation, alors ν est la seule mesure invariante, invariante par translation. Les preuves sont élémentaires. Des résultats similaires peuvent être démontrés dans le contexte de systèmes de particules avec un espace d'état $D^{\mathbb{Z}^d}$, D finie, avec des taux de saut uniformément positives, à portée finie et bornés. © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Version française abrégée

Considérons un processus de diffusion avec un espace d'état compact et de dimension finie. Si son générateur est elliptique, l'ensemble des mesures invariantes a un unique élément. Cela n'est plus vrai si l'espace d'état est de dimension infinie. En fait, il suffit de considérer un processus de diffusion qui représente la dynamique d'un système de spins dans un réseau de dimension d . Quand $d > 1$, on peut avoir plusieurs mesures de Gibbs, qui par construction sont réversibles par rapport au générateur du processus.

Une question naturelle est de connaître le genre de conditions qui assurent l'unicité de la mesure invariante pour des diffusions générales. Dans cette Note, nous montrons avec des méthodes élémentaires que, sous des conditions faibles pour les coefficients, ce qui suit est vrai pour des diffusions en dimension infinie dans le tore $T^{\mathbf{Z}^d}$: (i) si la mesure de Lebesgue est invariante, elle est la seule mesure invariante ; si $d = 1$ ou 2 et si il y a une mesure produit qui est invariante, alors elle est la seule mesure invariante ; (iii) si il y a une mesure produit invariante et invariante par translation, alors elle est la seule mesure invariante et invariante par translation. Ce théorème est une extension du théorème 4 de [5].

La preuve est basée sur l'analyse d'une fonctionnelle de Lyapounov par rapport à une mesure invariante donnée, qui est calculée sur des boîtes Λ_n de taille finie qui convergent quand $n \rightarrow \infty$ vers l'ensemble \mathbf{Z}^d . Il suffit de montrer que cette fonctionnelle est nulle pour démontrer les résultats d'unicité. La difficulté que l'on rencontre est de pouvoir contrôler les termes de cette fonctionnelle qui proviennent de la frontière de la boîte. La preuve de (i) est la plus difficile. Elle est inspirée des méthodes entropiques développées par Holley–Stroock [1].

Soit T le cercle. Considérons le semi-groupe S_t associé au générateur $L = \sum_{i \in \mathbf{Z}^d} (\frac{1}{2} \sigma_i \frac{\partial^2}{\partial \eta_i} + b_i \frac{\partial}{\partial \eta_i})$ qui définit un processus de Feller dans $\Omega := C([0, \infty); T^{\mathbf{Z}^d})$ [1]. Ici, $\sigma(\eta), b(\eta) : T^{\mathbf{Z}^d} \rightarrow R^{\mathbf{Z}^d}$ sont des fonctions Borel-mesurables, bornées ($\sup_{i, \eta} \{\sigma_i, b_i\} < \infty$) et à portée finie, σ_i, b_i sont leurs i -èmes composantes, $\eta \in T^{\mathbf{Z}^d}$ et η_i est la i -ème composante de η . Quand $\sup_{i, j, k, \eta} \left\{ \frac{\partial^2 \sigma_i}{\partial \eta_j \partial \eta_k}, \frac{\partial^2 b_i}{\partial \eta_j \partial \eta_k} \right\} < \infty$ on dira que les coefficients σ et b ont des dérivées partielles du deuxième ordre bornées. Quand $\inf_{i, \eta} \sigma_i(\eta) > 0$, on dira que cette diffusion est uniformément elliptique. Le résultat principal de cette note est le théorème suivant :

THÉORÈME 1. – *Considérons une diffusion en dimension infinie, uniformément elliptique avec des coefficients σ et b à portée finie, bornés et avec dérivées partielles du deuxième ordre bornées. Supposons que les coefficients sont C^∞ et que σ_i est une fonction seulement de η_i . Soit ν une mesure produit qui est invariante. Alors,*

- (i) *Si ν est invariante par translation, elle est l'unique mesure invariante, invariante par translation.*
 - (ii) *Si $d = 1, 2$, ν est l'unique mesure invariante.*
 - (iii) *Si ν est la mesure de Lebesgue, elle est l'unique mesure invariante.*
- D'autre part, si $d = 1$ et μ est une mesure de probabilité arbitraire dans \mathbf{T}^d on a $\lim_{t \rightarrow \infty} \mu S_t = \nu$.*

1. Introduction

Consider a diffusion process with a finite dimensional, compact state space. If the generator is elliptic, we know that the set of invariant measures has a unique element. If the state space is infinite dimensional this is no longer true. A natural question that arises is what are suitable conditions that ensure uniqueness of the invariant measure for general diffusions. Very few general results are known [2], and most are in a reversible context. In this Note, we show how elementary methods can give interesting information for more general diffusions.

Let T be the unit circle. Consider $\Omega := C([0, \infty); T^{\mathbf{Z}^d})$, the space of continuous functions from $[0, \infty)$ to $T^{\mathbf{Z}^d}$, with the topology of uniform convergence in compact subsets of $[0, \infty)$. Now let S_t be the

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unique semi-group associated to the generator $L = \sum_{i \in \mathbf{Z}^d} (\frac{1}{2} \sigma_i(\eta) \frac{\partial^2}{\partial \eta_i^2} + b_i(\eta) \frac{\partial}{\partial \eta_i})$, defined through its action on $C^2(T^{\mathbf{Z}^d})$ local functions. Here, $b, \sigma : T^{\mathbf{Z}^d} \rightarrow \mathbf{R}^{\mathbf{Z}^d}$ are Borel-measurable, σ_i, b_i are their i -th components, $\eta \in T^{\mathbf{Z}^d}$ and η_i are the i -th components of η . We say that the coefficients σ and b are bounded if $\sup_{i, \eta} \{\sigma_i, b_i\} < \infty$ and of finite range $R \in \mathbf{Z}^+$ if for each $i \in \mathbf{Z}^d$, $\sigma_i(\eta)$ and $b_i(\eta)$ depend only on coordinates η_j of η such that $|j - i| \leq R$. We say that the coefficients σ and b have bounded second order partial derivatives if $\sup_{i, j, k, \eta} \{ \frac{\partial^2 \sigma_i}{\partial \eta_j \partial \eta_k}, \frac{\partial^2 b_i}{\partial \eta_j \partial \eta_k} \} < \infty$. The generator defined above, with coefficients σ and b that are bounded, of finite range and with bounded second order partial derivatives defines a Feller semi-group S_t on the space of functions which are continuous and with values on $T^{\mathbf{Z}^d}$ [1]. Such a process will be called a finite range infinite dimensional diffusion family with bounded coefficients σ and b with bounded second order partial derivatives. If $\inf_{i, \eta} \sigma_i(\eta) > 0$, we will say that this diffusion is uniformly elliptic. In this Note we prove the following result which is a considerable improvement of Theorem 4 of [5].

THEOREM 1. – *Consider a finite range uniformly elliptic infinite dimensional diffusion family on $T^{\mathbf{Z}^d}$ with bounded coefficients b and σ . Assume that they are smooth, have bounded second order partial derivatives and that σ_i is a function only of η_i . Suppose that ν is a product measure which is invariant. Then,*

- (i) *If ν is translation invariant, it is the unique invariant, translation invariant measure.*
- (ii) *If $d = 1$ or 2 , ν is the unique invariant measure.*
- (iii) *If ν is the Lebesgue measure, it is the unique invariant measure.*

Furthermore, if $d = 1$ and μ is an arbitrary probability measure on $T^{\mathbf{Z}^d}$, then $\lim_{t \rightarrow \infty} \mu S_t = \nu$.

The last statement of this theorem is implied by part (ii) and Theorem 1 of [5]. Also, a simple consequence of part (iii) of the above theorem is the following corollary which gives a divergence free type condition which ensures that Lebesgue measure is invariant.

COROLLARY 1. – *Consider a finite range uniformly elliptic infinite dimensional diffusion family on $T^{\mathbf{Z}^d}$ with bounded coefficients b and σ . Assume that they are smooth, have bounded second order partial derivatives and that σ_i is a function only of η_i . Suppose there is an increasing sequence of finite subsets $\Lambda_n \subset \mathbf{Z}^d$ such that $\bigcup_{n=1}^{\infty} \Lambda_n = \mathbf{Z}^d$ and that $\lim_{n \rightarrow \infty} \int | \sum_{i \in \Lambda_n} (\frac{\partial^2 \sigma_i}{\partial \eta_i^2} - \frac{\partial b_i}{\partial \eta_i}) | dm = 0$. Then the Lebesgue measure is the unique invariant measure of the diffusion process.*

The proofs are based on an analysis of a Lyapunov functional related to a given invariant measure, which is computed on boxes Λ_n converging to the whole lattice \mathbf{Z}^d as $n \rightarrow \infty$. Showing that this functional vanishes is sufficient to prove a uniqueness result. The difficulty lies in the control of the boundary terms of the Lyapunov functional. An application of the method developed by Holley–Stroock [1] gives this control for $d \leq 2$, which in turn enables us to prove (ii). A different kind of argument provides a short proof of (ii). The proof of (iii) is harder.

It is possible to partially extend Theorem 1 to a discrete context. In fact, consider an interacting particle system defined by the action of its generator on local functions f of its state space $D^{\mathbf{Z}^d}$, D finite, as $Lf(\eta) = \sum_{T \subset \mathbf{Z}^d} \int_{D^T} c_T(d\zeta, \eta) (f(\eta^\zeta) - f(\eta))$. Here $c_T(d\zeta, \eta)$ describes the rates for Poisson events that change the current configuration η to a new configuration η^ζ that has been altered on the finite index set $T \subset \mathbf{Z}^d$ from η to ζ (see Liggett [3]).

THEOREM 2. – *Consider an interacting particle system on $X := D^{\mathbf{Z}^d}$ with bounded flip rates and of finite range. Suppose that $\inf_{T \subset \mathbf{Z}^d, \eta \in X, \zeta \in D^T} c_T(\zeta, \eta) > 0$. Let ν be a product measure that is invariant and call ν_i its restriction to site i . Then*

- (i) *If $d = 1$ or 2 and if $\inf_{i, \eta_i} \nu_i > 0$, then ν is the unique invariant measure.*
- (ii) *If ν is translation invariant, it is the unique invariant, translation invariant measure.*

Furthermore, if $d = 1$ and μ is an arbitrary probability measure on $D^{\mathbf{Z}^d}$, then $\lim_{t \rightarrow \infty} \mu S_t = \nu$.

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Note that the last statement is implied by Theorem 1 of Mountford [4] (*see also* [6]). The proof of parts (i) and (ii), which we omit in this Note, are the discrete context version of parts (i) and (ii) of Theorem 1.

2. Proof of Theorem 1

Define $\Lambda_N := [-NR, NR]^d$ and $\delta\Lambda_N := \Lambda_N \setminus \Lambda_{N-1}$, where R is the range of b . Let also $\Lambda_{N,i} := \Lambda_N \cup B_i$, where $B_i := \{j \in \mathbf{Z}^d : |i - j| \leq R\}$ is the ball of radius R centered at i . Now let μ be an arbitrary invariant measure. For a given finite subset F of \mathbf{Z}^d , we will denote by u_F the Radon–Nykodim derivative of μ restricted to the subset F with respect to Lebesgue measure m restricted to the same set. Also, we will use u_N for u_{Λ_N} . Similarly we will denote by v_F the Radon–Nykodim derivative of ν restricted to the subset F with respect to Lebesgue measure on F , and by v_N we will mean v_{Λ_N} . Using Malliavin calculus tools it is possible to show that these derivatives exist and are smooth (*see, for example, Holley–Stroock* [1]). Now, let ϕ be some local, smooth function depending on coordinates only on Λ_N . Then if $a_i = \sigma_i^2$ we have,

$$0 = \int \phi \frac{\partial u_N}{\partial t} dm = \sum_{i \in \Lambda_N} \int a_i \frac{1}{2} \frac{\partial^2 \phi}{\partial \eta_i^2} u_N dm + \sum_{i \in \Lambda_{N-1}} \int b_i \frac{\partial \phi}{\partial \eta_i} u_N dm + \sum_{i \in \delta\Lambda_N} \int b_i \frac{\partial \phi}{\partial \eta_i} u_{\Lambda_{N,i}} dm. \quad (1)$$

Let us call v_i the density of ν at site $i \in \mathbf{Z}^d$. Note that since ν is a product invariant measure, the measure on T with density v_i is invariant for the diffusion on T defined by the generator $L_i := a_i(x) \frac{d^2}{dx^2} + \bar{b}_i(x) \frac{d}{dx}$, where $\bar{b}_i(x) := \int b_i(\eta) d\nu_i$ with $\eta_i = x$, and ν_i is the restriction of ν to $\mathbf{Z}^d - \{i\}$. By standard arguments for one dimensional diffusions, it follows from the smoothness of $a_i(x)$ and $\bar{b}_i(x)$ and the ellipticity, that v_i is strictly positive on T . Compactness of T implies that $\inf_{\eta} v_i(\eta) > 0$ and therefore for each $N \geq 1$, $\inf_{\eta} v_N(\eta) > 0$. By Theorem 2.25 of Holley–Stroock we can now conclude that for each $N \geq 1$, $\inf_{\eta} u_N(\eta) > 0$. Hence, we can choose ϕ in (1) as the smooth function $\ln(u_N/v_N)$, so that using the equation $\sum_{i \in \Lambda_N} \int \left(\frac{a_i}{2} \frac{\partial^2}{\partial \eta_i^2} \frac{u_N}{v_N} + b_i \frac{\partial}{\partial \eta_i} \frac{u_N}{v_N} \right) v_{\Lambda_{N,i}} dm = 0$ (which is a consequence of the assumed invariance of ν), and using the fact that for each $i \in \mathbf{Z}^d$ the coefficient a_i depends only on the coordinate η_i we conclude that,

$$\frac{1}{2} \sum_{i \in \Lambda_N} \int a_i \frac{v_N}{u_N} \left(\frac{\partial(u_N/v_N)}{\partial \eta_i} \right)^2 d\nu = \sum_{i \in \delta\Lambda_N} \int b_i \frac{v_N}{u_N} \frac{\partial(u_N/v_N)}{\partial \eta_i} (u_{\Lambda_{N,i}} - u_N v_{\delta,N,i}) dm, \quad (2)$$

where $v_{\delta,N,i}$ is the density of ν on $\Lambda_{N,i} - \Lambda_N$. Now, the left-hand side of (2) can be lower bounded changing a_i by $a := \inf_{i,\eta} a_i(\eta)$. Also, the right-hand side can be upper-bounded by $2B \sum_{i \in \delta\Lambda_N} \int \frac{v_N}{u_N} \left| \frac{\partial(u_N/v_N)}{\partial \eta_i} \right| d\mu$, where $B := \sup_{i,\eta} |b_i(\eta)|$. Then, using Cauchy–Schwartz inequality twice in the integration and summation of this last quantity we get that,

$$\frac{a}{2} \sum_{i \in \Lambda_N} \int \frac{v_N}{u_N} \left(\frac{\partial(u_N/v_N)}{\partial \eta_i} \right)^2 d\nu \leq U_N \sqrt{\sum_{i \in \delta\Lambda_N} \int \frac{v_N}{u_N} \left(\frac{\partial(u_N/v_N)}{\partial \eta_i} \right)^2 d\nu}, \quad (3)$$

where $U_N := 2Bd(4R)^{d/2} N^{(d-1)/2}$ and $a := \inf_{i,\eta} a_i(\eta)$. At this point let us define for $n \geq 1$, $c_n := \sum_{i \in \delta\Lambda_n} \int \frac{v_n}{u_n} \left(\frac{\partial(u_n/v_n)}{\partial \eta_i} \right)^2 d\nu$. It follows from the inequality $c_n \leq \frac{1}{2} \sum_{i \in \Lambda_n} \int \frac{v_n}{u_n} \left(\frac{\partial(u_n/v_n)}{\partial \eta_i} \right)^2 d\nu$, the fact that $a > 0$ and inequality (3), that there is some constant $K > 0$ such that,

$$c_n \leq K n^{d-1} \quad (4)$$

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for every $n \geq 1$. We will complete the proof of Theorem 1 separately for each part. Before, we need to state a convexity property for the free energy, which will turn out to be crucial (see Lemma 3.3 of [1] for a similar statement). Let $x, y \in T$ and $f(x, y) : T^2 \rightarrow \mathbf{R}$ be bounded, and for each y a measurable function of x . For each $x, y \in T$ assume that the partial derivative $f_y := \partial f / \partial y$ exists and is bounded in T^2 . Then if $\alpha(x) : T \rightarrow [0, \infty)$ is a bounded measurable function with $\int_T \alpha dx = 1$ we have, $\left(\frac{\int_T \alpha f dx}{\int_T \alpha dx} \right)_y^2 \int_T \alpha f dx \leq \int_T \alpha \left(\frac{f_y}{f} \right)^2 f dx$, where $\left(\int_T \alpha f dx \right)_y$ is the partial derivative of $\int_T \alpha f dx$ with respect to y . It now follows from this property and the fact that ν is a product measure with smooth densities on boxes that for any i, F, N such that $i \in F \subset \Lambda_N$,

$$\int \frac{u_F}{v_F} \left(\frac{\partial(u_F/v_F)}{\partial \eta_i} \right)^2 d\nu \leq \int \frac{u_N}{v_N} \left(\frac{\partial(u_N/v_N)}{\partial \eta_i} \right)^2 d\nu. \quad (5)$$

2.1. *Proof of part (i).* – Assume that μ is translation invariant. Substituting (4) in (3) we conclude that $a \frac{1}{2} \sum_{i \in \Lambda_N} \int \frac{v_N}{u_N} \left(\frac{\partial(u_N/v_N)}{\partial \eta_i} \right)^2 d\nu \leq C_1 N^{d-1}$, for some constant $C_1 > 0$. Now, let F be a bounded subset of the lattice and $i \in F$. Define Λ_N^F as the points of Λ_N at a distance larger than $\text{diam}(F)$ to $\delta \Lambda_N$. Then, we can apply (5) to each $j \in \Lambda_N^F$ and use the translation invariance of μ and ν to conclude that $\frac{a}{2} |\Lambda_N^F| \int \frac{u_F}{v_F} \left(\frac{\partial(u_F/v_F)}{\partial \eta_i} \right)^2 dm \leq C_2 N^{d-1}$, where C_2 is a constant. Now, note that for N big enough, $|\Lambda_N^F| \geq C_2 N^d$ for some constant $C_2 > 0$. Hence we have that $\int \frac{u_F}{v_F} \left(\frac{\partial(u_F/v_F)}{\partial \eta_i} \right)^2 dm \leq C_2 N^{-1}$. Letting $N \rightarrow \infty$ we conclude that $\int \frac{u_F}{v_F} \left(\frac{\partial(u_F/v_F)}{\partial \eta_i} \right)^2 dm = 0$. The smoothness and positivity of u_F and v_F now imply that for every bounded $F \subset \mathbf{Z}^d$ and $i \in F$, it is true that $\partial(u_F/v_F)/\partial \eta_i = 0$. This implies that $\mu = \nu$.

2.2. *Proof of part (ii).* – Let us note that by the convexity property (5), we have $c_n \leq c_m$, whenever $n \leq m$. Hence, by inequality (3) we can conclude that $\sum_{n=1}^N c_n \leq C_3 N^{(d-1)/2} \sqrt{c_N}$, for some constant $C_3 > 0$. An application of Lemma 5.4 of Holley–Stroock [1] shows that $c_n = 0$ for $n \geq 1$ and $d = 1$ or 2. This implies that $\mu = \nu$.

2.3. *Proof of part (iii).* – The following lemma contains the main argument of the proof

LEMMA 1. – *Suppose that ν is the Lebesgue measure. Then for every $m \in \mathbf{Z}$ we have,*

$$c_{n+m} \geq \frac{c_m}{C} \left(1 + \frac{1}{C} \right)^{n-1}, \quad n \geq 1, \quad (6)$$

where $C = B((2R+1)^d + 1)^2/a$, $B = \sup_{i,\eta} |b_i(\eta)|$ and $a = \inf_{i,\eta} a_i(\eta)$.

Before proving this lemma we will show how part (iii) of Theorem 1 is implied by Lemma 1 and inequality (4). Assume that there is an $m \geq 1$ such that $c_m > 0$. Then Lemma 1 implies that the sequence c_{n+m} is bounded from below by an exponentially growing function in n . But this contradicts the polynomial upper bound of inequality (4). Thus, $\partial u_N / \partial \eta_i = 0$ whenever $i \in \Lambda_N$ and $N \geq 1$.

3. Proof of Lemma 1

Define $\delta \lambda_{N,i} := \Lambda_{N,i} \setminus \Lambda_N$ as the subset of $\delta \Lambda_{N+1}$ contained in the ball B_i . Then,

$$\left| \int b_i \frac{1}{u_N} \left(\frac{\partial u_N}{\partial \eta_i} \right) (u_{\Lambda_{N,i}} - u_N) dm \right| \leq \int \frac{1}{u_N} \left| \frac{\partial u_N}{\partial \eta_i} \right| \left| \int b_i (u_{\Lambda_{N,i}} - u_N) dm_{\delta \lambda_{N,i}} \right| dm_{\Lambda_N}. \quad (7)$$

Let us now call $M_{N,i}$ the cardinality of the set $\delta \lambda_{N,i}$. We will label the elements of this set by $j_1, j_2, \dots, j_{M_{N,i}}$. Next, we define a sequence of sets $\Lambda_N \subseteq F_{i,N,k} \subseteq \Lambda_{N,i}$, for $0 \leq k \leq M_{N,i}$, recursively

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as $F_{i,N,k+1} = F_{i,N,k} \cup \{j_{k+1}\}$, where $F_{i,N,0} = \Lambda_N$. Note that $F_{i,N,M_{N,i}} = \Lambda_{N,i}$. Then,

$$\left| \int b_i(u_{\Lambda_{N,i}} - u_N) dm_{\delta\lambda_{N,i}} \right| \leq \sum_{k=0}^{M_{N,i}-1} \left| \int b_i(u_{F_{i,N,k+1}} - u_{F_{i,N,k}}) dm_{F_{i,N,k+1} \setminus \Lambda_N} \right|. \quad (8)$$

Now, since $\int u_{F_{i,N,k+1}} dm_{\{j_{k+1}\}} = u_{F_{i,N,k}}$, we can use Cauchy–Schwartz inequality to deduce that,

$$\left| \int b_i(u_{F_{i,N,k+1}} - u_{F_{i,N,k}}) dm_{F_{i,N,k+1} \setminus \Lambda_N} \right| \leq B\sqrt{u_N} \left(\int \frac{1}{u_{F_{i,N,k+1}}} \left(\frac{\partial u_{F_{i,N,k+1}}}{\partial \eta_{j_{k+1}}} \right)^2 dm_{F_{i,N,k+1} \setminus \Lambda_N} \right)^{1/2}. \quad (9)$$

Using the bound (9) on (8) and substituting this estimate for the right hand side of (7), we can see that

$$\begin{aligned} \left| \int b_i \frac{1}{u_N} \left(\frac{\partial u_N}{\partial \eta_i} \right) (u_{\Lambda_{N,i}} - u_N) dm \right| &\leq B \sum_{k=0}^{M_{N,i}-1} \left(\frac{1}{2} \int \frac{1}{u_N} \left(\frac{\partial u_N}{\partial \eta_i} \right)^2 dm_{\Lambda_N} \right. \\ &\left. + \frac{1}{2} \int \frac{1}{u_{F_{i,N,k+1}}} \left(\frac{\partial u_{F_{i,N,k+1}}}{\partial \eta_{j_{k+1}}} \right)^2 dm_{F_{i,N,k+1}} \right) \leq \frac{B}{2} \sum_{k=0}^{M_{N,i}} \int \frac{1}{u_{F_{i,N,k}}} \left(\frac{\partial u_{F_{i,N,k}}}{\partial \eta_{j_k}} \right)^2 dm_{F_{i,N,k}}. \end{aligned}$$

Now, bounding the absolute value of the right-hand side of (2) by the above estimate and remarking that $M_{N,i} \leq (2R+1)^d$, we conclude that,

$$\begin{aligned} a \sum_{i \in \Lambda_N} \int \frac{1}{u_N^2} \left(\frac{\partial u_N}{\partial \eta_i} \right)^2 d\mu &\leq B \sum_{k=0}^{(2R+1)^d} \sum_{i \in \delta\Lambda_N} \theta_{M_{N,i}}(k) \int \frac{1}{u_{F_{i,N,k}}} \left(\frac{\partial u_{F_{i,N,k}}}{\partial \eta_{j_k}} \right)^2 dm_{\Lambda_{N+1}} \\ &\leq B((2R+1)^d + 1)^2 \sum_{i \in \delta\Lambda_{N+1}} \int \frac{1}{u_{N+1}^2} \left(\frac{\partial u_{N+1}}{\partial \eta_i} \right)^2 d\mu, \end{aligned} \quad (10)$$

where $\theta_y(x) : \mathbf{Z} \rightarrow \{0, 1\}$ equals 0 if $x > y$ and 1 otherwise, and we have used the convexity property (5) in the second inequality. Finally, from (10) and using again the convexity (5) we conclude that $\sum_{n=1}^N c_n \leq Cc_{N+1}$, where $C = B((2R+1)^d + 1)^2/a$. In turn, this inequality implies that $\sum_{n=1}^{N-1} c_{n-1+m} \leq Cc_{N-1+m}$ for $m \geq 1$, $N \geq 2$. Now, it is true that $c_{n-1+m} \geq \bar{c}_n$ where \bar{c}_n is the unique solution of the system of equations $\sum_{n=1}^{N-1} \bar{c}_n = C\bar{c}_N$, $N \geq 2$, with initial condition $\bar{c}_1 = c_m$. The lemma follows from the fact that $\bar{c}_n = \frac{c_m}{C} \left(1 + \frac{1}{C}\right)^{n-2}$, for $n \geq 2$.

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